REPRESENTATION OF SCHRÖDINGER OPERATOR OF A FREE PARTICLE VIA SHORT-TIME FOURIER TRANSFORM AND ITS APPLICATIONS

KEIICHI KATO, MASAHARU KOBAYASHI, AND SHINGO ITO

ABSTRACT. We propose a new representation of the Schrödinger operator of a free particle by using the short-time Fourier transform and give its applications.

1. INTRODUCTION

We consider the Schrödinger equation of a free particle,

$$\begin{cases} i\partial_t u + \frac{1}{2}\Delta u = 0, & (t,x) \in \mathbb{R}\times\mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{1}$$

where $i = \sqrt{-1}$, $u(t,x)$ is a complex valued function of $(t,x) \in \mathbb{R}\times\mathbb{R}^n$, $u_0(x)$ is a complex valued function of $x \in \mathbb{R}^n$, $\partial_t u = \partial u/\partial t$ and $\Delta u = \sum_{i=1}^n \partial^2 u/\partial x_i^2$.

When u_0 is a function in $\mathcal{S}(\mathbb{R}^n)$, the solution $u(t,x)$ of (1) can be written as

$$u(t,x) = (e^{\frac{1}{2}it\Delta}u_0)(x) = \mathcal{F}^{-1}_{\xi\to x}[e^{-\frac{1}{2}it|\xi|^2}\mathcal{F}u_0(\xi)](x).$$

Here we use the notation $\mathcal{F}f(\xi) = \int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}dx$ for the Fourier transform of f and $\mathcal{F}^{-1}f(x) = \int_{\mathbb{R}^n} f(\xi)e^{ix\cdot\xi}đ\xi$ with $đ\xi = (2\pi)^{-n}d\xi$ for the inverse Fourier transform of f.

The Schrödinger operator $e^{\frac{1}{2}it\Delta}$ and closely related operators such as

$$(e^{i|D|^\alpha}u_0)(x) = \mathcal{F}^{-1}_{\xi\to x}[e^{i|\xi|^\alpha}\mathcal{F}u_0(\xi)](x), \quad \alpha \in \mathbb{R}$$

have been studied extensively by many authors. Hörmander [8] has proved $e^{i|D|^2}$ is bounded on $L^p(\mathbb{R}^n)$ if and only if $p = 2$, and Miyachi [11] has proved the sharp end-point L^p-Sobolev estimates for $e^{i|D|^\alpha}, \alpha > 1$. We also remark that $e^{i|D|^2}$ is bounded on the Besov space $\dot{B}^{p,q}_s(\mathbb{R}^n)$ or $B^{p,q}_s(\mathbb{R}^n)$ if and only if $p = 2$ (Mizuhara [13] and Li [10]). On the other hand, a recent work by Bényi, Gröchenig, Okoudjou and Rogers [1] has shown $e^{i|D|^\alpha}, 0 \le \alpha \le 2$ is bounded on the modulation space $M^{p,q}$ for all $1 \le p,q \le \infty$, which means $e^{\frac{1}{2}it\Delta}$ preserves the $M^{p,q}$-norm (see the precise definition of $M^{p,q}$ in Section 2.2 below). For further developments in this direction we refer to Bényi-Okoudjou [2], Cordero-Nicola [3], Miyachi-Nicola-Rivetti-Tabacco-Tomita [12], Sugimoto [14], Wang-Zhao-Guo [15], Wang-Hudzik [17] and the reference therein.

2000 *Mathematics Subject Classification.* Primary 35C15; Secondary 42B35.

Key words and phrases. Schrödinger operator, Short-time Fourier transform, Modulation space.

In this paper, we propose a new representation of the solution $u(t,x)$ by using the short-time Fourier transform and give its applications. More precisely, let φ_0 be a function in $\mathcal{S}(\mathbb{R}^n)\setminus\{0\}$ and suppose $\varphi(t,x)=(e^{\frac{1}{2}it\Delta}\varphi_0)(x)$, which solves the initial value problem

$$\begin{cases} i\partial_t\varphi+\frac{1}{2}\Delta\varphi=0, \\ \varphi(0,x)=\varphi_0(x). \end{cases} \tag{2}$$

Then we have

$$u(t,x)=\frac{1}{\langle\varphi_0(\cdot),\varphi(t,\cdot)\rangle}V^*_{\varphi(t,\cdot)_{[(y,\xi)\to x]}}[e^{-\frac{1}{2}it|\xi|^2}V_{\varphi_0}u_0(y-\xi t,\xi)](x), \tag{3}$$

where $V_{\varphi_0}u_0$ denotes the short-time Fourier transform of u_0 with respect to the window φ_0 and $V^*_{\varphi(t,\cdot)}$ denotes the (informal) adjoint operator of the short-time Fourier transform $V_{\varphi(t,\cdot)}$ for fixed t, which are defined in Section 2.1.

