EXISTENCE OF SCATTERING STATES FOR WAVE EQUATIONS WITH DISSIPATIVE TERMS IN STRATIFIED MEDIA

Mitsuteru Kadowaki

1. Introduction.

Let $n \geqq 1$ and $x \in \mathbf{R}^n, y \in \mathbf{R}$. In this paper we study the following wave equations with dissipative terms :

$$\partial_t^2 u(x,y,t) + b(x,y)\partial_t u(x,y,t) - c_0(y)^2 \triangle u(x,y,t) = 0, \tag{1.1}$$

where $(x,y,t) \in \mathbf{R}^n \times \mathbf{R} \times [0,\infty)$, $\triangle$ is Laplacian for (x,y),

$$c_0(y) = \begin{cases} c_+ & (y \geqq h) \\ c_h & (0 < y < h) \\ c_- & (y \leqq 0), \end{cases}$$

and c_+, c_-, c_h and h are positive constants. Our aim is to show the existence of the scattering states to (1.1) for $t \to +\infty$. We deal with the following assumption for $b(x,y)$: There exist $C > 0$ and $\theta > 1$ such that

$$0 \leqq b(x,y) \leqq C(1 + |x|^2 + y^2)^{-\frac{\theta}{2}},$$

for any $(x,y) \in \mathbf{R}^n \times \mathbf{R}$.

Considering the case $c_h < \min(c_+, c_-)$ we find the guided waves (cf. Wilcox [8] or Weder [7]). In brief we deal with $c_h < c_+ \leqq c_-$ only.

$L^2(\mathbf{R}^{n+1})$ is the usual L^2 space defined on $\mathbf{R}^{n+1}$. For $s \in \mathbf{R}$, let be $L^2_s(\mathbf{R}^{n+1})$ the weighted L^2 space defined by

$$L^2_s(\mathbf{R}^{n+1}) = \{u(x) : X_{-s}u(x) \in L^2(\mathbf{R}^{n+1})\}, \quad X_{-s} = (1 + |x|^2 + y^2)^{\frac{s}{2}}.$$

Let E and F be Banach spaces. Then we denote by $\|\cdot\|_E$ the norm of E. If T is considered as an operator from E into F, we denote by $\|T\|_{E \to F}$ the operator norm. If in particular T is considered as an operator from $L^2(\mathbf{R}^{n+1})$ into itself, then its norm is denoted by the notation $\|T\|$.

We define the symmetric operator L_0 as :

$$L_0 = -c_0(y)^2 \triangle,$$

Then L_0 admits a unique self-adjoint realizations in $L^2(\mathbf{R}^{n+1}; c_0^{-2}(y)dxdy)$. Then L_0 is a non-negative operator (zero is not an eigenvalue) and the domain of L_0, $D(L_0)$, is given by $H^2(\mathbf{R}^{n+1})$, $H^s(\mathbf{R}^{n+1})$ being Sobolev space of order s over $\mathbf{R}^{n+1}$. We also denoted by $R(\zeta; L_0)$ the resolvent $(L_0 - \zeta)^{-1}$ of L_0 for $Im\zeta \neq 0$.

We remark that Weder [7] has showed the absence of eigenvalues and the limiting absorption principle for L_0.

To state our main results we introduce the free wave equations :

$$\partial_t^2 u(x,y,t) + L_0 u(x,y,t) = 0. \tag{1.2}$$

We put $f = (u(t,x,y), \partial_t u(t,x,y))$. Then (1.1) and (1.2) can be written as $\partial_t f = -iAf$ and $\partial_t f = -iA_0 f$ respectively, where

$$A = i\begin{pmatrix} 0 & 1 \\ -L_0 & -b(x,y) \end{pmatrix}, \qquad A_0 = i\begin{pmatrix} 0 & 1 \\ -L_0 & 0 \end{pmatrix}.$$

Let $\mathcal{H}$ be Hilbert spaces with inner products

$$< f, g >_{\mathcal{H}} = \int_{\mathbf{R}^{n+1}} (\nabla f_1(x,y)\overline{\nabla g_1(x,y)} + f_2(x,y)\overline{g_2(x,y)}c_0^{-2}(y))dxdy,$$

and the corresponding norms $\| \cdot \|_{\mathcal{H}}$ where $\nabla = (\partial/\partial x_1, \partial/\partial x_2, \cdots, \partial/\partial x_n, \partial/\partial y)$ and $f =^t (f_1, f_2), g =^t (g_1, g_2)$.

