Characterization of wave front sets in Fourier-Lebesgue spaces and its application

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Abstract

In this paper, we characterize the Fourier-Lebesgue type wave front set by using the wave packet transform. We apply this to the propagation of singularities for the first order hyperbolic partial differential equations with constant coefficient.

Key Words and Phrases. Wave front set, Fourier-Lebesgue space, Propagation of singularity.

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1 Introduction

The notion of wave front set, introduced by Hörmander [9], is a main tool of microlocal analysis. There are many kinds of wave front sets such as C^∞ type, analytic type, Sobolev type, Fourier-Lebesgue type and so on (see, for example, Sato-Kawai-Kashiwara [18], Hörmander [10], [11], Trèves [19], Pilipović-Teofanov-Toft [15], [16]). In this paper, we focus on the Fourier-Lebesgue type wave front sets.

For $1 \le q \le \infty$ and $s \in \mathbb{R}$, the Fourier-Lebesgue space $\mathcal{F}L^q_s(\mathbb{R}^n)$ is the set of all distributions $u \in \mathcal{S}'(\mathbb{R}^n)$ such that $\widehat{u}(\xi) = \int_{\mathbb{R}^n} u(x)e^{-ix\cdot\xi}dx$ is a function and $\|\langle\xi\rangle^s \widehat{u}(\xi)\|_{L^q_\xi} < \infty$ (see [8, Definition 2.2.1]). In the last several years, the Fourier-Lebesgue spaces have been studied extensively (Cordero-Nicola-Rodino [2], Okoudjou [14], Ruzhansky-Sugimoto-Toft-Tomita [17]). The Fourier-Lebesgue type wave front set $WF_{\mathcal{F}L^q_s}$ is defined as follows.

Definition 1.1. (Pilipović-Teofanov-Toft [15]) Let $1 \le q \le \infty$, $s \in \mathbb{R}$, $(x_0, \xi_0) \in \mathbb{R}^n \times (\mathbb{R}^n \backslash \{0\})$ and $u \in \mathcal{S}'(\mathbb{R}^n)$. Then $(x_0, \xi_0) \notin WF_{\mathcal{F}L^q_s}(u)$ means that there exist a conic neighborhood Γ of ξ_0 and a function $a \in C_0^\infty(\mathbb{R}^n)$ with $a(x_0) \neq 0$ satisfying that

$$\|\chi_\Gamma(\xi)\langle\xi\rangle^s \widehat{au}(\xi)\|_{L^q_\xi} < \infty, \tag{1}$$

where $\langle\xi\rangle = (1 + |\xi|^2)^{1/2}$ and χ_Γ is a characteristic function of Γ.

For $u \in \mathcal{S}'(\mathbb{R}^n)$, $WF_{\mathcal{F}L^2_s}(u)$ coincides with Sobolev type wave front set $WF_{H^s}(u)$. Pilipović, Teofanov and Toft have shown that $WF_{\mathcal{F}L^q_s}(u)$ coincides with modulation type wave front set $WF_{M^{p,q}_s}(u)$ ([15, Proposition 6]). We refer to Feichtinger [4] and Gröchenig [7] for the definition and basic properties on modulation spaces.

On the other hand, the wave packet transform has been introduced by Córdoba-Fefferman [1]. Let $u \in \mathcal{S}'(\mathbb{R}^n)$ and $\phi \in \mathcal{S}(\mathbb{R}^n)$ with $\phi(0) \neq 0$. The wave packet transform $W_\phi u$ of u with respect to ϕ is defined by

$$W_\phi u(x,\xi) = \int_{\mathbb{R}^n} \overline{\phi(y-x)} u(y) e^{-iy\cdot\xi} dy, \tag{2}$$

which has the information of frequency of u around x. Folland [5, Theorem 3.22] and Ōkaji [13, Theorem 2.2] gave a characterization of C^∞ type wave front set and Sobolev type wave front set in terms of the wave packet transform, respectively. In Gérard [6, Proposition 1.1], similar a characterization of Sobolev type wave front set is given by using the FBI transform (see also Delort [3, Theorem 1.2]). By using the wave packet transform, we give a characterization of the Fourier-Lebesgue type wave front set.

Proposition 1.2. *Let $1 \leq p, q \leq \infty$, $s \in \mathbb{R}$, $(x_0, \xi_0) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ and $u \in \mathcal{S}'(\mathbb{R}^n)$. The following conditions are equivalent.*
(i) $(x_0, \xi_0) \notin WF_{\mathcal{F}L^q_s}(u)$
(ii) There exist a neighborhood K of x_0, a conic neighborhood Γ of ξ_0 and a function $\phi \in C_0^\infty(\mathbb{R})$ with $\phi(0) \neq 0$ satisfying that

$$\| \| \chi_K(x) \chi_\Gamma(\xi) \langle \xi \rangle^s W_\phi u(x,\xi) \|_{L^p_x} \|_{L^q_\xi} < \infty. \tag{3}$$

Using the above proposition, we obtain the following theorem concerning the propagation of singularity in the framework of Fourier-Lebesgue type wave front set.

Theorem 1.3. *Let $1 \leq q \leq \infty$, $r \in \mathbb{R}$. Suppose that $u \in C(\mathbb{R}; \mathcal{S}'(\mathbb{R}^n))$ satisfies*

$$\begin{cases} (\partial_t \pm i|D|)u(t,x) = 0, & (t,x) \in \mathbb{R}^{n+1}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{4}$$

where $i = \sqrt{-1}$ and $|D| = \mathcal{F}^{-1}|\xi|\mathcal{F}$. If $(x_0, \xi_0) \notin WF_{\mathcal{F}L^q_r}(u_0)$, then $(x_0 \pm \frac{\xi_0}{|\xi_0|}t, \xi_0) \notin WF_{\mathcal{F}L^q_r}(u(t,\cdot))$ for all $t \in \mathbb{R}$.

Remark 1.4. (i) Let $a(\xi) \in C^\infty(\mathbb{R}^n \setminus \{0\})$ be a homogeneous function of degree 1, i.e., $a(\lambda\xi) = \lambda a(\xi)$ for $\lambda > 0$. For the equation $(\partial_t \pm ia(D))u(t,x) = 0$, the conclusion of Theorem 1.3 holds if we replace $(x_0 \pm \frac{\xi_0}{|\xi_0|}t, \xi_0) \notin WF_{\mathcal{F}L^q_r}(u(t,\cdot))$ with $(x_0 \pm \nabla a(\xi_0)t, \xi_0) \notin WF_{\mathcal{F}L^q_r}(u(t,\cdot))$ (refer to Appendix A).

(ii) Since the wave operator is factorized as $\partial_t^2 - \sum_{i=1}^n \partial_{x_i}^2 = (\partial_t - i|D|)(\partial_t + i|D|)$, we can apply Theorem 1.3 to the wave equation.

This paper is organized as follows. In Section 2, we prepare several lemmas to prove Proposition 1.2 and Theorem 1.3. In Section 3, we prove Proposition 1.2. In Section 4, we prove Theorem 1.3.