By using the representation (3), we have the following propositions.

Proposition 1.1. *Let $1\le p,q\le\infty$. Suppose $\varphi_0,\varphi(t,\cdot)\in\mathcal{S}(\mathbb{R}^n)\setminus\{0\}$ satisfy* (2). *Then*

$$\|u(t,\cdot)\|_{M^{p,q}_{\varphi(t,\cdot)}}=\|u_0\|_{M^{p,q}_{\varphi_0}} \tag{4}$$

holds for $u_0\in M^{p,q}(\mathbb{R}^n)$.

Proposition 1.2 (Bényi-Gröchenig-Okoudjou-Rogers [1]). *Let $1\le p,q\le\infty$ and $\varphi_0\in\mathcal{S}(\mathbb{R}^n)\setminus\{0\}$. Then there exists a positive constant C such that*

$$\|u(t,\cdot)\|_{M^{p,q}_{\varphi_0}}\le C(1+|t|)^{n/2}\|u_0\|_{M^{p,q}_{\varphi_0}} \tag{5}$$

for $u_0\in\mathcal{S}(\mathbb{R}^n)$.

Proposition 1.3 (Wang-Hudzik [17]). *Let $2\le p\le\infty, 1\le q\le\infty$ and $\varphi_0\in\mathcal{S}(\mathbb{R}^n)\setminus\{0\}$. Then there exists a positive constant C such that*

$$\|u(t,\cdot)\|_{M^{p,q}_{\varphi_0}}\le C(1+|t|)^{-n(1/2-1/p)}\|u_0\|_{M^{p',q}_{\varphi_0}} \tag{6}$$

for $u_0\in\mathcal{S}(\mathbb{R}^n)$ with $1/p+1/p'=1$.

Proposition 1.4 (Wang-Hudzik [17]). *Let $2\le p\le\infty, 1\le q\le\infty$ and $\varphi_0\in\mathcal{S}(\mathbb{R}^n)\setminus\{0\}$. Then there exists a positive constant C such that*

$$\|u(t,\cdot)\|_{M^{p,q}_{\varphi_0}}\le C(1+|t|)^{n(1/2-1/p)}\|u_0\|_{M^{p,q}_{\varphi_0}} \tag{7}$$

for $u_0\in\mathcal{S}(\mathbb{R}^n)$.

The paper is organized as follows. In Section 2, we recall the definitions and basic properties of the short-time Fourier transform and the modulation spaces. In Section 3, we prove the representation (3). In Section 4, we prove Propositions 1.1 – 1.4. Finally, in Section 5, we give local well-posedness result for the nonlinear Schrödinger equations with Cauchy data in modulation spaces $M^{p,1}$.

2. Preliminaries

Throughout this paper the letter C denotes a constant which may be different in each occasion.

2.1. **The Short-Time Fourier Transform.**

We recall the definitions of the short-time Fourier transform and its adjoint operator. Let $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\phi \in \mathcal{S}(\mathbb{R}^n)$. Then the short-time Fourier transform $V_\phi f$ of f with respect to the window ϕ is defined by

$$V_\phi f(x,\xi) = \langle f(y), \phi(y-x)e^{iy\cdot\xi}\rangle = \int_{\mathbb{R}^n} f(y)\overline{\phi(y-x)}e^{-iy\cdot\xi}dy. \tag{8}$$