$\mathcal{H}$ is the energy space for (1.1) and (1.2). Then A and A_0 generate a contraction semi-group $\{V(t)\}_{t \geqq 0}$ and a unitary group $\{U_0(t)\}_{t\in\mathbf{R}}$ on $\mathcal{H}$ respectively. Their domains are

$$D(A_0) = \{f \in \mathcal{H}; \triangle f_1 \in L^2(\mathbf{R}^{n+1}), f_2 \in H^1(\mathbf{R}^{n+1})\}$$

and $D(A) = D(A_0)$. Main results is as follows:

Theorem 1.1. *Assume that $c_+ = c_-$. Then the wave oprators*

$$W = s - \lim_{t\to+\infty} U_0(-t)V(t)$$

exists. Moreover W is not identically zero in $\mathcal{H}$.

Assuming $c_+ = c_-$ we show the existence of scattering states to (1.1) as an immediate consequence of Theorem 1.1.

Considering solutions to (1.1) in any low energy space we find scattering states for the case $c_+ \neq c_-$ too. This results is as follows :

Theorem 1.2. *Assume that $c_+ \neq c_-$. Then the local wave operators*

$$W_l = s - \lim_{t\to+\infty} E_0((-l,l))U_0(-t)V(t)$$

exists for any $l > 0$, where $E_0((-l,l))$ is the spectral projection of A_0 onto $(-l,l)$. Moreover W_l is not identically zero in $\mathcal{H}$.

It seems that there are no works dealing with scattering problem for dissipative wave equations in stratified media. Our proof of Theorem 1.1 and 1.2 are due to Mochizuki [4] (see section 4). He has proved existence of scattering states for wave equations with disspative terms in the case $c_h = c_+ = c_- = 1$. His idea is baesd on the combination Kato's smooth pertabation theory with a resolvent estimates for Laplacian in $\mathbf{R}^n (n \neq 2)$. To consider scattering problem for stratified media, we also need Mochizuki's estimates for L_0. That statement is as follows : There exist $\eta > 0$ and $C_\eta > 0$ such that for any $\zeta \in \mathbf{C}\backslash\mathbf{R}$ satisfing $|Im\zeta| \leqq \eta$

$$\sup_{|Im\zeta|\leqq\eta} \|X_{\frac{\theta}{2}} R(\zeta^2; L_0) X_{\frac{\theta}{2}}\| \leqq C_\eta |\zeta|^{-1}, \tag{1.3}$$

where $X_{\frac{\theta}{2}} = (1 + |x|^2 + y^2)^{-\theta/4}$.
(1.3) follows from low and high resolvent estimates. In section 2 and section 3 the idea of the proof for low and high estimates is given respectively.

2. Low energy resolvent estimates.

The idea is due to Kuroda [3] using the satandard trace operator. We deal with s, only $1/2 < s < 1$. This condition is sufficient to prove Theorem 1.1 and 1.2.

We state Weder's result [14]. That is needed to prove our results. a and b are used as follows :

$$c_+ = c_- \Longrightarrow 0 \leqq a < b < \frac{c_+^2}{h^2(\frac{c_+^2}{c_h^2} - 1)}\pi^2$$

or

$$c_+ \neq c_- \Longrightarrow 0 \leqq a < b < \frac{c_+^2}{h^2(\frac{c_+^2}{c_h^2} - 1)} \arctan^2 \frac{\sqrt{1 - \frac{c_+^2}{c_-^2}}}{\sqrt{\frac{c_+^2}{c_h^2} - 1}}.$$

We write S_r^n as the sphere of center zero and radius in $\mathbf{R}^n$. Then We define

$$S_c = S_+ \cup S_0 \cup S_-,$$

where

$$S_+ = \{\omega \in S^n_{\frac{1}{c_+}} : \omega_0 > (\frac{c_-^2}{c_+^2} - 1)^{1/2}|\overline{\omega}|\}$$

$$S_0 = \{\omega \in S^n_{\frac{1}{c_+}} : 0 < \omega_0 < (\frac{c_-^2}{c_+^2} - 1)^{1/2}|\overline{\omega}|\}$$

$$S_- = \{\omega \in S^n_{\frac{1}{c_-}} : \omega_0 < 0\},$$

$\omega = (\overline{\omega}, \omega_0)$ and $|\overline{\omega}|^2 = \omega_1^2 + \omega_2^2 + \cdots + \omega_n^2$.