Notation. For $x \in \mathbb{R}^n$ and $r > 0$, $B_r(x)$ stands $\{y \in \mathbb{R}^n; |y-x| \le r\}$. $\mathcal{F}[f](\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}dx$ is the Fourier transform of f. For a subset A of $\mathbb{R}^n$, we denote the complement of A by A^c, the set of all interior points of A by A° and the closure of A by $\overline{A}$. χ_A is the characteristic function of A, that is, $\chi_A(x) = 1$ for $x \in A$ and $\chi_A(x) = 0$ for $x \in A^c$. Throughout this paper, C and C_i $(i = 1, 2, 3, \ldots)$ serve as positive constants, if the precise value of which is not needed and C_N denote positive constants depending on N.

2 Preliminaries

In this section, we give some lemmas to prove Proposition 1.2 and Theorem 1.3.

Lemma 2.1. *Let ζ be a measurable function on $\mathbb{R}^n$ such that $\langle\cdot\rangle^k\zeta \in L^1(\mathbb{R}^n)$ for all $k \in \mathbb{R}$, $F \in \mathcal{S}'(\mathbb{R}^n)$, $1 \le q \le \infty$, and Γ, Γ' be open conic sets satisfying $\overline{\Gamma'} \subset \Gamma \subset \mathbb{R}^n$. Assume that $\|\chi_\Gamma(\xi)\langle\xi\rangle^s F(\xi)\|_{L^q_\xi} < \infty$ and $\|\langle\xi\rangle^{-N}F(\xi)\|_{L^q_\xi} < \infty$ for some $s \in \mathbb{R}$ and $N \in \mathbb{N}$. Then we have*

$$\|\chi_{\Gamma'}(\xi)\langle\xi\rangle^s(\zeta * F)(\xi)\|_{L^q_\xi} \le C_{s,N,\zeta}\left(\|\chi_\Gamma(\xi)\langle\xi\rangle^s F(\xi)\|_{L^q_\xi} + \left\|\frac{F(\xi)}{\langle\xi\rangle^N}\right\|_{L^q_\xi}\right) \tag{5}$$

for some positive constant $C_{s,N,\zeta}$.

Proof. To prove (5), we set

$$\begin{aligned}&\chi_{\Gamma'}(\xi)\langle\xi\rangle^s(\zeta * F)(\xi)\\ &\quad= \chi_{\Gamma'}(\xi)\langle\xi\rangle^s\int_\Gamma \zeta(\xi-\eta)F(\eta)d\eta + \chi_{\Gamma'}(\xi)\langle\xi\rangle^s\int_{\Gamma^c}\zeta(\xi-\eta)F(\eta)d\eta \equiv I_1 + I_2.\end{aligned}$$

For any $\xi, \eta \in \mathbb{R}^n$ and $s \in \mathbb{R}$, $\langle\xi\rangle^s\langle\xi-\eta\rangle^{-|s|}\langle\eta\rangle^{-s} \le C_s$ holds and thus Young's inequality yields

$$\begin{aligned}\|I_1\|_{L^q_\xi} &\le \left\|\int_{\mathbb{R}^n}\frac{\langle\xi\rangle^s}{\langle\xi-\eta\rangle^{|s|}\langle\eta\rangle^s}\langle\xi-\eta\rangle^{|s|}|\zeta(\xi-\eta)|\chi_\Gamma(\eta)\langle\eta\rangle^s|F(\eta)|d\eta\right\|_{L^q_\xi}\\ &\le C_s\big\|\langle\xi\rangle^{|s|}\zeta(\xi)\big\|_{L^1_\xi}\|\chi_\Gamma(\xi)\langle\xi\rangle^s F(\xi)\|_{L^q_\xi}.\end{aligned}$$

If $\eta \notin \Gamma$ and $\xi \in \Gamma'$, then $|\xi-\eta| \ge C|\xi|$ and $|\xi-\eta| \ge C|\eta|$ for some $C > 0$. So we have by Young's inequality

$$\begin{aligned}\|I_2\|_{L^q_\xi} &\le C_{s,N}\left\|\chi_{\Gamma'}(\xi)\langle\xi\rangle^s\int_{\Gamma^c}|\zeta(\xi-\eta)|\frac{\langle\xi-\eta\rangle^{|s|+N}}{\langle\xi\rangle^{|s|}\langle\eta\rangle^N}|F(\eta)|d\eta\right\|_{L^q_\xi}\\ &\le C_{s,N}\left\|\langle\xi\rangle^{|s|+N}\zeta(\xi)\right\|_{L^1_\xi}\left\|\frac{F(\xi)}{\langle\xi\rangle^N}\right\|_{L^q_\xi}.\end{aligned}$$

Therefore we obtain the conclusion. □

Lemma 2.2. *Let* $1\le p,q\le\infty$, $s\in\mathbb{R}$, $u\in\mathcal{S}'(\mathbb{R}^n)$ *and* $(x_0,\xi_0)\in\mathbb{R}^n\times(\mathbb{R}^n\setminus\{0\})$. *Assume that*

$$\|\|\chi_K(x)\chi_\Gamma(\xi)\langle\xi\rangle^s W_\phi u(x,\xi)\|_{L^p_x}\|_{L^q_\xi}<\infty$$

for a neighborhood K *of* x_0, *a conic neighborhood* Γ *of* ξ_0 *and a function* $\phi\in C_0^\infty(\mathbb{R}^n)$ *with* $\phi(0)\neq 0$. *If* $\psi\in C_0^\infty(\mathbb{R}^n)$ *and* Γ' *is a conic neighborhood of* ξ_0 *such that* $\psi(0)\neq0$, $\operatorname{supp}\psi\subset(\operatorname{supp}\phi)^\circ$ *and* $\overline{\Gamma'}\subset\Gamma$, *then*

$$\|\|\chi_K(x)\chi_{\Gamma'}(\xi)\langle\xi\rangle^s W_\psi u(x,\xi)\|_{L^p_x}\|_{L^q_\xi}<\infty.$$

Proof. Since $|\phi(x)|\ge C>0$ on $\operatorname{supp}\psi$, we have

$$\begin{aligned}W_\psi u(x,\xi)&=\int_{\mathbb{R}^n}\chi_{\operatorname{supp}\psi}(y-x)\overline{\left(\frac{\psi(y-x)}{\phi(y-x)}\right)}\overline{\phi(y-x)}u(y)e^{-iy\cdot\xi}dy\\ &=\int_{\mathbb{R}^n}\zeta(\xi-\eta)W_\phi u(x,\eta)d\eta,\end{aligned}\tag{6}$$

where $\zeta(\xi)=\mathcal{F}_{y\to\xi}[\chi_{\operatorname{supp}\psi}(y-x)\overline{\psi(y-x)}/\overline{\phi(y-x)}](\xi)$. By Lemma 2.1 and (6), we have

$$\begin{aligned}&\|\|\chi_K(x)\chi_{\Gamma'}(\xi)\langle\xi\rangle^s W_\psi u(x,\xi)\|_{L^p_x}\|_{L^q_\xi}\\ &\le\left\|\left\|\chi_{\Gamma'}(\xi)\langle\xi\rangle^s\int_{\mathbb{R}^n}|\zeta(\xi-\eta)|\,\|\chi_K(x)W_\phi u(x,\eta)\|_{L^p_x}d\eta\right\|_{L^p_x}\right\|_{L^q_\xi}\\ &\le C_{s,N,\zeta}\left(\|\|\chi_K(x)\chi_\Gamma(\xi)\langle\xi\rangle^s W_\phi u(x,\xi)\|_{L^p_x}\|_{L^q_\xi}+\left\|\frac{1}{\langle\xi\rangle^N}\|\chi_K(x)W_\phi u(x,\xi)\|_{L^p_x}\right\|_{L^q_\xi}\right).\end{aligned}$$