Let F be a function on $\mathbb{R}^n \times \mathbb{R}^n$ and $\phi \in \mathcal{S}(\mathbb{R}^n)$. Then the adjoint operator V_ϕ^* of V_ϕ is defined by

$$V_\phi^* F(x) = \iint_{\mathbb{R}^{2n}} F(y,\xi)\phi(x-y)e^{ix\cdot\xi}dy\,đ\xi$$

with $đ\xi = (2\pi)^{-n}d\xi$. It is known that for $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\phi \in \mathcal{S}(\mathbb{R}^n)$, $V_\phi f$ is a continuous function on $\mathbb{R}^n \times \mathbb{R}^n$ and

$$|V_\phi f(x,\xi)| \le C(1+|x|+|\xi|)^N, \quad \forall (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$$

for some constant $C, N \ge 0$ ([7, Theorem 11.2.3]). Moreover, for $\phi, \psi, \gamma \in \mathcal{S}(\mathbb{R}^n)$ satisfying $\langle \psi, \phi\rangle \ne 0$ and $\langle \gamma, \psi\rangle \ne 0$, we have the inversion formula

$$\frac{1}{\langle \psi, \phi\rangle} V_\psi^* V_\phi f = f, \quad f \in \mathcal{S}'(\mathbb{R}^n) \tag{9}$$

([7,Corollary 11.2.7]) and the following pointwise inequality

$$|V_\phi f(x,\xi)| \le \frac{C}{|\langle \gamma, \psi\rangle|}(|V_\psi f| * |V_\phi \gamma|)(x,\xi), \quad f \in \mathcal{S}'(\mathbb{R}^n), \tag{10}$$

for all $(x,\xi) \in \mathbb{R}^{2n}$ ([7, Lemma 11.3.3]).

2.2. **Modulation Spaces.**

We recall the definition of modulation spaces $M^{p,q}$ which were introduced by Feichtinger [5] to measure smoothness of a function or distribution in a way different from Besov spaces. Let $1 \le p, q \le \infty$ and $\phi \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$. Then the modulation space $M_\phi^{p,q}(\mathbb{R}^n) = M^{p,q}$ consists of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that the norm

$$\|f\|_{M_\phi^{p,q}} = \left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} |V_\phi f(x,\xi)|^p dx\right)^{q/p} d\xi\right)^{1/q} = \|V_\phi f(x,\xi)\|_{L_x^p L_\xi^q}$$

is finite (with usual modifications if $p = \infty$ or $q = \infty$).

The space $M_\phi^{p,q}(\mathbb{R}^n)$ is a Banach space, whose definition is independent of the choice of the window ϕ, i.e., $M_\phi^{p,q}(\mathbb{R}^n) = M_\psi^{p,q}(\mathbb{R}^n)$ for all $\phi, \psi \in \mathcal{S}(\mathbb{R}^n)\setminus\{0\}$ ([5, Theorem 6.1]). This property will be crucial in the sequel, since we will choose a suitable window ϕ to estimate the modulation space norm. If $1 \le p, q < \infty$ then $\mathcal{S}(\mathbb{R}^n)$ is dense in $M^{p,q}$ ([5, Theorem 6.1]). We also note $L^2 = M^{2,2}$, and $M^{p_1,q_1} \hookrightarrow M^{p_2,q_2}$ if $p_1 \le p_2, q_1 \le q_2$ ([5, Proposition 6.5]). Let us define by $\mathcal{M}^{p,q}(\mathbb{R}^n)$ the completion of $\mathcal{S}(\mathbb{R}^n)$ under the norm $\|\cdot\|_{M^{p,q}}$. Then

$\mathcal{M}^{p,q}(\mathbb{R}^n) = M^{p,q}(\mathbb{R}^n)$ for $1 \le p, q < \infty$. Moreover, the complex interpolation theory for these spaces reads as follows: Let $0 < \theta < 1$ and $1 \le p_i, q_i \le \infty, i = 1, 2$. Set $1/p = (1-\theta)/p_1 + \theta/p_2$, $1/q = (1-\theta)/q_1 + \theta/q_2$, then $(\mathcal{M}^{p_1,q_1}, \mathcal{M}^{p_2,q_2})_{[\theta]} = \mathcal{M}^{p,q}$ ([5, Theorem 6.1], [16, Theorem 2.3]). We refer to [5] and [7] for more details.