According to Lemma 2.3 in Weder [7] there exist trace operators, $T_0(\mu)$ and $T_1(\mu)$, from $L_s^2(\mathbf{R}^{n+1})$ into $L^2(\mathbf{S}_c)$ and $L^2(\mathbf{S}_1^{n-1})$ respectively such that

$$X_s E_0((a,b)) R(\zeta^2; L_0) X_s$$
$$= \begin{cases} \displaystyle\int_a^b \frac{1}{\mu-\zeta^2} X_s T_0^*(\mu) T_0(\mu) X_s d\mu + \int_a^b \frac{1}{\mu-\zeta^2} X_s T_1^*(\mu) T_1(\mu) X_s d\mu, (c_+ = c_-) \\ \displaystyle\int_a^b \frac{1}{\mu-\zeta^2} X_s T_0^*(\mu) T_0(\mu) X_s d\mu, (c_+ \neq c_-), \end{cases}$$

where $\zeta^2 \in \mathbf{C}\backslash\mathbf{R}$, $L^2(\mathbf{S}_c)$ and $L^2(\mathbf{S}_1^{n-1})$ are usual L^2 spaces defined on $\mathbf{S}_c$ and $\mathbf{S}_1^{n-1}$ respectively.

Analyzing $T_0(\mu)$ and $T_1(\mu)$ we obtain the following theorem.

Theorem 2.1. *As* $|\zeta| \to 0$, *one has*

$$\|X_s R(\zeta^2; L_0) X_s\| = \begin{cases} O(|\zeta|^{2s-2}) & (\textit{if} \quad n \geqq 2) \\ O(|\zeta|^{\gamma-1}) & (\textit{if} \quad n = 1), \end{cases}$$

for any $\gamma < s - 1/2$.

3. High energy resolvent estimates.

By a technical reason we deal with only the case $c_+ = c_-$ except for Lemma 3.4 and its introduction.

Theorem 3.1. *Let* $s > 1/2$. *Assume that* $c_+ = c_-$. *Then one has*

$$\|X_s R(\lambda \pm i\kappa; L_0) X_s\| = O(\lambda^{-\frac{1}{2}}) \quad (\lambda \to \infty),$$

uniformly in $\kappa > 0$, *where* $X_s = (1 + |x|^2 + y^2)^{-s/2}$.

It follows from Theorem 3.1 that

Corollary 3.2. *Assume that* $c_+ = c_-$. *Then there exist* $\lambda_1 >> 1, \eta_1 > 0$ *and* $C_{\lambda_1,\eta_1} > 0$ *such that for any* $\zeta \in \mathbf{C}\backslash\mathbf{R}$ *satisfing* $|Re\zeta| \geqq \lambda_1$ *and* $|Im\zeta| \leqq \eta_1$

$$\sup_{|Re\zeta| \geqq \lambda_1, |Im\zeta| \leqq \eta_1} \|X_{\frac{\theta}{2}} R(\zeta^2; L_0) X_{\frac{\theta}{2}}\| \leqq C_{\lambda_1,\eta_1} |\zeta|^{-1}.$$

To prove Theorem 3.1 we define the self-adjoint operator $L_0(\lambda)$ on $L^2(\mathbf{R}^{n+1})$,

$$\begin{cases} L_0(\lambda) = -\triangle - \lambda(c_0^{-2}(y) - c_+^{-2}) \\ D(L_0(\lambda)) = H^2(\mathbf{R}^{n+1}). \end{cases}$$

This operator has been introduced by Weder [7]. Theorem 3.1 is obtained as an immediate consequence of the following proposition

Proposition 3.3. *Assume that* $c_+ = c_-$. *Then one has*

$$\|X_s G_\kappa^{\pm}(0; \lambda) X_s\| = O(\lambda^{-\frac{1}{2}}) \quad (\lambda \to \infty),$$

uniformly in $\kappa > 0$*, where*

$$G_\kappa^\pm(0;\lambda) = (L_0(\lambda) - \lambda c_+^{-2} \mp i\kappa c_0^{-2}(y))^{-1}$$

for $\kappa > 0$.

