# THE PRINCIPLE OF LIMITING ABSORPTION FOR THE NON-SELFADJOINT SCHRÖDINGER OPERATOR WITH ENERGY DEPENDENT POTENTIAL

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## 1.Introduction

In this paper we shall show the principle of limiting absorption for the quadratic operator pencil

$$L(\kappa) = -\Delta - i\kappa B - \kappa^2 \tag{1.1}$$

in the $N$–dimensional euclidean space $\mathbb{R}^N$ with $N \neq 2$. Here $\Delta$ is the $N$–dimensional Laplacian, $\kappa \in \mathbb{C}$ and $B$ denotes the multiplication operator by a real-valued function $b(x)$ on $\mathbb{R}^N$. If we consider the solutions of the form $w(x,t) = u(x)e^{-i\kappa t}$ in (1.1), then $u(x)$ satisfies

$$L(\kappa)u(x) = (-\Delta - i\kappa b(x) - \kappa^2)u(x) = 0.$$

Throughout the paper we require the following condition:

$$\begin{aligned} &\textit{There exist constants } \ b_0 \ \textit{ and } \ \delta \in (0,1] \ \text{ such that } \ 0 < b_0 < \frac{2-\sqrt{2}}{2}\delta \\ &\textit{and} \quad |b(x)| \leq b_0(1+r)^{-1-2\delta} \quad \textit{for any} \quad x \in \mathbb{R}^N, \quad \textit{where} \quad r = |x|. \end{aligned} \tag{1.2}$$

The operator $L(\kappa)$ is derived from the wave equation

$$w_{tt}(x,t) - \Delta w(x,t) + b(x)w_t(x,t) = 0 \tag{1.3}$$

where $x \in \mathbb{R}^N$, $t \geq 0$, $w_t = \frac{\partial w}{\partial t}$, $w_{tt} = \frac{\partial^2 w}{\partial t^2}$ and $b(x) \in C^1(\mathbb{R}^N)$. If $b(x) \geq 0$, $b(x)w_t$ represents a friction of viscous type. As is known in Mochizuki [14], under a more general assumption than (1.2) with $b(x) \geq 0$, the solutions of (1.3) are asymptotically equal for $t \to +\infty$ to those of free wave equations

$$w_{0tt}(x,t) - \Delta w_0(x,t) = 0. \tag{1.4}$$

More precisely, he proved

Assume $N \neq 2$. If $b(x)$ satisfies $0 \leq b(x) \leq b_0(1+r)^{-1-\delta}$ for some $\delta > 0$, then the following statements hold:

(1) The Møller Weve operators exist:

$$^{\exists}W_+ = \lim_{t \to +\infty} U(t)U_0(-t), \quad ^{\exists}W_- = \lim_{t \to +\infty} U_0(-t)U(t),$$

where $\{U_0(t)\}_{t \in \mathbb{R}}$ and $\{U(t)\}_{t \geq 0}$ represent a one–parameter group of unitary operators and semi–group of contraction operators, respectively (lim means the strong limit in uniform operator topology).

(2) They both are not identically vanishing contraction operators in $\mathcal{H} \equiv \dot{H}^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$.

(3) $U_0(t)$ and $U(t)$ are intertwined by $W_\pm$, i.e.,

$$W_+ U_0(t) = U(t) W_+, \quad W_- U(t) = U_0(t) W_- \quad \text{for any} \quad t \geq 0.$$

(4) The scattering operator, defined by $S = W_- W_+$, commutes with $U_0(t)$:

$$S U_0(t) = U_0(t) S \quad \text{for any} \quad t \in \mathbb{R}.$$

If we consider the solutions of the form $w(x,t) = u(x) e^{-i\kappa t}$ in (1.3), then $u(x)$ satisfies

$$L(\kappa) u(x) = (-\Delta - i\kappa b(x) - \kappa^2) u(x) = 0. \tag{1.5}$$

The principle of limiting absorption states that there exists two Hilbert spaces

$$L^{2,(1+\delta)/2}(\mathbb{R}^N) \quad \text{and} \quad L^{2,-(1+\delta)/2}(\mathbb{R}^N)$$

such that

$$L^{2,(1+\delta)/2}(\mathbb{R}^N) \subset L^2(\mathbb{R}^N) \subset L^{2,-(1+\delta)/2}(\mathbb{R}^N)$$