3. Representation of the solution of free Schrödinger equation

In this section, we show that the solution $u(t,x)$ of (1) is represented by (3). Let $u(t,x)$ be the solution of (1) with $u(0,x) = u_0 \in \mathcal{S}(\mathbb{R}^n)$. Note that $u(t,x)$ is in $C^\infty(\mathbb{R}; \mathcal{S}(\mathbb{R}^n))$ in this case. Let $\varphi(t,x)$ be the solution of (2) with $\varphi(0,x) = \varphi_0(x) \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$, which is used as a window function. Using integration by parts, we have

$$
\begin{aligned}
&V_{\varphi(t,\cdot)}\left(\frac{1}{2}\Delta u(t,\cdot)\right)(x,\xi)\\
&= \frac{1}{2}\int_{\mathbb{R}^n} \overline{\varphi(t,y-x)}\Delta_y u(t,y) e^{-iy\cdot\xi} dy\\
&= \int_{\mathbb{R}^n} \overline{\frac{1}{2}\Delta_y \varphi(t,y-x)} u(t,y) e^{-iy\cdot\xi} dy + \int_{\mathbb{R}^n} (-i\xi\cdot\nabla_y)\overline{\varphi(t,y-x)} u(t,y) e^{-iy\cdot\xi} dy\\
&\quad - \frac{1}{2}|\xi|^2 \int \overline{\varphi(t,y-x)} u(t,y) e^{-iy\cdot\xi} dy\\
&= V_{\frac{1}{2}\Delta\varphi(t,\cdot)}(u(t,\cdot))(x,\xi) + \left(i\xi\cdot\nabla_x - \frac{1}{2}|\xi|^2\right) V_{\varphi(t,\cdot)}(u(t,\cdot))(x,\xi).
\end{aligned}
$$

Since $u(t,x)$ and $\varphi(t,x)$ are solutions of (1) and (2), and

$$
i\partial_t V_{\varphi(t,\cdot)}(u(t,\cdot))(x,\xi) = V_{-i\partial_t\varphi(t,\cdot)}(u(t,\cdot))(x,\xi) + V_{\varphi(t,\cdot)}(i\partial_t u(t,\cdot))(x,\xi),
$$

is valid, we obtain

$$
\begin{aligned}
&i\partial_t V_{\varphi(t,\cdot)}(u(t,\cdot))(x,\xi) + \left(i\xi\cdot\nabla_x - \frac{1}{2}|\xi|^2\right) V_{\varphi(t,\cdot)}(u(t,\cdot))(x,\xi)\\
&= V_{\varphi(t,\cdot)}\left(i\partial_t u(t,\cdot) + \frac{1}{2}\Delta u(t,\cdot)\right)(x,\xi) - V_{[i\partial_t\varphi(t,\cdot)+\frac{1}{2}\Delta\varphi(t,\cdot)]}(u(t,\cdot))(x,\xi)\\
&= 0.
\end{aligned}
$$

Hence the initial value problem (1) is transformed via the short-time Fourier transform with window function $\varphi(t,x)$ to

$$
\begin{cases}
(i\partial_t + i\xi\cdot\nabla_x - \frac{1}{2}|\xi|^2) V_{\varphi(t,\cdot)}(u(t,\cdot))(x,\xi) = 0,\\
V_{\varphi(0,\cdot)}(u(0,\cdot))(x,\xi) = V_{\varphi_0}u_0(x,\xi).
\end{cases}
\tag{11}
$$

It is easy to see that

$$
V_{\varphi(t,\cdot)}(u(t,\cdot))(x,\xi) = e^{-\frac{1}{2}it|\xi|^2} V_{\varphi_0}u_0(x-\xi t,\xi)
\tag{12}
$$

is the solution of (11). Applying the adjoint operator $V^*_{\varphi(t,\cdot)}$ of $V_{\varphi(t,\cdot)}$ to the both sides of (12), we have the representation (3) by the inversion formula (9). It is easy to check that the above argument is valid for $u_0(x) \in \mathcal{S}'(\mathbb{R}^n)$.

4. Proof of Propositions

In this section, we prove Propositions 1.1 – 1.4.