We note that the support of χ_K is compact and $|W_\phi u(x,\xi)|$ is majored by a constant times $\langle\xi\rangle^{N_0}$ for sufficiently large N_0. By taking $N>N_0$ sufficiently large, we have the conclusion. □

Lemma 2.3. *Let* $1\le q\le\infty$, $s\in\mathbb{R}$, $u\in\mathcal{S}'(\mathbb{R}^n)$ *and* $(x_0,\xi_0)\in\mathbb{R}^n\times(\mathbb{R}^n\setminus\{0\})$. *Assume that* $(x_0,\xi_0)\notin WF_{\mathcal{F}L^q_s}(u)$, *i.e.,* $\|\chi_\Gamma(\xi)\langle\xi\rangle^s\widehat{au}(\xi)\|_{L^q_\xi}<\infty$ *for some open conic set* Γ *and a function* a *satisfying the condition of Definition 1.1. If* $b\in C_0^\infty(\mathbb{R}^n)$ *satisfies* $b(x_0)\neq0$ *and* $\operatorname{supp}b\subset(\operatorname{supp}a)^\circ$, *then* $\|\chi_\Gamma(\xi)\langle\xi\rangle^s\widehat{bu}(\xi)\|_{L^q_\xi}<\infty$.

We can prove Lemma 2.3 in the same way as in the proof of Lemma 2.2, but we give a proof in Appendix B for reader's convenience.

3 Proof of Proposition 1.2

In this section, we prove Proposition 1.2.

Proof of Proposition 1.2. We prove (i) implies (ii). Since $(x_0, \xi_0) \notin WF_{\mathcal{F}L^q_s}(u)$, there exist a function $a \in C_0^\infty(\mathbb{R}^n)$ with $a(x_0) \neq 0$ and a conic neighborhood Γ of ξ_0 satisfying that $\|\chi_\Gamma(\xi)\langle\xi\rangle^s \widehat{au}(\xi)\|_{L^q_\xi} < \infty$. Take $r > 0$ and $b \in C_0^\infty(\mathbb{R}^n)$ satisfying $\operatorname{supp} b \subset B_{4r}(x_0) \subset \operatorname{supp} a$ and $b \equiv 1$ in $B_{2r}(x_0)$. It follows from Lemma 2.3 that $\|\chi_\Gamma(\xi)\langle\xi\rangle^s \widehat{bu}(\xi)\|_{L^q_\xi} < \infty$. Take a neighborhood K of x_0 and a function $\phi \in C_0^\infty(\mathbb{R}^n)$ satisfying $K \subset B_r(x_0)$, $\phi(0) \neq 0$ and $\operatorname{supp} \phi \subset B_r(0)$. Note that $x \in K$ and $y - x \in B_r(0)$ imply $y \in B_{2r}(x_0)$. So $\chi_K(x)\overline{\phi(y-x)}u(y) = \chi_K(x)\overline{\phi(y-x)}b(y)u(y)$. Let Γ' be a conic neighborhood of ξ_0 such that $\overline{\Gamma'} \subset \Gamma$. Since $W_\phi(bu)(x,\xi) = (e^{-ix\cdot\xi}\mathcal{F}[\overline{\phi}](\xi)) \underset{\xi}{*} \mathcal{F}[bu](\xi)$ we have by Lemma 2.1

$$
\begin{aligned}
&\|\|\chi_K(x)\chi_{\Gamma'}(\xi)\langle\xi\rangle^s W_\phi u(x,\xi)\|_{L^p_x}\|_{L^q_\xi} \\
&= \|\|\chi_K(x)\chi_{\Gamma'}(\xi)\langle\xi\rangle^s W_\phi(bu)(x,\xi)\|_{L^p_x}\|_{L^q_\xi} \\
&\leq \left\|\left\|\chi_K(x)\chi_{\Gamma'}(\xi)\langle\xi\rangle^s(|\mathcal{F}[\overline{\phi}]| * |\mathcal{F}[bu]|)(\xi)\right\|_{L^p_x}\right\|_{L^q_\xi} \\
&\leq C_{s,N,\phi}\,\|\chi_K(x)\|_{L^p_x}\left(\left\|\chi_\Gamma(\xi)\langle\xi\rangle^s\widehat{bu}(\xi)\right\|_{L^q_\xi} + \left\|\frac{\widehat{bu}(\xi)}{\langle\xi\rangle^N}\right\|_{L^q_\xi}\right).
\end{aligned}
$$

Since $|\widehat{bu}(\xi)|$ has at most polynomial growth we obtain the desired result if we take an integer N sufficiently large.

Conversely, we prove (ii) implies (i). By Lemma 2.2, we can choose Γ being a conic neighborhood of ξ_0, $R \in \mathbb{R}$ and $\phi \in C_0^\infty(\mathbb{R}^n)$ which satisfy $\phi \equiv 1$ in $B_{2R}(0)$ and $\|\|\chi_{B_R(x_0)}(x)\chi_\Gamma(\xi)\langle\xi\rangle^s W_\phi u(x,\xi)\|_{L^p_x}\|_{L^q_\xi} < \infty$. Put $K = B_R(x_0)$ and take $a \in C_0^\infty(\mathbb{R}^n)$ satisfying $a(x_0) \neq 0$ and $\operatorname{supp} a \subset B_R(x_0)$. Since $\phi(y-x) \equiv 1$ for $x \in K$ and $y \in \operatorname{supp} a$, we have

$$
\begin{aligned}
\chi_K(x)\widehat{au}(\xi) &= \chi_K(x)\int_{\mathbb{R}^n}\overline{\phi(y-x)}a(y)u(y)e^{-iy\cdot\xi}dy \\
&= \chi_K(x)\int_{\mathbb{R}^n}\widehat{a}(\xi-\eta)W_\phi(x,\eta)d\eta.
\end{aligned}
$$

So we obtain by Lemma 2.1

$$
\begin{aligned}
&\|\chi_K(x)\|_{L^p_x}\,\|\chi_{\Gamma'}(\xi)\langle\xi\rangle^s\widehat{au}(\xi)\|_{L^q_\xi} \\
&\leq \left\|\chi_{\Gamma'}(\xi)\langle\xi\rangle^s\int_{\mathbb{R}^n}|\widehat{a}(\xi-\eta)|\,\|\chi_K(x)W_\phi u(x,\eta)\|_{L^p_x}\,d\eta\right\|_{L^q_\xi} \\
&\leq C_{s,N,a}\left(\|\|\chi_K(x)\chi_\Gamma(\xi)\langle\xi\rangle^s W_\phi u(x,\xi)\|_{L^p_x}\|_{L^q_\xi} + \left\|\frac{1}{\langle\xi\rangle^N}\,\|\chi_K(x)W_\phi(x,\xi)\|_{L^p_x}\right\|_{L^q_\xi}\right)
\end{aligned}
$$

for a conic neighborhood Γ' of ξ_0 satisfying $\overline{\Gamma'} \subset \Gamma$. Since χ_K has compact support and $|W_\phi u(x,\xi)|$ is majored by a constant times $\langle \xi \rangle^{N_0}$ for sufficiently large N_0, we obtain the desired result if we take an integer $N > N_0$ sufficiently large. $\square$

4 Proof of Theorem 1.3

In the sequel, for a function $f(t,x)$ on $\mathbb{R}\times\mathbb{R}^n$, we denote $\widehat{f}(t,\xi) = \int_{\mathbb{R}^n} f(t,x)e^{-ix\cdot\xi}dx$ and $W_\phi f(t,x,\xi) = W_\phi(f(t,\cdot))(x,\xi)$.

