extriangulated cat localization

拡大三角圏の局所化について

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§0 Intro

Representation theory of algebras

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \parallel$

fin. dim. k-alg (k : field)

$$\begin{array}{ccc} \operatorname{mod} A & \subseteq & \operatorname{Mod} A \\ \left(\begin{array}{c}\text{cat of fin. dim} \\ A\text{-modules}\end{array}\right) & & \left(\begin{array}{c}\text{cat of all} \\ A\text{-modules}\end{array}\right) \\ \cap\! | & & \cap\! | \\ D^b(\operatorname{mod} A) & \subseteq & D(\operatorname{Mod} A) \\ \left(\begin{array}{c}\text{bounded} \\ \text{derived cat.}\end{array}\right) & & \left(\begin{array}{c}\text{unbounded} \\ \text{der. cat.}\end{array}\right) \end{array}$$

abel. cat.

triang. cat.

(1) $A \overset{\text{Morita}}{\sim} B$ Morita equiv $:\Leftrightarrow$ $\operatorname{mod} A \simeq \operatorname{mod} B$ equiv

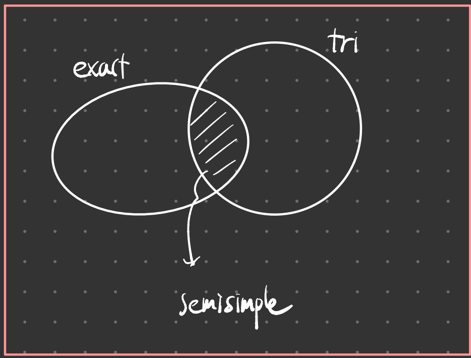
(2) $A \overset{\text{der}}{\sim} B$ derived. eq $:\Leftrightarrow$ $D^b(\operatorname{mod} A) \simeq D^b(\operatorname{mod} B)$ tri. equiv.

Prob 1

different framework $\left\{\begin{array}{l} \text{abel. cat.} \subseteq \text{exact. cat.} \\ \text{tri. cat.} \end{array}\right\} \subseteq$ extri. cat. (ET cat)

[Nakaoka – Palu]

cf



ET cat

Prob2

Mod A $\subseteq$ $D(\mathrm{Mod}A)$ too large to investigate!

$\rightsquigarrow$ (1) Focus on nice subcat's + approximation theory $\Big\}$ contrav. finite (co) subcat.

(2) Divide them in two subcat's.

(co) torsion pair

(3) Divide them in sub / factor.

Localization !

Ex A alg $\ni$ e idemp

$\rightsquigarrow$ eAe , A/AeA

sub factor

(1) abel. cat.

$$\begin{tikzcd}
 & X \arrow[r, mapsto] \arrow[d, phantom, "\cap" rotate=90] & X_e \arrow[d, phantom, "\cap" rotate=90] \\
\boxed{\mathrm{mod}\, A/AeA} \arrow[r, hook] \arrow[dr, "0"', ""{name=Z}] & \mathrm{mod}\, A \arrow[r, "Q"] \arrow[d, "F"', "\text{exact}"] & \mathrm{mod}\, eAe \arrow[dl, dashed, "\exists!\, F'\ \text{exact}"] \\
 & \mathcal{B} &
\end{tikzcd}$$

Serre

$\mathcal{B}$ abel.

Serre localization

exact seq of abel. cat.

(2) tri. cat

$$\{X \mid H^i(X) \in \mathrm{mod}\, A/AeA\}$$

$$\begin{tikzcd}
\boxed{D^b_{\mathrm{mod}\, A/AeA}(\mathrm{mod}\, A)} \arrow[r, hook] \arrow[dr, "0"'] & D^b(\mathrm{mod}\, A) \arrow[r, "Q"] \arrow[d, "F"', "\text{exact}"] & D^b(\mathrm{mod}\, eAe) \arrow[dl, dashed, "\exists!\, F'\ \text{exact}"] \\
 & \mathcal{B} &
\end{tikzcd}$$

thick

$\mathcal{B}$ tri

Verdier localization

exact seq of tri. cat.

Constructions of der. cat. $\mathrm{Mod}\,A \rightsquigarrow D(\mathrm{Mod}\,A)$

(1) via Ver. quot.

$$
\begin{array}{c}
\mathrm{Mod}\,A \\
\downarrow \\
C(\mathrm{Mod}A) \quad \text{Frobenius exact str.} \\
\left(\begin{array}{c}\text{the cat of} \\ \text{cpx's in } \mathrm{Mod}\,A\end{array}\right) \\
\downarrow /\text{homotopy}
\end{array}
$$

$$
\underset{\text{thick}}{K^{ac}(\mathrm{Mod}A)} \hookrightarrow \underset{\text{tri. cat.}}{K(\mathrm{Mod}A)} \longrightarrow D(\mathrm{Mod}\,A) \quad \text{Ver. loc.}
$$

(2) via model. str.