(the definition of these spaces is given in section 3), and each of $R\left((\sigma \pm i\tau)^2\right) \equiv L(\sigma \pm i\tau)^{-1}$ is continuously extended to $\tau = 0$ as an operator from $L^{2,(1+\delta)/2}(\mathbb{R}^N)$ to $L^{2,-(1+\delta)/2}(\mathbb{R}^N)$. More precisely,

**Main Theorem.** *Assume $N \neq 2$ and (1.2). Let*

$$K_+^+ = \left\{ \kappa = \sigma + i\tau \in \mathbb{C} \setminus \{0\} \,\middle|\, \sigma \in I = (a_1, a_2),\ \tau \in (0,1) \right\}$$

*with $0 < a_1 < a_2 < \infty$. For $\sigma + i\tau \in K_+^+$, and $f \in L^{2,(1+\delta)/2}(\mathbb{R}^N)$ $(0 < \delta \leq 1)$, we define*

$$u(\sigma, \tau) = R\left((\sigma + i\tau)^2\right) f.$$

*Then $u(\sigma,\tau) \in L^{2,-(1+\delta)/2}(\mathbb{R}^N)$ converges in $L^{2,-(1+\delta)/2}(\mathbb{R}^N)$ as $\tau \downarrow 0$. Moreover this convergence is uniform with respect to $\sigma \in I$. If we denote this limit by $u(\sigma)$, then $u(\sigma)$ is a solution of $L(\sigma)u(x) = f(x)$ and is uniformly continuous with respect to $\sigma \in I$ in $L^{2,-(1+\delta)/2}(\mathbb{R}^N)$.*

**Remark.** *In this theorem, we can replace $K_+^+$ by $K_-^+$, $K_+^-$ or $K_-^-$:*

$$\begin{aligned}
K_-^+ &= \left\{ \kappa = \sigma - i\tau \in \mathbb{C} \setminus \{0\} \,\middle|\, \sigma \in I = (a_1, a_2),\ \tau \in (0,1) \right\}, \\
K_+^- &= \left\{ \kappa = \sigma + i\tau \in \mathbb{C} \setminus \{0\} \,\middle|\, \sigma \in I = (-a_2, -a_1),\ \tau \in (0,1) \right\}, \\
K_-^- &= \left\{ \kappa = \sigma - i\tau \in \mathbb{C} \setminus \{0\} \,\middle|\, \sigma \in I = (-a_2, -a_1),\ \tau \in (0,1) \right\}.
\end{aligned}$$

The principle of limiting absorption has been studied for the Schrödinger equation

$$\left(-\Delta + V(x) - \kappa^2\right) u(x) = f(x) \quad \text{in} \quad \mathbb{R}^N. \tag{1.6}$$

After the pioneering work of Éĭdus [2], Agmon [1], Ikebe–Saito [4] and Mochizuki [11] extended it to a wider class of real valued potentials. The spectral theory is also developed for complex potentials by Kato [6], Mochizuki [9], [10] and Saito [14], [15], and Saito extended the principle there to complex potentials.

The methods developed in these works are not directly applied to our problem since $i\kappa b(x)$ depends on the energy $\kappa$. In order to overcome the difficulty we suppose the smallness condition (1.2). Then Mochizuki's method [12] directly applies to obtain the key a-priori estimate ( Theorem 3.4 (3.5) ). From this follows the uniqueness of solutions for $L(\kappa)u(x) = f(x)$ $(\mathrm{Im}\kappa \gtrless 0)$ which plays an important role in the proof of Main Theorem (Corollary 3.5 (1)).

