# Concentration phenomena of solutions to Chemotaxis models on compact metric graphs and related results

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**Abstract.** In this talk, we investigate concentration phenomena of the following stationary chemotaxis model on compact metric graphs as $k \to \infty$:

$$0 = (u_x - ku(\phi(v))_x)_x, \quad 0 = v_{xx} - v + u \quad on \quad \mathcal{G},$$

with the Kirchhoff-Neumann boundary condition:

$$\sum_{e \succ z} \partial u^{(e)}(z) = 0, \quad \sum_{e \succ z} \partial v^{(e)}(z) = 0 \quad (\forall z \in V)$$

and

$$\int_{\mathcal{G}} u\, dx = M,$$

where $\mathcal{G} = (V, E)$ is a compact metric graph, $k$ and $M$ are positive constants, $\phi(v) = v$ or $\phi(v) = \log v$, $\partial u^{(e)}(z)$ means the outward derivative on each edge $e \in E$ incident to the vertex $z \in V$. In the case $\phi(v) = v$, this problem can be reduced to find a solution $v$ of

$$0 = v_{xx}(x) - v(x) + M \frac{e^{kv(x)}}{\int_{\mathcal{G}} e^{kv}(z)\, dz} \quad \text{on} \quad \mathcal{G}$$

with

$$\sum_{e \succ z} \partial v^{(e)}(z) = 0 \quad (\forall z \in V).$$

Here, $u$ can be obtain as

$$u(x) = M \frac{e^{kv(x)}}{\int_{\mathcal{G}} e^{kv}(z)\, dz}.$$

To solve $v$, define the corresponding energy as follows:

$$E_k(w) = \frac{1}{2} \int_{\mathcal{G}} ((w_x)^2 + w^2)\, dx - \frac{M}{k} \log\left( \int_{\mathcal{G}} e^{kw}\, dx \right)$$

for $w \in H^1(\mathcal{G}) := \{u \in C(\mathcal{G}) \mid u^{(e)} = u|_e \in H^1(e)\ (\forall e \in E)\}$. We consider the minimization problem:

$$\sigma(k) := \inf\{E_k(w) \mid w \in H^1(\mathcal{G})\}.$$

To state our main result, for a given $y \in \mathcal{G}$, we define the Green function $G_y(x)$ as the unique solution to

$$-w_{xx} + w = M\delta_y \quad \text{in } \mathcal{G}$$

with

$$\sum_{e \succ z} \partial w^{(e)}(z) = 0 \quad (\forall z \in V).$$

We denote $H(y) := G_y(y)$ $(y \in \mathcal{G})$. Then, we have the following main result.

**Theorem 0.1** *There exists a minimizer $v_k$ to $\sigma(k)$. Furthermore, as $k \to \infty$, we have*

$$\sigma(k) \to -\frac{M}{2} H(y^*),$$

*and, taking a subsequence if necessary,*

$$v_k \to G_{y^*} \quad \textit{in} \quad H^1(\mathcal{G}),$$

*where $H(y^*) := \max_{y \in \mathcal{G}} H(y)$. Moreover, as $u_k(x) = M \frac{e^{kv_k(x)}}{\int_{\mathcal{G}} e^{kv_k(z)}\, dz}$, we have*

$$u_k \to M\delta_{y^*} \quad (k \to \infty),$$

*i.e. $u_k$ satisfies*

$$\int_{\mathcal{G}} u_k \phi \, dx \to M\phi(y^*) \quad (\forall \phi \in H^1(\mathcal{G})).$$

*Namely, $u_k$ concentrates near the maximum point of the function $H(z) = G_z(z)$.*

**Remark 0.1** *In the case $\phi(v) = \log v$, this problem can be reduced to find a solution $v$ of*

$$0 = v_{xx}(x) - v(x) + M \frac{v^k(x)}{\int_{\mathcal{G}} v^k(z)\, dz} \quad \textit{on} \quad \mathcal{G}$$

*with the same Kirchhoff-Neumann boundary condition. For the corresponding least energy solution, a similar concentration phenomena occurs and that $u_k(x) = M \frac{v^k(x)}{\int_{\mathcal{G}} v^k(z)\, dz}$ concentrates near the maximum point of the function $H(z) = G_z(z)$.*

**Remark 0.2** *We also have similar results for other chemotaxis models( e.g. Jäger-Luckhaus model), in which solutions concentrate near the optimization point of the another Green function.*

**Remark 0.3** *For several typical metric graphs $\mathcal{G}$, we can compute the corresponding Green functions $G_y(x)$ on $\mathcal{G}$, respectively, and determine the maximum point of $H(y) = G_y(y)$.*