$$
\begin{array}{c}
\mathrm{Mod}\,A \\
\downarrow
\end{array}
$$

$$
\underset{\text{model. str.}}{C(\mathrm{Mod}\,A)} \longrightarrow \underset{\text{homotopy cat of the model. cat.}}{\mathrm{Ho}(C(\mathrm{Mod}A))} = D(\mathrm{Mod}A)
$$

Q

$$
\begin{array}{ccccc}
\overset{\text{exact}}{C^{ac}(\mathrm{Mod}A)} & \hookrightarrow & \overset{\text{abel}}{C(\mathrm{Mod}A)} & \longrightarrow & \overset{\text{tri}}{D(\mathrm{Mod}A)} \quad ? \\
\downarrow & & \downarrow & & \| \\
K^{ac}(\mathrm{Mod}A) & \hookrightarrow & K(\mathrm{Mod}A) & \longrightarrow & D(\mathrm{Mod}A) \quad \text{exact seq of tri. cat's}
\end{array}
$$

$\rightsquigarrow$ ET cat provides a conceptual understanding of ?

§1 Interplay between abel. cat and tri. cat

(0) Construction of $D(\operatorname{Mod} A)$

$$\begin{cases} \text{Ver. quot.} \\ \text{The homotopy cat of "admissible" model cat's} \end{cases}$$

(1) A self-injective alg

$$\cap| \qquad \underset{\text{abel}}{\operatorname{mod} A} \longrightarrow \underset{\text{tri}}{\underline{\operatorname{mod}} A} := \operatorname{mod} A \Big/ \left\langle \begin{smallmatrix} \bullet & \longrightarrow & \bullet \\ & \searrow \; P \; \nearrow & \\ & (\text{proj}) & \end{smallmatrix} \right\rangle$$

(1') A Gorenstein alg

$$\cap| \qquad \operatorname{mod} A \supseteq \underset{\text{exact}}{\operatorname{CM} A} := \{ X \mid \operatorname{Ext}^1(X, A) = 0 \} \longrightarrow \underset{\text{tri}}{\underline{\operatorname{CM}} A}$$

(1'') $\mathcal{C}$ exact cat $\supseteq \mathcal{N}$ biresolv. cat

$$\underset{\text{exact}}{\mathcal{C}} \longrightarrow \underset{\text{tri}}{\mathcal{C}/\mathcal{N}} \qquad \text{Rump's loc.} \qquad \text{[Rump '21]}$$

cf A Gor. alg $\rightsquigarrow$ $\left.\begin{matrix} (\operatorname{CM} A, \underset{\text{coresolv.}}{P^{<\infty}}) \\ (\underset{\text{resolv.}}{I^{<\infty}}, \operatorname{co}\operatorname{CM} A) \end{matrix}\right\}$ Cotorsion pairs with $\overset{\text{biresolv.}}{P^{<\infty}} = I^{<\infty} = \mathcal{N}$ (i.e. Cotor. triple)

$$\underset{\text{abel.}}{\operatorname{mod} A} \longrightarrow \underset{\text{tri}}{\operatorname{mod} A / \mathcal{N}} \simeq \underline{\operatorname{CM}} A$$

(2) $\mathcal{C}$ tri. cat $\ni$ $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ t-structure

$$\mathcal{C} \xrightarrow{H} \mathcal{C}^{\leq 0} \cap \mathcal{C}^{\geq 0} =: \mathcal{H} \quad \text{cohom.}$$

tri heart abel.

[Beilinson - Berenstein - Deligne]

(2') $\mathcal{C}$ tri $\ni$ $(\mathcal{U}, \mathcal{V})$ cotorsion pair

$$\mathcal{C} \xrightarrow{H} \underline{\mathcal{H}} \quad \text{cohom}$$

tri heart abel

[Abe - Nakaoka]

(3) $\mathcal{C}$ tri $\ni$ X rigid + α

$$\mathcal{C} \xrightarrow{(X,-)} \operatorname{mod} \operatorname{End}(X)$$

tri abel

[Buan - Marsh]

The above phenomena can be regarded as Loc. of ET cat's.

§2 Localization of ET cat's

Def $\mathcal{C} := (\mathcal{C}, \mathbb{E}, \mathfrak{s})$ ET cat

$$\begin{cases} \mathcal{C} & \text{add. cat} \\ \mathbb{E}(-,-) : \mathcal{C} \times \mathcal{C}^{op} \to Ab & \text{bifunc} \\ \mathfrak{s} : \mathbb{E}(C,A) \to \{\text{"triangle"}\} & (\text{or } \{\text{"s.e.s"}\}) \end{cases} + \boxed{\text{Some axioms}}$$

ET	tri	exact
$\mathcal{C}$	$\mathcal{C}$	$\mathcal{C}$
$\mathbb{E}(-,-)$	$\mathcal{C}(-,-[1])$	$\mathrm{Ext}^1(-,-)$
$\mathfrak{s}$ realization	$h \in \mathcal{C}(C, A[1]) \overset{\mathfrak{s}}{\longmapsto} A \to B \to C \xrightarrow{h} A[1]$	$\delta \in \mathrm{Ext}^1(C,A) \overset{\mathfrak{s}}{\longmapsto} 0 \to A \to B \to C \to 0$ via Yoneda ext.