Proof of Proposition 1.1. Taking $L^p_x L^q_\xi$ norm of the both sides of (12), we have

$$\begin{aligned}\|u(t,\cdot)\|_{M^{p,q}_{\varphi(t,\cdot)}} &= \|V_{\varphi(t,\cdot)}(u(t,\cdot))(x,\xi)\|_{L^p_x L^q_\xi}\\ &= \|e^{-\frac{1}{2}it|\xi|^2} V_{\varphi_0} u_0(x-\xi t,\xi)\|_{L^p_x L^q_\xi}\\ &= \|V_{\varphi_0} u_0(x,\xi)\|_{L^p_x L^q_\xi}\\ &= \|u_0\|_{M^{p,q}_{\varphi_0}}.\end{aligned}$$

□

Proof of Proposition 1.2. By Proposition 1.1 and the pointwise inequality (10), it suffices to estimate $\|u_0\|_{M^{p,q}_{\varphi_0}}$ above by $C(1+|t|)^{n/2}\|u_0\|_{M^{p,q}_{\varphi(t,\cdot)}}$, where $\varphi_0(x)$ and $\varphi(t,x)$ satisfy equation (2). We note that

$$V_{\varphi_0} u_0 = V_{\varphi_0}\left[\frac{1}{\langle \varphi_0(\cdot), \varphi(t,\cdot)\rangle} V^*_{\varphi_0} V_{\varphi(t,\cdot)} u_0\right]$$

by the inversion formula (9). Then, we have

$$\begin{aligned}&V_{\varphi_0} u_0(x,\xi)\\ &= V_{\varphi_0[y\to(x,\xi)]}\left[\frac{1}{\langle \varphi_0(\cdot), \varphi(t,\cdot)\rangle}\iint_{\mathbb{R}^{2n}} V_{\varphi(t,\cdot)} u_0(z,\eta)\varphi_0(y-z)e^{iy\cdot\eta}dz đ\eta\right](x,\xi)\\ &= \frac{1}{\langle \varphi_0(\cdot), \varphi(t,\cdot)\rangle}\iiint_{\mathbb{R}^{3n}} \overline{\varphi_0(y-x)}e^{-iy\cdot\xi} V_{\varphi(t,\cdot)} u_0(z,\eta)\varphi_0(y-z)e^{iy\cdot\eta}dydz đ\eta\\ &= \frac{1}{\langle \varphi_0(\cdot), \varphi(t,\cdot)\rangle}\iint_{\mathbb{R}^{2n}}\left(\int_{\mathbb{R}^n}\overline{\varphi_0(y-x)}\varphi_0(y-z)e^{-i(\xi-\eta)\cdot y}dy\right) V_{\varphi(t,\cdot)} u_0(z,\eta)dz đ\eta\\ &= \frac{1}{\langle \varphi_0(\cdot), \varphi(t,\cdot)\rangle}\iint_{\mathbb{R}^{2n}} V_{\varphi_0}\varphi_0(x-z,\xi-\eta)e^{-i(\xi-\eta)\cdot z} V_{\varphi(t,\cdot)} u_0(z,\eta)dz đ\eta.\end{aligned}$$

By Young's inequality, we have

$$\|u_0\|_{M^{p,q}_{\varphi_0}} \le \frac{C}{|\langle \varphi_0(\cdot), \varphi(t,\cdot)\rangle|}\|\varphi_0\|_{M^{1,1}_{\varphi_0}}\|u_0\|_{M^{p,q}_{\varphi(t,\cdot)}}.$$

Since

$$|\langle \varphi_0(\cdot), \varphi(t,\cdot)\rangle| = |\langle \widehat{\varphi_0}, e^{-\frac{1}{2}it|\xi|^2}\widehat{\varphi_0}\rangle| = \left|\int_{\mathbb{R}^n} e^{\frac{1}{2}it|\xi|^2}|\widehat{\varphi_0}(\xi)|^2 d\xi\right|,$$

the stationary phase method yields that

$$|\langle \varphi_0(\cdot), \varphi(t,\cdot)\rangle| \sim C|\widehat{\varphi_0}(0)|^2|t|^{-n/2} \quad (\text{as } |t|\to\infty).$$

