On thermal convection equations of Oberbeck-Boussinesq type with the dissipation function

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In this presentation we consider the stability of the motionless state and the bifurcation problem in the Rayleigh-Bénard convection problem applying a model equation of the Oberbeck-Boussinesq type including the dissipation function.

The Oberbeck-Boussinesq equation is often used as a simplified model equation for the thermomechanical response of linear viscous fluids undergoing isochoric motions in isothermal processes but not necessarily isochoric ones in non-isothermal processes. Its justification from the point of view of continuum mechanics was given by Rajagopal, Růžička and Srinivasa [1] and there is no doubt concerning the usefulness of the Oberbeck-Boussinesq equation. However, there are some phenomena such as the earth's upper mantle convetcion, convection in fast rotating configurations and etc., in which the Oberbeck-Boussinesq equation seems inappropriate due to the fact that the effect of dissipative heating is not taken into account in the equation. It is thus desirable to derive a model equation including the effect of the dissipative heating.

In [2] we derived a model system of equations including the effect of dissipative heating and investigated the derived system in the context of the Rayleigh-Bénard convection. The system derived in [2] takes the form

$$\begin{aligned} \operatorname{div} \mathbf{v} &= 0, \\ \varepsilon^3(\partial_t \mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} - \tfrac{1}{\sqrt{\mathrm{Gr}}}\Delta\mathbf{v}) + \nabla p &= (1-\varepsilon^3\theta)\mathbf{b}, \\ \partial_t\theta + \mathbf{v}\cdot\nabla\theta - \zeta(\theta + \tfrac{\theta^b+\theta^t}{2(\theta^b-\theta^t)})\mathbf{v}\cdot\mathbf{b} - \tfrac{1}{\mathrm{Pr}\sqrt{\mathrm{Gr}}}\Delta\theta &= \tfrac{2\zeta}{\sqrt{\mathrm{Gr}}}\mathbf{D}(\mathbf{v})\cdot\mathbf{D}(\mathbf{v}). \end{aligned} \tag{1}$$

Here $\mathbf{v} = (v_1, v_2, v_3)$ is the velocity field; p is the pressure; θ is the temperature; $\mathbf{b} = (0, 0, -1)$; Gr , Pr and ζ are positive physical parameters called the Grashof, Prandtl and dissipation numbers, respectively; and $\varepsilon > 0$ is a small non-dimensional parameter. θ^t denotes the temperature at the upper boundary and θ^b at the lower boundary. The function $2\mathbf{D}(\mathbf{v})\cdot\mathbf{D}(\mathbf{v})$ denotes the dissipation function : $2\mathbf{D}(\mathbf{v})\cdot\mathbf{D}(\mathbf{v}) = \frac{1}{2}\sum_{i,j=1}^{3}(\partial_{x_j}v_i + \partial_{x_i}v_j)^2$.

As a first step of the mathematical analysis of (1), we consider the stability of the motionless state in the Rayleigh-Bénard convection. After a suitable change of variables we have the following system in the infinite layer $\{(x_1,x_2,x_3)\in\mathbf{R}^3 \ ; \ 0 < x_3 < 1\}$:

$$\begin{aligned} \operatorname{div}\mathbf{v} &= 0, \\ \partial_t\mathbf{v} - \Delta\mathbf{v} - \lambda\theta\mathbf{e}_3 + \nabla p + \mathbf{v}\cdot\nabla\mathbf{v} &= 0, \\ \partial_t\theta - \tfrac{1}{\mathrm{Pr}}\Delta\theta - \tfrac{\lambda}{\mathrm{Pr}}v_3 + \tfrac{\lambda\zeta}{\mathrm{Pr}}(\Theta - x_3)v_3 + \zeta\theta v_3 + \mathbf{v}\cdot\nabla\theta &= \tfrac{2\zeta}{\lambda}\mathbf{D}(\mathbf{v})\cdot\mathbf{D}(\mathbf{v}), \end{aligned} \tag{2}$$

where $\mathbf{e}_3 = (0,0,1)$; $\Theta = \frac{\theta^b+\theta^t}{2(\theta^b-\theta^t)} + \frac{1}{2}$; and $\lambda > 0$ is defined by $\lambda^2 = \mathrm{R} = \mathrm{Gr}\,\mathrm{Pr}$. Here R is the Rayleigh number. The unknown $\{\mathbf{v}, p, \theta\}$ now denotes the deviation of the velocity, pressure and temperature from the motionless state $\{\bar{\mathbf{v}}, \bar{p}, \bar{\theta}\} = \{\mathbf{0}, -x_3 + \frac{\varepsilon^3}{2}x_3(1-x_3), \frac{1}{2} - x_3\}$. The boundary conditions on $\{x_3 = 0, 1\}$ are then given by

$$\mathbf{v} = \mathbf{0} \quad \text{and} \quad \theta = 0 \quad \text{on} \quad \{x_3 = 0, 1\}.$$