## 2. Spectral Structure for $L(\kappa)$

In this section, for the sake of simplicity, we assume $b(x) \geq 0$. We define the linear operator $H_0$ by $\mathfrak{D}(H_0) = \mathbf{H}^2(\mathbb{R}^N)$, $H_0 = -\Delta$. As is well known, $H_0$ is a selfadjoint operator and it holds that $\rho(H_0) = \mathbb{C} \setminus [0,\infty)$, $\sigma(H_0) = \sigma_{ess}(H_0) = \sigma_c(H_0) = [0,\infty)$, $\sigma_r(H_0) = \sigma_p(H_0) = \emptyset$, where $\rho(H_0)$, $\sigma(H_0)$, $\sigma_{ess}(H_0)$, $\sigma_c(H_0)$, $\sigma_r(H_0)$ and $\sigma_p(H_0)$ denote the resolvent set, spectrum, essential spectrum, continuous spectrum, residual spectrum, and point spectrum, respectively.

For $\kappa \in \mathbb{C} \setminus \{0\}$, we define the linear operator $L(\kappa)$ by $L(\kappa) = H_0 - i\kappa B - \kappa^2$, $\mathfrak{D}(L(\kappa)) = \mathbf{H}^2(\mathbb{R}^N)$. This is a non-selfadjoint but closed operator. The adjoint operator $L^*(\kappa)$ of $L(\kappa)$ is defind by $L^*(\kappa) = H_0 + i\overline{\kappa}B - \overline{\kappa}^2$, $\mathfrak{D}(L^*(\kappa)) = \mathbf{H}^2(\mathbb{R}^N)$.

The resolvent set, the spectrum, the essential spectrum, the continuous spectrum, the residual spectrum and the point spectrum of $L(\kappa)$ are defined as follows ( cf. Markus [7] pp.56, Saito [14] pp.406 ).

$$\rho(L(\kappa)) = \Big\{ \kappa^2 \in \mathbb{C} \;\Big|\; Ker(L(\kappa)) = \{0\},\ \overline{\Re(L(\kappa))} = L^2(\mathbb{R}^N),\ (L(\kappa))^{-1} \in \mathfrak{L}\left(L^2(\mathbb{R}^N)\right) \Big\},$$

$$\sigma(L(\kappa)) = \mathbb{C} \setminus \rho(L(\kappa)),$$

$$\sigma_{ess}(L(\kappa)) = \Big\{ \kappa^2 \in \sigma(L(\kappa)) \;\Big|\; {}^{\exists}\{f_n\} \subset \mathfrak{D}(L(\kappa)) \text{ s.t } \|f_n\|_{L^2} = 1,\ L(\kappa)f_n \underset{S}{\to} 0\ (n\to\infty),\ f_n \underset{w}{\to} 0\ (n\to\infty) \Big\},$$

(where $\underset{S}{\to}$ and $\underset{w}{\to}$ denote the strong and weak convergence in $L^2$ respectively),

$$\sigma_c(L(\kappa)) = \Big\{ \kappa^2 \in \sigma(L(\kappa)) \;\Big|\; Ker(L(\kappa)) = \{0\},\ \overline{\Re(L(\kappa))} = L^2(\mathbb{R}^N),\ (L(\kappa))^{-1} \notin \mathfrak{L}\left(L^2(\mathbb{R}^N)\right) \Big\},$$

$$\sigma_r(L(\kappa)) = \Big\{ \kappa^2 \in \sigma(L(\kappa)) \;\Big|\; Ker(L(\kappa)) = \{0\},\ \overline{\Re(L(\kappa))} \neq L^2(\mathbb{R}^N) \Big\},$$

$$\sigma_p(L(\kappa)) = \big\{ \kappa^2 \in \sigma(L(\kappa)) \;\big|\; Ker(L(\kappa)) \neq \{0\} \big\},$$