Hence we have (5). $\square$

Proof of Proposition 1.3. Firstly, we prove (6) for $p = 2$. By Proposition 1.1, it suffices to show that $\|V_{\varphi_0}u_0(x,\xi)\|_{L^2_x L^q_\xi} = \|V_{\varphi(t,\cdot)}u_0(x,\xi)\|_{L^2_x L^q_\xi}$. By Plancherel theorem, we have

$$\begin{aligned}
\|V_{\varphi(t,\cdot)}u_0(x,\xi)\|_{L^2_x L^q_\xi} &= \left\| \left\| \int_{\mathbb{R}^n} \overline{\varphi(t,y-x)} u_0(y) e^{-iy\cdot\xi} dy \right\|_{L^2_x} \right\|_{L^q_\xi} \\
&= \left\| \left\| \int_{\mathbb{R}^n} \overline{\widehat{\varphi}(t,\eta)} e^{-iy\cdot\eta} u_0(y) e^{-iy\cdot\xi} dy \right\|_{L^2_\eta} \right\|_{L^q_\xi} \\
&= \left\| \left\| \int_{\mathbb{R}^n} e^{\frac{1}{2}it|\eta|^2} \overline{\widehat{\varphi_0}(\eta)} e^{-iy\cdot\eta} u_0(y) e^{-iy\cdot\xi} dy \right\|_{L^2_\eta} \right\|_{L^q_\xi} \\
&= \left\| \left\| \int_{\mathbb{R}^n} \overline{\widehat{\varphi_0}(\eta)} e^{-iy\cdot\eta} u_0(y) e^{-iy\cdot\xi} dy \right\|_{L^2_\eta} \right\|_{L^q_\xi} \\
&= \left\| \left\| \int_{\mathbb{R}^n} \overline{\varphi_0(y-x)} u_0(y) e^{-iy\cdot\xi} dy \right\|_{L^2_x} \right\|_{L^q_\xi} \\
&= \|V_{\varphi_0}u_0(x,\xi)\|_{L^2_x L^q_\xi}.
\end{aligned}$$

Secondly, we prove (6) for $p = \infty$. The same argument as in the proof of Proposition 1.2, we have

$$\begin{aligned}
&\|u_0\|_{M^{\infty,q}_{\varphi_0}} \\
&= \left\| V_{\varphi_0} \left[\frac{1}{\|\varphi(t,\cdot)\|^2_{L^2}} V^*_{\varphi(t,\cdot)} V_{\varphi(t,\cdot)} u_0(x,\xi) \right] \right\|_{L^\infty_x L^q_\xi} \\
&= \frac{1}{\|\varphi_0\|^2_{L^2}} \left\| \iint_{\mathbb{R}^{2n}} V_{\varphi_0}(\varphi(t,\cdot))(x-z,\xi-\eta) e^{-iz\cdot(\xi-\eta)} V_{\varphi(t,\cdot)}(u(t,\cdot))(z,\eta) dz \bar{d}\eta \right\|_{L^\infty_x L^q_\xi} \\
&\le \frac{C}{\|\varphi_0\|^2_{L^2}} \|V_{\varphi_0}(\varphi(t,\cdot))(x,\xi)\|_{L^\infty_x L^1_\xi} \|V_{\varphi(t,\cdot)}(u(t,\cdot))(x,\xi)\|_{L^1_x L^q_\xi} \\
&= \frac{C}{\|\varphi_0\|^2_{L^2}} \|\varphi(t,\cdot)\|_{M^{\infty,1}_{\varphi_0}} \|u_0\|_{M^{1,q}_{\varphi_0}}.
\end{aligned}$$

Since

$$\begin{aligned}
V_{\varphi_0}(\varphi(t,\cdot))(x,\xi) &= \int_{\mathbb{R}^n} \overline{\varphi_0(y-x)} \varphi(t,y) e^{-iy\cdot\xi} dy \\
&= e^{-ix\cdot\xi} \int_{\mathbb{R}^n} \widehat{\varphi}(t,\eta) \overline{\widehat{\varphi_0}(\eta-\xi)} e^{ix\cdot\eta} d\eta \\
&= e^{-ix\cdot\xi} \int_{\mathbb{R}^n} e^{-\frac{1}{2}it|\eta|^2} \widehat{\varphi_0}(\eta) \overline{\widehat{\varphi_0}(\eta-\xi)} e^{ix\cdot\eta} d\eta,
\end{aligned}$$

the stationary phase method (see [6], [9]) yields that

$$\begin{aligned}&|V_{\varphi_0}(\varphi(t,\cdot))(x,\xi)|\\&\le C|t|^{-\frac{n}{2}}\left|\widehat{\varphi_0}\left(-\frac{x}{t}\right)\widehat{\varphi_0}\left(-\frac{x}{t}-\xi\right)\right|\\&+C|t|^{-n/2-1}\sum_{|\alpha|\le 2(1+n)}\int_{\mathbb{R}^n}\left|\left(\frac{\partial}{\partial\eta}\right)^{\alpha}\left[\widehat{\varphi_0}(\eta)\overline{\widehat{\varphi_0}(\eta-\xi)}\right]\right|d\eta \quad (\text{as } |t|\to\infty).\end{aligned}$$