Proof of Theorem 1.3. Here, we only treat the initial value problem

$$\begin{cases} (\partial_t - i|D|)u(t,x) = 0, & (t,x) \in \mathbb{R}^{n+1}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{7}$$

since we can treat the case $(\partial_t + i|D|)u(t,x) = 0$ in the same way. Let $\phi \in C_0^\infty(\mathbb{R}^n)$ with $\phi(0) \neq 0$. Since $|\eta| = |\xi| + (\eta-\xi)\cdot\xi/|\xi| + |\eta| - \eta\cdot\xi/|\xi|$, we have by integration by parts

$$\begin{aligned} W_\phi(|D|u)(t,x,\xi) &= \int_{\mathbb{R}^n} \overline{\phi(y-x)} \int_{\mathbb{R}^n} |\eta|\, \widehat{u}(t,\eta)e^{iy\cdot\eta} đ\eta\, e^{-iy\cdot\xi}dy \\ &= \left(|\xi| + \frac{\xi}{i|\xi|}\cdot\nabla_x\right) W_\phi u(t,x,\xi) + R_\phi(u;t,x,\xi), \end{aligned}$$

where $đ\eta = (2\pi)^{-n}d\eta$ and

$$R_\phi(u;t,x,\xi) = \iint_{\mathbb{R}^{2n}} \overline{\phi(y-x)} \left(|\eta| - \frac{\xi\cdot\eta}{|\xi|}\right) \widehat{u}(t,\eta)e^{iy\cdot(\eta-\xi)} đ\eta dy.$$

So the initial value problem (7) is transformed to

$$\begin{cases} \left(\partial_t - \frac{\xi}{|\xi|}\cdot\nabla_x - i|\xi|\right) W_\phi u(t,x,\xi) = iR_\phi(u;t,x,\xi), \\ W_\phi u(0,x,\xi) = W_\phi u_0(x,\xi). \end{cases} \tag{8}$$

It is easy to see that (8) is equivalent to the integral equation

$$\begin{aligned} W_\phi u(t,x,\xi) = e^{it|\xi|}W_\phi u_0 &\left(x + \frac{\xi}{|\xi|}t, \xi\right) \\ &+ i\int_0^t e^{i(t-\theta)|\xi|} R_\phi\left(u;\theta, x + \frac{\xi}{|\xi|}(t-\theta), \xi\right) d\theta. \end{aligned} \tag{9}$$

Let T be a positive number. We show by induction $(x_0 - \frac{\xi_0}{|\xi_0|}t, \xi_0) \notin WF_{\mathcal{F}L^q_r}(u(t,\cdot))$ for $t \in [-T,T]$.

Since $u(t,\cdot) \in \mathcal{S}'(\mathbb{R}^n)$, there exists $s \in \mathbb{R}^n$ such that $\|\langle\cdot\rangle^s \widehat{au}(t,\cdot)\|_{L^q} < \infty$ for all $a \in C_0^\infty(\mathbb{R}^n)$ and $t \in [-T,T]$. Thus $(x_0 - \frac{\xi_0}{|\xi_0|}t, \xi_0) \notin WF_{\mathcal{F}L^q_s}(u(t,\cdot))$ holds for all $t \in [-T,T]$.

Next we show $(x_0 - \frac{\xi_0}{|\xi_0|}t, \xi_0) \notin WF_{\mathcal{F}L^q_{\sigma+1}}(u(t,\cdot))$ for all $t \in [-T,T]$ and $s \le \sigma \le r-1$ under the assumption $(x_0 - \frac{\xi_0}{|\xi_0|}t, \xi_0) \notin WF_{\mathcal{F}L^q_\sigma}(u(t,\cdot))$ for all $t \in [-T,T]$. It is clear that there exist a neighborhood K of $x_0 - \frac{\xi_0}{|\xi_0|}t$, a conic neighborhood Γ of ξ_0, $\phi \in C_0^\infty(\mathbb{R}^n)$ with $\phi(0) \neq 0$ such that

$$\|\|\chi_K(x)\chi_{\Gamma\cap\{|\xi|\le 1\}}(\xi)\langle\xi\rangle^{\sigma+1}W_\phi u(t,x,\xi)\|_{L^p_x}\|_{L^q_\xi} < \infty.$$

Put $\widetilde{\Gamma} = \Gamma \cap \{|\xi| \ge 1\}$. From the equation (9), it is enough to show that

$$I^{(1)}_{K,\widetilde{\Gamma},\phi} \equiv \left\|\left\|\chi_K(x)\chi_{\widetilde{\Gamma}}(\xi)\langle\xi\rangle^\sigma|\xi|W_\phi u_0\Big(x + \frac{\xi}{|\xi|}t, \xi\Big)\right\|_{L^p_x}\right\|_{L^q_\xi} < \infty, \tag{10}$$

$$I^{(2)}_{K,\widetilde{\Gamma},\phi,\psi} \equiv \left\|\left\|\chi_K(x)\chi_{\widetilde{\Gamma}}(\xi)\langle\xi\rangle^\sigma|\xi|\int_0^t \left|R_\phi\Big(\psi u; \theta, x + \frac{\xi}{|\xi|}(t-\theta), \xi\Big)\right| d\theta\right\|_{L^p_x}\right\|_{L^q_\xi} < \infty \tag{11}$$

and

$$I^{(3)}_{K,\widetilde{\Gamma},\phi,\psi} \equiv \Bigg\|\Bigg\|\chi_K(x)\chi_{\widetilde{\Gamma}}(\xi)\langle\xi\rangle^\sigma|\xi| \times \int_0^t \left|R_\phi\Big((1-\psi)u; \theta, x + \frac{\xi}{|\xi|}(t-\theta), \xi\Big)\right| d\theta\Bigg\|_{L^p_x}\Bigg\|_{L^q_\xi} < \infty \tag{12}$$

for some $\psi \in C_0^\infty(\mathbb{R}^n)$ and all $t \in [-T,T]$.