where $Ker(L(\kappa)) = \{ f \in \mathfrak{D}(L(\kappa)) \mid L(\kappa)f = 0 \}$, $\Re(L(\kappa))$ is the range of the operator $L(\kappa)$, $\overline{\Re(L(\kappa))}$ denotes the closure of $\Re(L(\kappa))$ in $L^2(\mathbb{R}^N)$, and $\mathfrak{L}\left(L^2(\mathbb{R}^N)\right)$ is the set of the bounded linear operator on $L^2(\mathbb{R}^N)$. For $\kappa^2 \in \rho(L(\kappa))$, we put $R(\kappa^2) = (L(\kappa))^{-1}$ and call it resolvent of $L(\kappa)$. We can define the spectrum sets for $L^*(\kappa)$ similarly as above. As in the case of the usual definition, we see

$$\kappa^2 \in \sigma_r(L(\kappa)) \Leftrightarrow \overline{\kappa}^2 \in \sigma_p(L^*(\kappa)),\ \kappa^2 \notin \sigma_p(L(\kappa)). \tag{2.1}$$

From this we obtain the absence of the residual spectrum of $L(\kappa)$.

In these definitions, noting that $i\kappa B$ is $H_0$– and $(H_0 - i\kappa B)$–compact, we can follow the same line of proof as Schechter [16] Chapter 3 to show the stability theorem of essential spectrum.

Moreover, if we assume $N \neq 2$ and the following condition:

$$\begin{aligned} &\textit{There exist positive constants } b_0 \textit{ and } \delta \text{ such that } 0 < b_0 < \delta \leq 1 \\ &\textit{and} \quad 0 \leq b(x) \leq b_0(1+r)^{-1-2\delta} \quad \textit{for any} \quad x \in \mathbb{R}^N, \end{aligned} \tag{2.2}$$

then the absence of positive eigenvalue of $L(\kappa)$ is obtaind by the modifications of the proof of (3.5) of Theorem 3.4 below. Therefore, we obtain

**Theorem 2.1.** *Assume that $N \neq 2$ and (2.2). Then the following assertions hold:*

$$\sigma(L(\kappa)) = \sigma_{ess}(L(\kappa)) = \sigma_c(L(\kappa)) = [0,\infty),$$
$$\sigma_p(L(\kappa)) = \sigma_r(L(\kappa)) = \emptyset,\ \rho(L(\kappa)) = \mathbb{C} \setminus [0,\infty).$$

## 3. Some Inequalities-Resolvent Estimate

In this section we shall show some inequalities for $L^2$-solutions of the Schrödinger equation of the form

$$-\Delta u - i\kappa b(x)u - \kappa^2 u = f(x) \qquad \text{in} \quad \mathbb{R}^N, \tag{3.1}$$

where $\kappa = \sigma + i\tau\,(\sigma(\neq 0), \tau \in \mathbb{R}) \in \mathbb{C} \setminus \{0\}$ and $f(x) \in L^{2,(1+\delta)/2}(\mathbb{R}^N)$, where

$$L^{2,\alpha}(G) = \left\{ f(x) \;\middle|\; \|f\|_{\alpha,G} = \left( \int_G (1+|x|)^{2\alpha} |f(x)|^2 dx \right)^{\frac{1}{2}} < \infty \right\}.$$

If $G = \mathbb{R}^N$, we shall omit the second symbol $G$.

We define two operators $\mathcal{D}^{\pm}$ and $\mathcal{D}_r^{\pm}$ as follows ( cf. Ikebe–Saito [4], Mochizuki [11], [12], [13], Saito [14], [15] ):

$$\mathcal{D}^{\pm} u = \nabla u + \frac{N-1}{2r} u \frac{x}{r} \mp i\kappa u \frac{x}{r} \quad \left(\mathrm{Im}\kappa \gtreqless 0\right),$$

$$\mathcal{D}_r^{\pm} u = \frac{x}{r} \cdot \mathcal{D}^{\pm} u = u_r + \frac{N-1}{2r} u \mp i\kappa u \quad \left(\mathrm{Im}\kappa \gtreqless 0\right).$$