Hence we have (6) for $p=\infty$. By the complex interpolation method, we have (6) for all $1\le p\le\infty$.

□

Proof of Proposition 1.4. By using the complex interpolation method between $p=\infty$ for (5) and $p=2$ for (6), we have the conclusion.

□

5. NONLINEAR SCHRÖDINGER EQUATION

Next we consider the following initial value problem of the nonlinear Schrödinger equation,

$$\begin{cases} i\partial_t u+\frac{1}{2}\Delta u=f(u),\\ u(0,x)=u_0(x),\end{cases} \tag{13}$$

where $f(u)$ is a polynomial of u and $\overline{u}$ with $f(0)=0$.

The following result is already known, but we obtain it as a corollary of our representation (3).

PROPOSITION 5.1 (Bényi-Okoudjou [2]). *For $u_0\in M^{p,1}(\mathbb{R}^n)$ with $1\le p\le\infty$, there exists a positive constant T and a unique solution of* (13) *such that $u\in C([0,T];M^{p,1}(\mathbb{R}^n))$.*

PROOF. Using the representation (3) of the Schrödinger operator $e^{\frac{1}{2}it\Delta}$, we have the integral equation associated to (13),

$$\begin{aligned}&V_{\varphi(t,\cdot)}(u(t,\cdot))(x,\xi)\\&=e^{-\frac{1}{2}it|\xi|^2}V_{\varphi_0}u_0(x-t\xi,\xi)+\int_0^t e^{-\frac{1}{2}i(t-s)|\xi|^2}V_{\varphi(s,\cdot)}[f(u)](x-(t-s)\xi,\xi)ds.\end{aligned}$$

We recall that $M^{p,1}$ is a Banach algebra, i.e., for $\phi\in\mathcal{S}(\mathbb{R}^n)\setminus\{0\}$, there exists a positive constant C such that

$$\|u_1u_2\|_{M^{p,1}_\phi}\le C\|u_1\|_{M^{p,1}_\phi}\|u_2\|_{M^{p,1}_\phi} \tag{14}$$

for all $u_1,u_2\in M^{p,1}(\mathbb{R}^n)$ ([2, Corollary 2.7], [15, Corollary 4.2]).

We define the mapping $F(u)$ from $C([0,T];M^{p,1}(\mathbb{R}^n))$ to itself as follows:

$$F(u) = V^*_{\varphi(t,\cdot)}\left[e^{-\frac{1}{2}it|\xi|^2}V_{\varphi_0}u_0(x-t\xi,\xi) + \int_0^t e^{-\frac{1}{2}i(t-s)|\xi|^2}V_{\varphi(s,\cdot)}[f(u)](x-(t-s)\xi,\xi)ds\right].$$

Putting $A = \|u_0\|_{M^{p,1}_{\varphi_0}}$ and $X_T = C([0,T];M^{p,1}(\mathbb{R}^n))$, we define a closed subspace $X_{T,A}$ of $C([0,T];M^{p,1}(\mathbb{R}^n))$ as follows:

$$X_{T,A} = \left\{u \in C([0,T];M^{p,1}(\mathbb{R}^n)) \;\middle|\; \|u\|_{X_T} = \sup_{t\in[0,T]}\|u(t,\cdot)\|_{M^{p,1}_{\varphi(t,\cdot)}} \le 2A\right\}.$$

The mapping F is well defined on $X_{T,A}$ for small $T>0$. In fact, the above fact (14) for multiplication on $M^{p,1}(\mathbb{R}^n)$ yields that

$$\|F(u)\|_{X_T} \le \|u\|_{M^{p,1}} + \int_0^T C(s)\widetilde{f}(\|u\|_{M^{p,1}_{\varphi(s,\cdot)}})ds,$$

where $C(s)$ is a positive continuous function of s and $\widetilde{f}(u)$ is a polynomial of u and $\overline{u}$ which is made from $f(u)$ replacing all the coefficients to their absolute value. Hence we have

$$\|F(u)\|_{X_T} \le A + \widetilde{f}(A)C_1T$$

with $C_1 = \sup_{s\in[0,T]}C(s)$, which implies $F(u) \in X_{T,A}$ for small $T>0$.