From the assumption $(x_0,\xi_0) \notin WF_{\mathcal{F}L^q_r}(u_0)$ and Proposition 1.2, there exist a constant $\varepsilon > 0$, a function $\phi_1 \in C_0^\infty(\mathbb{R}^n)$ with $\phi_1(0) \neq 0$ and a conic neighborhood Γ' of ξ_0 such that $\|\|\chi_{B_{2\varepsilon}(x_0)}(x)\chi_{\Gamma'}(\xi)\langle\xi\rangle^r W_{\phi_1}u_0(x,\xi)\|_{L^p_x}\|_{L^q_\xi} < \infty$. Let $K_1 = B_\varepsilon(x_0 - \frac{\xi_0}{|\xi_0|}t)$ and Γ_1 be a conic neighborhood of ξ_0 satisfying $\varepsilon T^{-1} > d_1 = \sup_{\xi\in\Gamma_1}\mathrm{dist}(\frac{\xi}{|\xi|}, \frac{\xi_0}{|\xi_0|})$ and $\overline{\Gamma}_1 \subset \Gamma'$. If $x \in K_1$ and $\xi \in \Gamma_1$ then $x + \frac{\xi}{|\xi|}t \in B_{2\varepsilon}(x_0)$. Thus we have

$$I^{(1)}_{K_1,\widetilde{\Gamma}_1,\phi_1} \le \left\|\left\|\chi_{B_{2\varepsilon}(x_0)}(x)\chi_{\Gamma'}(\xi)\langle\xi\rangle^r W_{\phi_1}u_0(x,\xi)\right\|_{L^p_x}\right\|_{L^q_\xi} < \infty,$$

where $\widetilde{\Gamma}_1 = \Gamma_1 \cap \{|\xi| \ge 1\}$

Next we show (11). By the assumption of induction we can take a conic neighborhood Γ'' of ξ_0 and $\psi_t \in C_0^\infty(\mathbb{R}^n)$ such that $\psi_t \equiv 1$ near $x_0 - \frac{\xi_0}{|\xi_0|}t$ and $\|\chi_{\Gamma''}(\cdot)\langle\cdot\rangle^\sigma \widehat{\psi_t u}(t,\cdot)\|_{L^q_\xi} < \infty$ for all $t \in [-T,T]$. Take $\varepsilon' > 0$ satisfying $\psi_t \equiv 1$ on $B_{6\varepsilon'}(x_0 - \frac{\xi_0}{|\xi_0|}t)$. Let $\phi_2 \in C_0^\infty(\mathbb{R}^n)$ with $\phi_2(0) \neq 0$ and $\operatorname{supp}\phi_2 \subset B_{2\varepsilon'}(0)$, $K_2 =$

$B_{\varepsilon'}(x_0 - \frac{\xi_0}{|\xi_0|}t)$ and Γ_2 be a conic neighborhood of ξ_0 satisfying $\overline{\Gamma}_2 \subset \Gamma''$ and $\varepsilon' T^{-1} > d_2 = \sup_{\xi\in\Gamma_2} \mathrm{dist}(\frac{\xi}{|\xi|}, \frac{\xi_0}{|\xi_0|})$. Since $(1-\Delta_y)^{N+1} e^{iy\cdot(\eta-\xi)} = \langle \eta-\xi\rangle^{2N+2} e^{iy\cdot(\eta-\xi)}$ for an integer $N>0$, we have by integration by parts

$$\begin{aligned} &R_{\phi_2}\left(\psi_\theta u; \theta, x + \frac{\xi}{|\xi|}(t-\theta), \xi\right) \\ &= \iint_{\mathbb{R}^{2n}} (1-\Delta_y)^{N+1} \overline{\phi_2\left(y - x - \frac{\xi}{|\xi|}(t-\theta)\right)} \left(|\eta| - \frac{\xi\cdot\eta}{|\xi|}\right) \frac{\widehat{\psi_\theta u}(\theta,\eta) e^{iy\cdot(\eta-\xi)}}{\langle\eta-\xi\rangle^{2N+2}} đ\eta dy. \end{aligned}$$

Put $\widetilde{\Gamma}_2 = \Gamma_2 \cap \{|\xi| \geq 1\}$. Using the fact that

$$\left(|\eta| - \frac{\xi\cdot\eta}{|\xi|}\right)\langle\eta-\xi\rangle^{-2} = \frac{|\xi||\eta| - \xi\cdot\eta}{|\xi|(|\eta|^2 + |\xi|^2 - 2\xi\cdot\eta)} \leq \frac{|\xi||\eta| - \xi\cdot\eta}{|\xi|(2|\xi||\eta| - 2\xi\cdot\eta)} = \frac{1}{2|\xi|},$$

we have

$$\begin{aligned} I^{(2)}_{K_2,\widetilde{\Gamma}_2,\phi_2,\psi_\theta} &= \left\| \left\| \chi_{K_2}(x)\chi_{\widetilde{\Gamma}_2}(\xi)\langle\xi\rangle^\sigma |\xi| \int_0^t \left| R_{\phi_2}\left(\psi_\theta u; \theta, x + \frac{\xi}{|\xi|}(t-\theta), \xi\right)\right| d\theta \right\|_{L^p_x} \right\|_{L^q_\xi} \\ &\leq C\left\|\chi_{K_2}\right\|_{L^p} \left\|(1-\Delta)^{N+1}\phi_2\right\|_{L^1} \int_0^T \left\| \int_{\mathbb{R}^n} \frac{\chi_{\widetilde{\Gamma}_2}(\xi)\langle\xi\rangle^\sigma}{\langle\eta-\xi\rangle^{2N}} |\widehat{\psi_\theta u}(\theta,\eta)| đ\eta \right\|_{L^q_\xi} d\theta \\ &\leq C_{K_2,\phi_2}(J_1 + J_2), \end{aligned}$$

where

$$J_1 = \int_0^T \left\| \int_{\Gamma''} \frac{\langle\xi\rangle^\sigma}{\langle\eta-\xi\rangle^{2N}} |\widehat{\psi_\theta u}(\theta,\eta)| d\eta \right\|_{L^q_\xi} d\theta$$

and

$$J_2 = \int_0^T \left\| \int_{(\Gamma'')^c} \frac{\chi_{\widetilde{\Gamma}_2}(\xi)\langle\xi\rangle^\sigma}{\langle\eta-\xi\rangle^{2N}} |\widehat{\psi_\theta u}(\theta,\eta)| d\eta \right\|_{L^q_\xi} d\theta.$$

Since $\langle\xi\rangle \leq 2\langle\eta-\xi\rangle$ or $\langle\xi\rangle \leq 2\langle\eta\rangle$ hold, we have

$$\frac{\langle\xi\rangle^\sigma}{\langle\eta-\xi\rangle^{2N}\langle\eta\rangle^\sigma} \leq \frac{C}{\langle\eta-\xi\rangle^{2N-|\sigma|}} \tag{13}$$

for $2N > |\sigma|$. Thus if we take an integer N sufficiently large, then Young's inequality, (13) and the assumption of induction yield

$$\begin{aligned} J_1 &= \int_0^T \left\| \int_{\mathbb{R}^n} \frac{\langle\xi\rangle^\sigma}{\langle\eta-\xi\rangle^{2N}\langle\eta\rangle^\sigma} \chi_{\Gamma''}(\eta)\langle\eta\rangle^\sigma |\widehat{\psi_\theta u}(\theta,\eta)| d\eta \right\|_{L^q_\xi} d\theta \\ &\leq C\left\| \frac{1}{\langle\cdot\rangle^{2N-|\sigma|}} \right\|_{L^1} \int_0^T \left\| \chi_{\Gamma''}(\xi)\langle\xi\rangle^\sigma \widehat{\psi_\theta u}(\theta,\xi) \right\|_{L^q_\xi} d\theta < \infty. \end{aligned}$$

