UNIFORM DISPERSIVE ESTIMATES FOR THE SEMI-CLASSICAL HARTREE EQUATION WITH LONG-RANGE INTERACTION

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0.1. **Setup of the problem.** In this talk, we consider the Hartree equation in three-dimensional space:

$$i\hbar\partial_t\gamma^\hbar = \left[-\frac{\hbar^2}{2}\Delta_x + w * \rho^\hbar(\gamma^\hbar), \gamma^\hbar\right], \quad \gamma^\hbar : \mathbb{R}_{\geq 0} \to \mathcal{B}_{\mathrm{sa}}(L^2(\mathbb{R}^3)), \tag{NLH}$$

where $w : \mathbb{R}^3 \to \mathbb{R}$ is a given function, $\mathcal{B}_{\mathrm{sa}}(L^2(\mathbb{R}^3))$ is the space of all bounded self-adjoint operators on $L^2(\mathbb{R}^3)$, $[A, B] = AB - BA$ is the commutator, and $\hbar \in (0, 1]$ is the reduced Planck constant. We often write $A(x, x')$ for the integral kernel of the linear operator A on $L^2(\mathbb{R}^3)$. We formally define the (semi-classically scaled) density function of A by $\rho^\hbar(A)(x) = (2\pi\hbar)^3 A(x, x)$. It is well-known that the Hartree equation (NLH) appears in many-body problems of quantum mechanics.

The global well-posedness of (NLH) is well-known. For long-time behavior of solutions, a small-data scattering result was given in [3] for short-range interactions. In [3], the optimal decay of the density function of the solution $\gamma^\hbar(t)$

$$\|\rho^\hbar(\gamma^\hbar(t))\|_{L^\infty_x} \lesssim \frac{1}{\langle t\rangle^3} \tag{0.1}$$

is given when $\hbar = 1$ and the initial data is small, where $\langle t\rangle := (1 + |t|^2)^{1/2}$. However, if we directly apply the argument in [3] to general $\hbar \in (0, 1]$, then the smallness condition for initial data gets stronger as $\hbar \to 0$. More precisely, for any $\hbar \in (0, 1]$, we can actually prove (0.1) when the initial data $\gamma_0^\hbar$ is small. However, we immediately find that $\|\gamma_0^\hbar\|_{\mathcal{X}_\hbar} \to 0$ as $\hbar \to 0$ for any acceptable family of initial data $(\gamma_0^\hbar)_{\hbar\in(0,1]}$, where $\|\cdot\|_{\mathcal{X}_\hbar}$ is a natural norm in the semi-classical regime.

The recent work by Smith [4] proved (0.1) for regular interaction around a certain class of translation-invariant stationary solutions for all $\hbar \in (0, 1]$ *when* $\|\gamma_0^\hbar\|_{\mathcal{X}_\hbar} \leq \varepsilon_0$*, where* $\varepsilon_0 > 0$ *is a small but* $\hbar$*–independent constant.* Moreover, Hong and the speaker obtained a similar result in [1] near vacuum when $w(x) = \pm|x|^{-a}$ for $1 < a < 5/3$.

Although both [3] and [1, 4] dealt with short-range interaction w, a modified scattering result for the Coulomb interaction was proved in [2], and the same optimal decay rate (0.1) is given when $\hbar = 1$. In this talk, we prove (0.1) *for long-range interaction* w *when* $\|\gamma_0^\hbar\|_{\mathcal{X}_\hbar} \leq \varepsilon_0$*, where* $\varepsilon_0 > 0$ *is a small but* $\hbar$*–independent constant.*

0.2. **Main result.** Before stating the main result, we prepare some notation. First of all, if we write $A \lesssim B$, then it means $A \leq CB$, where $C > 0$ is a $\hbar$-independent constant. Let $\mathcal{B} = \mathcal{B}(L^2(\mathbb{R}^3))$ be the space of all bounded linear operators. Define the Schatten–r norm by $\|A\|_{\mathfrak{S}^r} = \{\mathrm{Tr}\,(|A|^r)\}^{1/r}$ for $r \in [1, \infty)$ and $\|A\|_{\mathfrak{S}^r} := \|A\|_{\mathcal{B}}$ for $r = \infty$. Moreover, we define the semi-classically scaled Schatten–r norm by $\|A\|_{\mathfrak{S}^r_\hbar} := (2\pi\hbar)^{3/r}\|A\|_{\mathfrak{S}^r}$. Throughout this talk, we use the following class of operators for initial data:

$$\mathcal{X}_\hbar^\sigma := \Big\{ A \in \mathcal{B}(L^2(\mathbb{R}^3)) \mid A \text{ is self-adjoint}, \\ \langle \hbar\nabla\rangle^\sigma A \langle \hbar\nabla\rangle^\sigma \text{ is compact, and } \|A\|_{\mathcal{X}_\hbar^\sigma} < \infty \Big\},$$

where the norm $\|\cdot\|_{\mathcal{X}_\hbar^\sigma}$ is defined by

$$\|A\|_{\mathcal{X}_\hbar^\sigma} := \|A\|_{\mathfrak{S}_\hbar^1} + \sum_{j=0,1} \|\langle \mathcal{D}_j\rangle^\sigma A \langle \mathcal{D}_j\rangle^\sigma\|_{\mathcal{B}} + \sum_{j=0,1} \|\mathfrak{D}_j A\|_{\mathfrak{S}_\hbar^1} + \sum_{j=0,1} \|\langle \mathcal{D}_j\rangle^\sigma \mathfrak{D}_j[A] \langle \mathcal{D}_j\rangle^\sigma\|_{\mathfrak{S}_\hbar^2 \cap \mathcal{B}} \\ + \sum_{j,k,\ell=0,1} \|\langle \mathcal{D}_j\rangle^\sigma \mathfrak{D}_k[\mathfrak{D}_\ell[A]] \langle \mathcal{D}_j\rangle^\sigma\|_{\mathcal{B}}, \tag{0.2}$$

where $\mathcal{D}_0 = \hbar\nabla_x$, $\mathcal{D}_1 := x$ and $\mathfrak{D}_j A := [\mathcal{D}_j/\hbar, A]$ for $j = 0, 1$.

Remark 0.1. The norm $\|\cdot\|_{\mathcal{X}_\hbar^\sigma}$ defined in (0.2) is "natural" in the semi-classical regime.

For interaction $w(x)$, we assume the following condition:

$$w \in C^3(\mathbb{R}^3), \quad \text{and} \quad |\partial^\alpha w(x)| \lesssim \frac{1}{\langle x\rangle^{1+|\alpha|}} \text{ for all } |\alpha| \le 3. \tag{$\mathbf{A}$}$$

Example 0.2. The Coulomb interaction $w(x) = \pm|x|^{-1}$ does not satisfy ($\mathbf{A}$). However, the Coulomb interaction regularized around the origin $w(x) = \pm\langle x\rangle^{-1}$ satisfies ($\mathbf{A}$).

Theorem 0.3. *Let $d = 3$ and $3/2 < \sigma < 2$. Assume that $w(x)$ satisfies* ($\mathbf{A}$). *Then, there exists $\varepsilon_0 > 0$ such that the following holds: If the family of self-adjoint initial data $(\gamma_0^\hbar)_{\hbar\in(0,1]} \subset \mathcal{X}_\hbar^\sigma$ satisfies $\sup_{\hbar\in(0,1]} \|\gamma_0^\hbar\|_{\mathcal{X}_\hbar^\sigma} \le \varepsilon_0$, then, there exists a unique global solution $\gamma^\hbar(t) \in C(\mathbb{R}_\ge; \mathfrak{S}_\hbar^1)$ to* (NLH) *such that*

$$\sup_{\hbar\in(0,1]} \sup_{t\ge 0} \Big\{ \|\rho^\hbar(\gamma^\hbar(t))\|_{L_x^1} + \langle t\rangle^3 \|\rho^\hbar(\gamma^\hbar(t))\|_{L_x^\infty} \Big\} \lesssim 1.$$

Remark 0.4. The solution obtained in Theorem 0.3 exhibits modified scattering. Namely, there exists $\gamma_+^\hbar \in \mathfrak{S}_\hbar^1$ such that

$$\lim_{t\to\infty} \|e^{i\Psi(t,-i\hbar\nabla_x)} \mathcal{U}^\hbar(t)^* \gamma^\hbar(t) \mathcal{U}^\hbar(t) e^{-i\Psi(t,-i\hbar\nabla_x)} - \gamma_+^\hbar\|_{\mathcal{B}} = 0, \tag{0.3}$$

where

$$\Psi(t,\xi) := \frac{1}{\hbar}\int_0^t V(\tau, \tau\xi)\, d\tau, \quad V(t) := w * \rho^\hbar(\gamma^\hbar(t)).$$

However, we could not remove the $\hbar$–dependence from the rate of convergence (0.3).

References

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[4] M. Smith, *Phase mixing for the Hartree equation and Landau damping in the semiclassical limit*, arXiv Preprint (2024), arXiv:2412.14842v2.