We require $\{\mathbf{v}, p, \theta\}$ to be $\frac{2\pi}{l_j}$-periodic in x_j-direction ($j = 1, 2$). The initial boundary value problem for (2) is now posed under the initial condition

$$\mathbf{v}|_{t=0} = \mathbf{v}_0, \quad \theta|_{t=0} = \theta_0.$$

Notation : We set $\Omega = \mathbf{T}_{l_1,l_2}\times(0,1)$, $\mathbf{T}_{l_1,l_2} = \mathbf{R}^2/(\frac{2\pi}{l_1}\mathbf{Z}\times\frac{2\pi}{l_2}\mathbf{Z})$; $(\cdot,\cdot)$ denotes the scalar product of $L^2(\Omega)$; $H^m(\Omega)$ denotes the m-th order L^2-Sobolev space on Ω.

In the case of the Oberbeck-Boussinesq equation ($\zeta = 0$) the stability of the motionless state is known to be controlled by the critical Rayleigh number $\lambda_c^2 > 0$ which is given by

$$\frac{1}{\lambda_c} \equiv \sup\left\{\frac{2(\mathbf{v}\cdot\mathbf{e}_3,\theta)}{\|\nabla\mathbf{v}\|_2^2 + \|\nabla\theta\|_2^2} \ ; \ \{\mathbf{v},\theta\}\in H_0^1(\Omega)^4 - \{0\}, \ \operatorname{div}\mathbf{v} = 0\right\};$$

and the motionless state is uncondtionally stable if $\lambda < \lambda_c$ and unstable if $\lambda > \lambda_c$.

In case $\zeta > 0$ we show that the motionless state is (conditionally) asymptotically stable even when λ is slightly beyond λ_c for sufficiently small $\zeta > 0$.

Theorem ([2]). *(i) For each* $\{\mathbf{v}_0, \theta_0\} \in H_0^1(\Omega)^3 \times L^2(\Omega)$ *with* $\operatorname{div} \mathbf{v}_0 = 0$ *there exist* $T > 0$ *and a unique solution* $\{\mathbf{v}(t), \theta(t)\}$ *of (2) on* $[0, T]$ *in the class*

$$\mathbf{v} \in C([0,T]; (H_0^1)^3) \cap L^2(0,T; (H^2)^3), \quad \theta \in C([0,T]; L^2) \cap L^2(0,T; H_0^1).$$

(ii) There exist $\zeta_0 > 0$ *and* $\lambda_c(\zeta)$ *such that if* $0 \leq \zeta \leq \zeta_0$ *and* $\lambda < \lambda_c(\zeta)$*, then the motionless state is asymptotically stable, namely, there exists* $\delta > 0$ *such that for each* $\{\mathbf{v}_0, \theta_0\} \in H_0^1(\Omega)^3 \times L^2(\Omega)$ *with* $\operatorname{div} \mathbf{v}_0 = 0$ *and* $\|\mathbf{v}_0\|_{H^1} + \|\theta_0\|_{L^2} < \delta$*, the solution* $\{\mathbf{v}(t), \theta(t)\}$ *exists on* $[0, \infty)$ *and satisfies*

$$\|\mathbf{v}(t)\|_{H^1} + \|\theta(t)\|_{L^2} \leq Ce^{-\alpha t}(\|\mathbf{v}_0\|_{H^1} + \|\theta_0\|_{L^2})$$

for some constants C*,* $\alpha > 0$*. If* $\lambda > \lambda_c(\zeta)$*, then the motionless state is unstable.*

The number $\lambda_c(\zeta)$ *satisfies*

$$\lambda_c(0) = \lambda_c \quad \textit{and} \quad \lambda_c(\zeta) > \lambda_c \quad \textit{for} \quad 0 < \zeta \leq \zeta_0.$$

The next question is that what happens after the motioness state becomes unstable when λ is close to the critical number $\lambda_c(\zeta)$. It is known that in case $\zeta = 0$ various stationary solutions bifurcate from the motionless state at the critical Rayleigh number λ_c. In this case, due to the unconditional stability of the motionless state, bifurcation occurs only supercritically. In contrast to the case $\zeta = 0$, there are some transcritical bifurcation branches when $\zeta > 0$, and, in particular, stationary solutions with hexagonal patterns bifurcate transcritically at $\lambda_c(\zeta)$. The details will be given in the talk.

References

[1] Rajagopal, K. R., Růžička, M., Srinivasa, A. R.: On the Oberbeck-Boussinesq Approximation. Math. Models Methods Appl. Sci. **6**, 1157–1167 (1996).

[2] Kagei, Y., Růžička, M., Thäter, G.: Natural convection with dissipative heating, preprint (1999).