**Proposition 3.1.** *If $u$ is the solution of* (3.1)*, then the following inequality holds*:

$$\begin{aligned} &\int_{\mathbb{R}^N} (1+r)^{-1-\delta} \left| u_r + \frac{N-1}{2r} u \right|^2 dx + |\kappa|^2 \int_{\mathbb{R}^N} \left\{ (1+r)^{-1-\delta} - \frac{2|b(x)|}{\delta} \right\} |u|^2 dx \\ &\leqq \int_{\mathbb{R}^N} (1+r)^{-1-\delta} |\mathcal{D}_r^{\pm} u|^2 dx + \frac{2}{\delta} \int_{\mathbb{R}^N} |f \overline{i\kappa u}| dx \qquad \left(\mathrm{Im}\kappa \gtreqless 0\right). \end{aligned} \tag{3.2}$$

**Lemma 3.2.** *Let $\varphi = \varphi(r)$ be a real-valued $C^1$ -function. If $u$ is a solution of* (3.1)*, then the following identity holds*:

$$\begin{aligned} &\int_{\mathbb{R}^N} \left\{ \left( \pm\tau\varphi + \frac{\varphi_r}{2} \right) |\mathcal{D}^{\pm} u|^2 + \left( \frac{\varphi}{r} - \varphi_r \right) \left( |\mathcal{D}^{\pm} u|^2 - |\mathcal{D}_r^{\pm} u|^2 \right) \right\} dx \\ &\pm \int_{\mathbb{R}^N} \frac{b(x)\varphi}{2} \left( |\mathcal{D}_r^{\pm} u|^2 + |\kappa|^2 |u|^2 - \left| u_r + \frac{N-1}{2r} u \right|^2 \right) dx \\ &+ \int_{\mathbb{R}^N} \left\{ -\frac{(N-1)(N-3)}{8} \left( \frac{\varphi}{r^2} \right)_r |u|^2 \pm \frac{\tau(N-1)(N-3)}{4r^2} \varphi |u|^2 \right\} dx \\ &= \mathrm{Re} \int_{\mathbb{R}^N} f\varphi \overline{\mathcal{D}_r^{\pm} u} dx \quad \left(\mathrm{Im}\kappa \gtreqless 0\right). \end{aligned} \tag{3.3}$$

From this, we obtain

**Proposition 3.3.** *Assume that $N \neq 2$. If $u$ is a solution of* (3.1)*, then the following inequality holds*:

$$\begin{aligned} &\int_{\mathbb{R}^N} (1+r)^{-1+\delta} |\mathcal{D}^{\pm} u|^2 dx \\ &- \frac{1}{\delta} \int_{\mathbb{R}^N} |b(x)| (1+r)^{\delta} \left( |\mathcal{D}^{\pm} u|^2 + |\kappa|^2 |u|^2 + \left| u_r + \frac{N-1}{2r} u \right|^2 \right) dx \\ &\leq \frac{2}{\delta} \int_{\mathbb{R}^N} (1+r)^{\delta} |f \overline{\mathcal{D}_r^{\pm} u}| dx \qquad \left(\mathrm{Im}\kappa \gtreqless 0\right). \end{aligned} \tag{3.4}$$

By Proposition 3.1 and 3.3, we have

**Theorem 3.4.** *Assume that $N \neq 2$ and (1.2). If $u$ is a solution of (3.1) then there exist positive constants $C_1$ and $C_2$ independent of $\kappa$ such that the following inequalities hold:*

$$|\kappa|\|u\|_{-(1+\delta)/2} \leq C_1\|f\|_{(1+\delta)/2} \quad \left(\mathrm{Im}\kappa \gtreqless 0\right), \tag{3.5}$$

$$\|\mathcal{D}^{\pm}u\|_{(-1+\delta)/2} \leq C_2\|f\|_{(1+\delta)/2} \quad \left(\mathrm{Im}\kappa \gtreqless 0\right). \tag{3.6}$$

From these, we find

**Corollary 3.5.** *Assume that $N \neq 2$ and (1.2).*

(1) *There exists a unique solution $u$ of (3.1) in $L^{2,-(1+\delta)/2}(\mathbb{R}^N)$ for $\mathrm{Im}\kappa \gtreqless 0$.*