On the other hand, if $\eta \notin \Gamma''$, $\xi \in \widetilde{\Gamma}_2$ and $2N > |\sigma|$ then we have

$$\frac{\langle \xi \rangle^{\sigma}}{\langle \eta - \xi \rangle^{2N}} \leq \frac{C}{\langle \eta - \xi \rangle^{2N-|\sigma|}} \leq \frac{C}{\langle \eta - \xi \rangle^{N_1} \langle \eta \rangle^{N_2}}, \tag{14}$$

where $N_1 + N_2 = 2N - |\sigma|$. Since $|\widehat{\psi_\theta u}(\theta, \xi)|$ has at most polynomial growth with respect to ξ, Young's inequality and the inequality (14) yield

$$J_2 \leq C \left\| \frac{1}{\langle \cdot \rangle^{N_1}} \right\|_{L^1} \int_0^T \left\| \frac{\widehat{\psi_\theta u}(\theta, \xi)}{\langle \xi \rangle^{N_2}} \right\|_{L^q_\xi} d\theta < \infty,$$

if we take integers N_1 and N_2 sufficiently large. Thus we obtain $I^{(2)}_{K_2, \widetilde{\Gamma}_2, \phi_2, \psi_\theta} < \infty$.

Finally we show (12). Let ζ_1 be a C^∞ function on $\mathbb{R}^n$ satisfying $\zeta_1(\eta) = 0$ for $|\eta| \leq 1$ and $\zeta_1(\eta) = 1$ for $|\eta| \geq 2$ and put $\zeta_2(\eta) = 1 - \zeta_1(\eta)$. It suffices to show that

$$I^{(3)}_{K_2, \widetilde{\Gamma}_2, \phi_2, \psi_\theta} \leq \sum_{j=1,2} \left\| \left\| \chi_{K_2}(x) \chi_{\widetilde{\Gamma}_2}(\xi) \langle \xi \rangle^{\sigma} |\xi| \int_0^t |R_j| d\theta \right\|_{L^p_x} \right\|_{L^q_\xi} < \infty,$$

where

$$R_1 = \lim_{h_1, h_2 \to 0} \iiint_{\mathbb{R}^{3n}} \overline{\phi_2\Big(y - x - \frac{\xi}{|\xi|}(t - \theta)\Big)} \Big(|\eta| - \frac{\xi \cdot \eta}{|\xi|}\Big) b(h_1 \eta) \zeta_1(\eta) \\ \times (1 - \psi_\theta(\tilde{x})) u(\theta, \tilde{x}) b(h_2 \tilde{x}) e^{-i(\tilde{x} \cdot \eta - y \cdot \eta + y \cdot \xi)} d\tilde{x} đ\eta dy.$$

and

$$R_2 = \lim_{h_2 \to 0} \iiint_{\mathbb{R}^{3n}} \overline{\phi_2\Big(y - x - \frac{\xi}{|\xi|}(t - \theta)\Big)} \Big(|\eta| - \frac{\xi \cdot \eta}{|\xi|}\Big) \zeta_2(\eta) \\ \times (1 - \psi_\theta(\tilde{x})) u(\theta, \tilde{x}) b(h_2 \tilde{x}) e^{-i(\tilde{x} \cdot \eta - y \cdot \eta + y \cdot \xi)} d\tilde{x} đ\eta dy$$

for $b \in \mathcal{S}(\mathbb{R}^n)$ with $b(0) = 1$. From the structure theorem of $\mathcal{S}'$ (see, for example, [12, Theorem 2.14]), there exist constants $l, m \geq 0$ and functions $f_\alpha(\theta, \cdot) \in L^2(\mathbb{R}^n)$ for multi-indices α such that

$$u(\theta, \tilde{x}) = \langle \tilde{x} \rangle^l \sum_{|\alpha| \leq m} D^\alpha f_\alpha(\theta, \tilde{x}). \tag{15}$$

Since

$$e^{-i(\tilde{x} - y) \cdot \eta} = \frac{(-\Delta_\eta)^{N_3} e^{-i(\tilde{x} - y) \cdot \eta}}{|\tilde{x} - y|^{2N_3}} \quad \text{and} \quad e^{iy \cdot (\eta - \xi)} = \frac{(1 - \Delta_y)^{N_4} e^{iy \cdot (\eta - \xi)}}{\langle \eta - \xi \rangle^{2N_4}}$$

for positive integers N_3 and N_4, we have by (15) and integration by parts

$$R_1 = \lim_{h_1, h_2 \to 0} \sum_{|\alpha| \leq m} \iiint_{\mathbb{R}^{3n}} (1 - \Delta_y)^{N_4} \left\{ \frac{\overline{\phi_2(y - x - \frac{\xi}{|\xi|}(t - \theta))}}{|\tilde{x} - y|^{2N_3}} \right\} \\ \times \frac{1}{\langle \eta - \xi \rangle^{2N_4}} (-\Delta_\eta)^{N_3} \left\{ \left(|\eta| - \frac{\eta \cdot \xi}{|\xi|} \right) b(h_1 \eta) \zeta_1(\eta) \right\} \\ \times (1 - \psi_\theta(\tilde{x})) \langle \tilde{x} \rangle^l D^\alpha f_\alpha(\theta, \tilde{x}) b(h_2 \tilde{x}) e^{-i(\tilde{x} \cdot \eta - y \cdot \eta + y \cdot \xi)} d\tilde{x} đ\eta dy.$$

We note that

$$(1-\Delta_y)^{N_4}\left\{\frac{\overline{\phi_2(y-x-\frac{\xi}{|\xi|}(t-\theta))}}{|\tilde{x}-y|^{2N_3}}\right\}=\frac{\widetilde{\phi}_2(y-x-\frac{\xi}{|\xi|}(t-\theta))}{|\tilde{x}-y|^{2N_3}}+A, \tag{16}$$

where $\widetilde{\phi}_2(y)=(1-\Delta_y)^{N_4}\overline{\phi_2(y)}$ and A is a finite sum of $o(|\tilde{x}-y|^{-2N_3})$ times C_0^∞ function of y. Put

$$\begin{aligned}R_1' = \lim_{h_1,h_2\to 0}\sum_{|\alpha|\le m}\iiint_{\mathbb{R}^{3n}} &|\tilde{x}-y|^{-2N_3}\widetilde{\phi}_2\Big(y-x-\frac{\xi}{|\xi|}(t-\theta)\Big)\\ &\times\frac{1}{\langle\eta-\xi\rangle^{2N_4}}(-\Delta_\eta)^{N_3}\left\{\left(|\eta|-\frac{\eta\cdot\xi}{|\xi|}\right)b(h_1\eta)\zeta_1(\eta)\right\}\\ &\times(1-\psi_\theta(\tilde{x}))\langle\tilde{x}\rangle^l b(h_2\tilde{x})D^\alpha f_\alpha(\theta,\tilde{x})e^{-i(\tilde{x}\cdot\eta-y\cdot\eta+y\cdot\xi)}d\tilde{x}đ\eta dy.\end{aligned}$$