By using Mourre's method (cf. Mourre's [5]) Proposition 3.3 can be proven. Concering with $L_0(\lambda)$ there exist $F_0(\lambda), F_1(\lambda)$ and $G_j(\lambda)(j = 1, 2, 3 \cdots Q(\lambda))$ which are partially isometric operators from $L^2(\mathbf{R}^{n+1})$ onto $L^2(\mathbf{R}_k^n), L^2(\Omega_0)$ and $L^2(\mathbf{R}_{\overline{k}}^n)$ respectively, where $k = (\overline{k}, k_0) \in \mathbf{R}^n \times \mathbf{R}$, $\Omega_0 = \{k \in \mathbf{R}^{n+1}; 0 < k_0 < \sqrt{q_-(\lambda)} = \sqrt{\lambda(c_+^{-2} - c_-^{-2})}\}$ and $Q(\lambda)$ is finite number in $\mathbf{N}$. Using $F_0(\lambda), F_1(\lambda)$ and $G_j(\lambda)$ we construct the conjugate operators and show Mourre's estimates (3.1) (cf. Mourre [5]). First we define conjugate operator, $D(\lambda)$, as follows (cf. Kadowaki[1]):

$$F_0(\lambda)^*(-D_{n+1})F_0(\lambda) + F_1(\lambda)^*(-D_n)F_1(\lambda) + \sum_{j=1}^{Q(\lambda)} G_j(\lambda)^*(-D_n)G_j(\lambda),$$

where

$$D_{n+1} = \frac{1}{2i}(k \cdot \nabla_k + \nabla_k \cdot k), \quad D_n = \frac{1}{2i}(\overline{k} \cdot \nabla_{\overline{k}} + \nabla_{\overline{k}} \cdot \overline{k}).$$

We consider the commutator $i[L_0(\lambda), D(\lambda)]$ as a form on $H^2(\mathbf{R}^{n+1}) \cap D(D(\lambda))$. Moreover we can extented the commutator as a bounded operators,$i[L_0(\lambda), D(\lambda)]^0$, from $H^1(\mathbf{R}^{n+1})$ to $H^{-1}(\mathbf{R}^{n+1})$ and obtain Mourre's estimates as follows

Lemma 3.4. *Let* $\lambda > 1$*, take* $f_\lambda(r) \in C_0^\infty(\mathbf{R}), 0 \leqq f_\lambda \leqq 1$ *such that* f_λ *has support in* $((c_+^{-2} - c_-^{-2}/2)\lambda, 2c_+^{-2}\lambda)$ *and* $f_\lambda = 1$ *on* $[(c_+^{-2} - c_-^{-2}/4)\lambda, 3c_+^{-2}\lambda/2]$*. Then there exists a positive constant* C *which is independent of* λ *such that*

$$f_\lambda(L_0(\lambda)) i[L_0(\lambda), D(\lambda)]^0 f_\lambda(L_0(\lambda)) \geqq C\lambda f_\lambda(L_0(\lambda))^2 \tag{3.1}$$

in the form sense.

It follows from (4.3) that $f_\lambda(L_0(\lambda))i[L_0(\lambda), D(\lambda)]^0 f_\lambda(L_0(\lambda))$ is non-negative and hence we define an operator, $G_\kappa^\pm(\epsilon;\lambda)$, on $L^2(\mathbf{R}^{n+1})$ by

$$G_\kappa(\epsilon;\lambda) = (L_0(\lambda) - \lambda c_+^{-2} \mp i\kappa c_0^{-2}(y) \mp i\epsilon M_0(\lambda))^{-1}$$

for $\kappa > 0$ and $\epsilon > 0$, where $M_0(\lambda) = f_\lambda(L_0(\lambda))i[L_0(\lambda), D(\lambda)]^0 f_\lambda(L_0(\lambda))$.