(2) *There exists a positive constant $C_3$ independent of $\kappa$ such that the following inequality holds:*

$$|\kappa|^2\|u\|^2_{-(1+\delta)/2,E_R} \leq C_3(1+R)^{-\delta}\|f\|^2_{(1+\delta)/2} \qquad \left(\mathrm{Im}\kappa \gtreqless 0\right), \tag{3.7}$$

*where $E_R = \{x \in \mathbb{R}^N | |x| > R\}$ for $R > 0$.*

From Theorem3.4, Corollary 3.5 and the elliptic estimate for (3.1), we can show Main Theorem.

## 4. The Principle of Limiting Absorption for $i\begin{pmatrix} 0 & 1 \\ \Delta & -B \end{pmatrix}$.

First of all, note that (1.3) with initial data $w(x,0) = w_1$, $w_t(x,0) = w_2$ is equivalent to

$$i\begin{pmatrix} w \\ w_t \end{pmatrix}_t - i\begin{pmatrix} 0 & 1 \\ \Delta & -B \end{pmatrix}\begin{pmatrix} w \\ w_t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} w(0) \\ w_t(0) \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}. \tag{4.1}$$

We define the Hilbert space $\mathcal{H} \equiv \dot{H}^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ ( $\dot{H}^1(\mathbb{R}^N)$ being the Beppo-Levi space ) with inner product

$$(\widetilde{f}, \widetilde{g})_{\mathcal{H}} = \int_{\mathbb{R}^N} \left(\nabla f_1 \cdot \overline{\nabla g_1} + f_2 \cdot \overline{g_2}\right) dx$$

for $\widetilde{f} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$, $\widetilde{g} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in \mathcal{H}$.

If $b(x) \geq 0$ then the solution of (4.1) is given by

$$\begin{pmatrix} w \\ w_t \end{pmatrix} = e^{-iWt}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

where $W = i\begin{pmatrix} 0 & 1 \\ \Delta & -B \end{pmatrix}$ is the generator of the contraction semi-group $\left\{e^{-iWt}\right\}_{t\geq 0}$ on $\mathcal{H}$ (cf. Mochizuki [12] pp.384).

The limiting absorption principle for $W$ can be stated as follows:

**Theorem 4.1.** *Assume that $N \neq 2$, (1.2) and $\widetilde{f} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \mathcal{H}_1 \equiv \dot{H}^1(\mathbb{R}^N) \cap L^{2,(1+\delta)/2}(\mathbb{R}^N) \times L^{2,(1+\delta)/2}(\mathbb{R}^N)\,(\subset \mathcal{H})$. If $\widetilde{u} = \widetilde{u}_{\sigma\pm i\tau} = \begin{pmatrix} u_{1,\sigma\pm i\tau} \\ u_{2,\sigma\pm i\tau} \end{pmatrix}$ is a solution of the equation:*

$$W\widetilde{u} - (\sigma \pm i\tau)\widetilde{u} = \widetilde{f} \tag{4.2}$$

($\sigma \in \mathbb{R} \setminus \{0\}$, $\tau > 0$), *then there exists the limit*

$$\lim_{\tau \downarrow 0} \widetilde{u}_{\sigma \pm i\tau} = \widetilde{u_{\pm}}$$

*in* $\mathcal{H}_2 \equiv H^{1,-(1+\delta)/2}(\mathbb{R}^N) \times L^{2,-(1+\delta)/2}(\mathbb{R}^N)\,(\supset \mathcal{H})$ *and* $\widetilde{u_{\pm}}$ *solves the equation*

$$W\widetilde{u_{\pm}} - \sigma\widetilde{u_{\pm}} = \widetilde{f},$$

*where*

$$H^{1,-(1+\delta)/2}(\mathbb{R}^N) = \left\{ f(x) \,\middle|\, \int_{\mathbb{R}^N} (1+r)^{-1-\delta} \left( |\nabla f(x)|^2 + |f(x)|^2 \right) dx < \infty \right\}.$$

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