Since $\operatorname{supp}\zeta_1\cap B_1(0)=\emptyset$ we have

$$\lim_{h_1\to 0}\left|(-\Delta_\eta)^{N_3}\left\{\left(|\eta|-\frac{\eta\cdot\xi}{|\xi|}\right)b(h_1\eta)\zeta_1(\eta)\right\}\right|\le\frac{C'}{|\eta|^{2N_3-1}}\le\frac{C''}{\langle\eta\rangle^{2N_3-1}} \tag{17}$$

on the support of ζ_1. If $x\in K_2=B_{\varepsilon'}(x_0-\frac{\xi_0}{|\xi_0|}t)$, $\xi\in\widetilde{\Gamma}_2$ and $y-x-(t-\theta)\xi/|\xi|\in B_{2\varepsilon'}(0)$ then $y\in B_{5\varepsilon'}(x_0-\frac{\xi_0}{|\xi_0|}\theta)$ and if $\widetilde{x}\in\operatorname{supp}(1-\psi_\theta(\tilde{x}))$ then $\widetilde{x}\notin B_{6\varepsilon'}(x_0-\frac{\xi_0}{|\xi_0|}\theta)$. Thus $|\tilde{x}-y|\ge\varepsilon'>0$ holds on the support of $\chi_{K_2}(x)\chi_{\Gamma_2}(\xi)\overline{\phi(y-x-\frac{\xi}{|\xi|}(t-\theta))}(1-\psi_\theta(\tilde{x}))$. Put $g_{y,h_2}(x)=(1-\psi_\theta(x))\langle x\rangle^l b(h_2x)|x-y|^{-2N_3}$. If $|\tilde{x}-y|\ge\varepsilon'>0$ and $y\in B_{5\varepsilon'}(x_0-\frac{\xi_0}{|\xi_0|}\theta)$ then we have $|\tilde{x}-y|\ge C_1\langle\tilde{x}-y\rangle\ge C_2\langle\tilde{x}\rangle$ and, hence,

$$\lim_{h_2\to 0}\|\langle\cdot\rangle^\alpha\widehat{g}_{y,h_2}(\cdot)\|_{L^1}\le\lim_{h_2\to 0}\left\|\frac{1}{\langle\cdot\rangle^{N-\alpha}}\int|(1-\Delta_x)^N g_{y,h_2}(x)|dx\right\|_{L^1}\le\widetilde{C} \tag{18}$$

for $N>\alpha+n$ and a positive constant $\widetilde{C}$ which is not depending on y and $\theta\in[0,T]$. By (17) and (18) we have

$$\begin{aligned}|R_1'|&\le C\lim_{h_2\to 0}\sum_{\alpha\le m}\iint_{\mathbb{R}^{2n}}\frac{|\widetilde{\phi}_2(y-x-\frac{\xi}{|\xi|}(t-\theta))|}{\langle\eta-\xi\rangle^{2N_4}\langle\eta\rangle^{2N_3-1}}\\ &\qquad\times\int_{\mathbb{R}^n}|\widehat{g}_{y,h_2}(\eta-\eta')||\eta'|^{|\alpha|}|\widehat{f}_\alpha(\theta,\eta')|d\eta'đ\eta dy\\ &\le C\lim_{h_2\to 0}\sum_{|\alpha|\le m}\iint_{\mathbb{R}^{2n}}\frac{|\widetilde{\phi}_2(y-x-\frac{\xi}{|\xi|}(t-\theta))|}{\langle\eta-\xi\rangle^{2N_4}\langle\eta\rangle^{2N_3-1-\alpha}}\left\|\langle\cdot\rangle^{|\alpha|}\widehat{g}_{y,h_2}\right\|_{L^1}\left\|\widehat{f_\alpha}(\theta,\cdot)\right\|_{L^2}đ\eta dy\\ &\le C\int_{\mathbb{R}^n}\frac{\|\widehat{f}_\alpha(\theta,\cdot)\|_{L^2}}{\langle\eta\rangle^{2N_3-1-|\alpha|}\langle\xi-\eta\rangle^{2N_4}}đ\eta.\end{aligned} \tag{19}$$

Thus, from (16) and (19), we have

$$|R_1| \le C \int_{\mathbb{R}^n} \frac{\|\widehat{f}_\alpha(\theta,\cdot)\|_{L^2}}{\langle\eta\rangle^{2N_3-1-|\alpha|}\langle\xi-\eta\rangle^{2N_4}} đ\eta.$$

On the other hand, since $\zeta_2 \in C_0^\infty(\mathbb{R}^n)$ we have

$$(1-\Delta_\eta)^N \left\{ \Big(\eta - \frac{\eta\cdot\xi}{|\xi|}\Big)\zeta_2(\eta) \right\} \le \frac{C}{\langle\eta\rangle^{2N-1}}. \tag{20}$$

For $g'_{y,h_2}(x) = (1-\psi_\theta(x))\langle x\rangle^l b(h_2x)\langle x-y\rangle^{-2N_3}$,

$$\lim_{h_2\to 0} \|\langle\cdot\rangle^\alpha \widehat{g'}_{y,h_2}(\cdot)\|_{L^1} \le \widetilde{C'}, \tag{21}$$

where $\widetilde{C'}$ is not depending on y and $\theta \in [0,T]$. Since

$$e^{-i(\tilde{x}-y)\cdot\eta} = \frac{(1-\Delta_\eta)^{N_3} e^{-i(\tilde{x}-y)\cdot\eta}}{\langle\tilde{x}-y\rangle^{2N_3}} \quad \text{and} \quad e^{iy\cdot(\eta-\xi)} = \frac{(1-\Delta_y)^{N_4} e^{iy\cdot(\eta-\xi)}}{\langle\eta-\xi\rangle^{2N_4}}$$

for positive integers N_3 and N_4, we have by (15), (20), (21) and integration by parts

$$|R_2| \le C \int_{\mathbb{R}^n} \frac{\|\widehat{f}_\alpha(\theta,\cdot)\|_{L^2}}{\langle\eta\rangle^{2N_3-1-|\alpha|}\langle\xi-\eta\rangle^{2N_4}} đ\eta.$$

Since

$$\frac{\langle\xi\rangle^\sigma|\xi|}{\langle\eta\rangle^{2N_3-1-|\alpha|}\langle\xi-\eta\rangle^{2N_4}} \le \frac{C}{\langle\eta\rangle^{2N_3-2-|\alpha|-\sigma}\langle\xi-\eta\rangle^{2N_4-\sigma-1}}$$

for $N_3 \ge (2+|\alpha|+\sigma)/2$ and $N_4 \ge (\sigma+1)/2$, we have by Young's inequality

$$\left\| \int_{\mathbb{R}^n} \frac{\langle\xi\rangle^\sigma|\xi|}{\langle\eta\rangle^{2N_3-1-|\alpha|}\langle\xi-\eta\rangle^{2N_4}} đ\eta \right\|_{L^q_\xi} \le \left\| \frac{1}{\langle\cdot\rangle^{2N_3-2-|\alpha|-\sigma}} \right\|_{L^1} \left\| \frac{1}{\langle\cdot\rangle^{2N_4-\sigma-1}} \right\|_{L^q}.$$