We write

$$F_\kappa^\pm(\epsilon;\lambda) = \lambda^{\frac{1}{2}} Z_s(\epsilon, \lambda) G_\kappa^\pm(\lambda^{-\frac{1}{2}}\epsilon;\lambda) Z_s(\epsilon, \lambda),$$

where $Z_s(\epsilon, \lambda) = (\lambda^{\frac{1}{2}} + |D(\lambda)|)^{-s}(\lambda^{\frac{1}{2}} + \epsilon|D(\lambda)|)^{s-1}$.

This is due to Yafaev [9]. But we do not use the scaling argument for λ (see the definition of $G_\kappa^\pm(0;\lambda)$ and $D(\lambda)$). Using (3.1) and differeting $F_\kappa^\pm(\epsilon;\lambda)$ for $\epsilon > 0$ we obtain Proposition 3.3. (cf. Kikuchi-Tamura[2]).

4. Proof of Theorem 1.1 and 1.2.

The proof of Theorem 1.1 and 1.2 are same as in §3 in Mochizuki[4]. Here we give only a sketch of the proof of Theorem 1.1.

proof of Theorem 1.1. Noting that the local boundedness of $R(\zeta'; L_0)$ for $\zeta' \in \mathbf{C}_\pm$(see Weder[14]) we obtain (1.3) by Theorem 2.1 and Corollary 3.2. It follows from (1.3) that

$$\sup_{|Im\zeta| \leqq \eta} \|\Lambda_{\frac{\theta}{2}}(A_0 - \zeta)^{-1}\Lambda_{\frac{\theta}{2}}\|_{\mathcal{H}\to\mathcal{H}} \leqq C_\eta,$$

where C_η is a positive constant which depend on η only and

$$\Lambda_{\frac{\theta}{2}} = \begin{pmatrix} 0 & 0 \\ 0 & (1+|x|^2+y^2)^{-\frac{\theta}{4}} \end{pmatrix}.$$

This means that $\Lambda_{\frac{\theta}{2}}$ is A_0-smooth (cf. Reed-Simon [6]). Therefore we have that $\{U_0(-t)V(t)f\}_{t \geqq 0}$ is Chauchy sequence as $t \to \infty$ in $\mathcal{H}$ for $f \in \mathcal{H}$.

To prove $W \not\equiv 0$ we assume that

$$\lim_{t\to\infty} \|V(t)f\| = 0,$$

for any $f \in \mathcal{H}$. Then we can show a contradiction. □

References

[1] M. Kadowaki, *Low energy resolvent estimstes for acoustic propagators in perturbed stratified media*, J. Differential Equations. **141** (1997), 25-53.

[2] K. Kikuchi - H. Tamura, *The limiting amplilude principle for acoustic propagators in perturbed stratified fluids*, J. Differential Equations. **93** (1991), 260-282.

[3] S.T. Kuroda, *An Introduction to Scattering Theory*, Lecture Note Series N^o 51, Matematisk Institut. Aarhus University, 1980.

[4] K. Mochizuki, *Scattering theory for wave equations with dissipative terms*, Publ. RIMS, Kyoto Univ **12** (1976), 383-390.

[5] E. Mourre, *Absence of singular continuous spectrum for certain self-adjoint operators*, Comm. Math. Phys **78** (1981), 391-408.

[6] M. Reed - B. Simon, *Methods of Modern Mathematical Physics, Vol.3*, Academic Press, New York, San Francisco, London, 1979.

[7] R. Weder, *Spectral and scattering theory for wave propagation in perturbed stratified media*, Applied Mathematical Sciences 87, Springer-Verlag, New york, Berlin, Heidelberg, 1991.

[8] C. Wilcox, *Sound propagation in stratified fluids*, Applied Mathematical Sciences 50, Springer-Verlag, New york, Berlin, Heidelberg, 1984.

[9] D.R. Yafaev, *Eikonal approximation and the aymptotics of the total scattering cross-section for the Schrödinger equation*, Ann. Inst. Henri Poincaré, Phys. théor **44** (1986), 397-425.

Tokyo Metoropolitan College of Aeronautical Eingineering, 8-52-1, Minamisenju, Arakawaku, Tokyo 116-0003 Japan

E-mail address: kadowaki@kouku-k.ac.jp