The same argument as above yields that F is a contraction mapping from $X_{T,A}$ to itself for small $T>0$. Picard's fixed point theorem for a contraction mapping on $X_{T,A}$ implies the conclusion.

□

References

[1] Á Bényi, K. Gröchenig, K. Okoudjou and L.G. Rogers, *Unimodular Fourier multipliers for modulation spaces,* J. Funct. Anal. 246 (2007), 366-384.

[2] Á. Bényi and K. Okoudjou, *Local well-posedness of nonlinear dispersive equations on modulation spaces,* Bull. London Math. Soc. 41(3) (2009), 549-558.

[3] E. Cordero and F. Nicola, *Sharpness of some properties of Wiener amalgam and modulation spaces,* Bull. Aust. Math. Soc. 80 (2009), 105-116.

[4] A. Córdoba and C. Fefferman, *Wave packets and Fourier integral operators,* Comm. partial Differential Equations, 3, (1978), 979-1005.

[5] H.G. Feichtinger, *Modulation spaces on locally compact abelian groups*, in: M. Krishna, R. Radha and S. Thangavelu (Eds.), Wavelets and their Applications, Chennai, India, Allied Publishers, New Delhi, 2003, pp. 99-140, Updated version of a technical report, University of Vienna, 1983.

[6] D. Fujiwara, *Asymptotic methods in Linear Partial Differential Equations*, Iwanami, Tokyo, (1977) (in Japanese).

[7] K. Gröchenig, *Foundations of Time-Frequency Analysis,* Birkhäuser, 2001.

[8] L. Hörmander, *Estimates for translation invariant operators in L^p-spaces,* Acta Math. 104 (1960), 93-140.

[9] L. Hörmander, *The analysis of linear partial differential operators. I. Distribution theory and Fourier analysis.* Springer-Verlag, Berlin, 1983.

[10] L. Li, *Some remarks on a multiplier theorem of Hörmander,* Math. Appl. (Wuhan) 15 (2002), 152-154.

[11] A. Miyachi, *On some Fourier multipliers for* $H^p(R^n)$, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 27 (1980), 157–179.

[12] A. Miyachi, F. Nicola, S. Rivetti, A. Tabacco and N. Tomita, *Estimates for unimodular Fourier multipliers on modulation spaces*, Proc. Amer. Math. Soc. 137 (2009), 3869–3883.

[13] T. Mizuhara, *On Fourier multipliers of homogeneous Besov spaces,* Math. Nachr. 133 (1987), 155-161.

[14] M. Sugimoto, *On some Lp-estimates for hyperbolic equations,* Ark.Mat. 30 (1992), 149-163.

[15] B. Wang, L. Zhao, B. Guo, *Isometric decomposition operators, function spaces* $E^{\lambda}_{p,q}$ *and applications to nonlinear evolution equations,* J. Funct. Anal. 233 (2006) 1-39.

[16] B. Wang and C. Huang, *Frequency-uniform decomposition method for the generalized BO, KdV and NLS equations,* J. Differential Equations 239 (2007), 213-250.

[17] B. Wang and H. Hudzik, *The global Cauchy problem for the NLS and NLKG with small rough data*, J. Differential Equations 232 (2007), 36-73.

Department of Mathematics
Tokyo University of Science
Kagurazaka 1-3, Shinjuku-ku
Tokyo 162-8601, Japan
E-mail address: kato@ma.kagu.tus.ac.jp

Department of Mathematics
Tokyo University of Science
Kagurazaka 1-3, Shinjuku-ku
Tokyo 162-8601, Japan
E-mail address: kobayashi@jan.rikadai.jp

Department of Mathematics
Tokyo University of Science
Kagurazaka 1-3, Shinjuku-ku
Tokyo 162-8601, Japan
E-mail address: ito@ma.kagu.tus.ac.jp