Thus if we take integers N_3 and N_4 sufficiently large, we obtain

$$\begin{aligned} I^{(3)}_{K_2,\widetilde{\Gamma}_2,\phi_2,\psi_\theta} &\le C \left\| \left\| \chi_{K_2}(x)\chi_{\widetilde{\Gamma}_2}(\xi) \int_0^t \int_{\mathbb{R}^n} \frac{\langle\xi\rangle^\sigma|\xi|\|\widehat{f}_\alpha(\theta,\cdot)\|_{L^2}}{\langle\eta\rangle^{2N_3-1-|\alpha|}\langle\xi-\eta\rangle^{2N_4}} đ\eta d\theta \right\|_{L^p_x} \right\|_{L^q_\xi} \\ &\le C\|\chi_{K_2}\|_{L^p} \int_0^T \|\widehat{f}_\alpha(\theta,\cdot)\|_{L^2} d\theta \left\| \int_{\mathbb{R}^n} \frac{\langle\xi\rangle^\sigma|\xi|}{\langle\eta\rangle^{2N_3-1-|\alpha|}\langle\xi-\eta\rangle^{2N_4}} đ\eta \right\|_{L^q_\xi} \\ &\le C_{K_2,N_3,N_4} \int_0^T \left\| \widehat{f}_\alpha(\theta,\cdot) \right\|_{L^2} d\theta < \infty. \end{aligned}$$

Hence we get the inequality (12). Taking $K \subset K_1 \cap K_2$, $\Gamma \subset \Gamma_1 \cap \Gamma_2$ and $\phi \in C_0^\infty(\mathbb{R}^n)$ with $\phi(0) \ne 0$ and $\operatorname{supp}\phi \subset \operatorname{supp}\phi_1 \cap \operatorname{supp}\phi_2$, we obtain $(x_0 - \xi_0 t/|\xi_0|, \xi_0) \notin WF^q_{\sigma+1}(u)$ for $t \in [-T,T]$. Since T is an arbitrary positive number, we obtain the desired result. □

Appendix A

We give a sketch of proof on Remark 1.4 (i). Let $\phi \in C_0^\infty(\mathbb{R}^n)$ with $\phi(0) \neq 0$. Applying the wave packet transform to the equation $(\partial_t - ia(D))u(t,x) = 0$, we have

$$\partial_t W_\phi u(t,x,\xi) - i\int_{\mathbb{R}^n} \overline{\phi(y-x)} \int_{\mathbb{R}^n} a(\eta)\widehat{u}(\eta)e^{iy\cdot\eta} đ\eta\, e^{-iy\cdot\xi} dy = 0.$$

Since $a(\eta)$ is a homogeneous function of degree 1, $\nabla a(\eta)\cdot\eta = a(\eta)$ holds. So we have

$$a(\eta) = a(\xi) + \nabla a(\xi)\cdot(\eta-\xi) + r(\xi,\eta), \tag{22}$$

where $r(\xi,\eta) = a(\eta) - \nabla a(\xi)\cdot\eta$. By Taylor's theorem, we have

$$r(\xi,\eta) = \sum_{i,j=1}^n C_{ij}(\xi_i-\eta_i)(\xi_j-\eta_j)\int_0^1 \partial_{\xi_i}\partial_{\xi_j} a(\xi+\theta(\eta-\xi))(1-\theta)d\theta.$$

From the equation (22) we have by integration by parts

$$\begin{aligned}&\int_{\mathbb{R}^n} \overline{\phi(y-x)} \int_{\mathbb{R}^n} a(\eta)\widehat{u}(\eta)e^{iy\cdot\eta} đ\eta\, e^{-iy\cdot\xi} dy \\ &\qquad = a(\xi)W_\phi u(t,x,\xi) - i\nabla a(\xi)\cdot\nabla_x W_\phi u(t,x,\xi) + R(u(t,\cdot);x,\xi),\end{aligned}$$

where

$$R(u(t,\cdot);x,\xi) = \int_{\mathbb{R}^n} \overline{\phi(y-x)} \int_{\mathbb{R}^n} r(\xi,\eta)\hat{u}(\eta)e^{iy\cdot(\eta-\xi)} đ\eta dy.$$

If $|\xi|/2 \le |\xi-\eta|$, then $|\eta| \le |\xi-\eta| + |\xi| \le 3|\xi-\eta|$. Since $a(\eta)$ is a homogeneous function of degree 1, if $|\xi|/2 \le |\xi-\eta|$, then we have by Cauchy-Schwarz's inequality

$$\begin{aligned} r(\xi,\eta)\langle\xi-\eta\rangle^{-2} &= \frac{|\xi|a(\eta) - |\xi|\nabla a(\xi)\cdot\eta}{\langle\xi-\eta\rangle^2|\xi|} \\ &= \frac{|\xi|(\nabla a(\eta) - \nabla a(\xi))\cdot\eta}{\langle\xi-\eta\rangle^2|\xi|} \\ &\le \frac{|\eta||\nabla a(\eta) - \nabla a(\xi)|}{\langle\xi-\eta\rangle^2} \\ &\le \frac{C}{|\xi|}. \end{aligned}$$

If $|\xi-\eta| \le |\xi|/2$, then we have

$$\begin{aligned} r(\xi,\eta)\langle\xi-\eta\rangle^{-2} &\le \sum_{i,j=1}^n C_{ij}\int_0^1 |\partial_{\xi_i}\partial_{\xi_j} a(\xi+\theta(\eta-\xi))|(1-\theta)d\theta \\ &\le \sum_{i,j=1}^n C_{ij}\int_0^1 \frac{1-\theta}{|\xi+\theta(\eta-\xi)|}d\theta \\ &\le \frac{C}{|\xi|}. \end{aligned}$$

So we can regard $R(u(t,\cdot);x,\xi)$ as lower order term with respect to ξ. Thus the equation $(\partial_t - ia(\xi) - \nabla a(\xi)\cdot\nabla_x)W_\phi u(t,x,\xi) = R(u(t,\cdot);x,\xi)$ implies $(x_0 - \nabla a(\xi_0)t, \xi_0) \notin WF_{\mathcal{F}L^q_r}(u(t,\cdot))$ in the same way as in the proof of Theorem 1.3.

Appendix B

We prove Lemma 2.3. Since $|a(x)| \geq C > 0$ on $\operatorname{supp} b$, we have

$$\begin{aligned}\widehat{bu}(\xi) &= \int_{\mathbb{R}^n} \chi_{\mathrm{supp} b}(x)\frac{b(x)}{a(x)}a(x)u(x)e^{-ix\cdot\xi}dx \\ &= \int_{\mathbb{R}^n} \zeta(\xi-\eta)\widehat{au}(\xi)d\eta,\end{aligned}$$

where $\zeta(\xi) = \mathcal{F}[\chi_{\mathrm{supp} b}(x)b(x)/a(x)](\xi)$. By Lemma 2.1, we have

$$\|\chi_{\widetilde{\Gamma}}(\xi)\langle\xi\rangle^s\widehat{bu}(\xi)\| \leq C_{s,N,\zeta}\left(\|\chi_\Gamma(\xi)\langle\xi\rangle^s\widehat{au}(\xi)\|_{L^q_\xi} + \left\|\frac{\widehat{au}(\xi)}{\langle\xi\rangle^N}\right\|_{L^q_\xi}\right). \tag{23}$$

Since $\widehat{au}(\xi)$ has at most polynomial growth, we obtain the conclusion if we take an integer N sufficiently large.

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