Rmk $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ ET cat

$$\delta \in \mathbb{E}(C,A) \underset{\text{Yoneda}}{\cong} \{ (-,C) \xrightarrow{\delta_\#} \mathbb{E}(-,A) \}$$

$$\downarrow$$

$A \xrightarrow{f} B \xrightarrow{g} C$

$\mathfrak{s}$ - conflation

$$\left.\begin{array}{l} (-,A) \xrightarrow{f} (-,B) \xrightarrow{g} (-,C) \\ \xrightarrow{\delta_\#} \mathbb{E}(-,A) \xrightarrow{f} \mathbb{E}(-,B) \xrightarrow{g} \mathbb{E}(-,C) \end{array}\right\} \text{exact in } Ab$$

$$\begin{cases} f & \mathfrak{s}\text{ - inflation} \\ g & \mathfrak{s}\text{ - deflation} \end{cases}$$

Def $(F, \phi) : (\mathcal{C}, \mathbb{E}, \mathfrak{s}) \to (\mathcal{C}', \mathbb{E}', \mathfrak{s}')$ exact func.

$$:\Leftrightarrow \begin{cases} F : \mathcal{C} \to \mathcal{C}' & \text{add func} \\ \phi : \mathbb{E} \to \mathbb{E}' \circ (F^{op} \times F) & \end{cases}$$

s.t.

$$\begin{tikzcd} \mathbb{E}(C,A) \arrow[r, "\mathfrak{s}"] \arrow[d, "\phi"'] \arrow[dr, phantom, "\circlearrowright"] & {[A \to B \to C]} \arrow[d, "F"] \\ \mathbb{E}'(FC, FA) \arrow[r, "\mathfrak{s}'"'] & {[FA \to FB \to FC]} \end{tikzcd}$$

Rmk (A benefit of ET cat)

ET cat is closed under basic categ. operations.

$\mathcal{C}$: given ET cat

(1) [NP] $\mathcal{C} \supseteq \mathcal{N}$ extension-closed

$$\rightsquigarrow \begin{tikzcd} (\mathcal{N}, \mathbb{E}|_{\mathcal{N}}, \mathfrak{s}|_{\mathcal{N}}) \arrow[d, hook, "\text{exact}"] \\ \mathcal{C} \end{tikzcd} \quad \text{ET cat}$$

(2) [NP] $\mathcal{C} \supseteq \mathcal{N}$ a subcat of proj-inj.

$$\rightsquigarrow \mathcal{C} \xrightarrow[\text{ideal quot.}]{} \mathcal{C}/[\mathcal{N}] \quad \text{exact}$$

ET cat

(3) [Herschend - Liu - Nakaoka]

$\mathcal{C} \supseteq \mathcal{N}$ any subcat

$$\rightsquigarrow \quad \mathbb{E}^{\mathcal{N}}(C,A) := \left\{ \begin{tikzcd} A \arrow[r] \arrow[d, equal] & B' \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{Pb}"] & N \arrow[d] \\ A \arrow[r] & B \arrow[r] & C \end{tikzcd} \right\}$$

$\forall \in \mathcal{N}$ split

$(\mathcal{C}, \mathbb{E}^{\mathcal{N}}, \mathfrak{s}^{\mathcal{N}}) \xrightarrow{\mathrm{id}} \mathcal{C}$ exact func

(ET cat which makes $\mathcal{N}$ projective)

e.g.

(a) $\operatorname{mod} A \supseteq \mathcal{N}$ a subcat

$\mathbb{E}^{\mathcal{N}}(C,A) := \{ \quad (1) \quad \}$

$(\operatorname{mod} A, \mathbb{E}^{\mathcal{N}}, \mathfrak{s}^{\mathcal{N}})$ exact cat

$\rightsquigarrow$ [Auslander - Solberg] relative cotilting.

(b) $\mathcal{C}$: cpt. gen. tri. cat. $\supseteq \mathcal{N} := \{\text{cpt}\}$

$\mathbb{E}^{\mathcal{N}}(C,A) := \{ \text{pure - exact triangle} \}$

$\rightsquigarrow$ [Beligiannis, Krause] purity in tri. cat.

(4) [Nakaoka - O - Sakai] Today's aim!

$\mathcal{C} \supseteq \mathcal{N}$ thick subcat $+ \alpha$

$\rightsquigarrow \quad \mathcal{N} \longrightarrow \mathcal{C} \longrightarrow \mathcal{C}/\mathcal{N}$ localization of ET cat

<u>Def</u> $\mathcal{C}$ ET cat $\supseteq \mathcal{N}$ full subcat

(1) $\mathcal{N}$ <u>thick</u> $:\Leftrightarrow$ 2-out-of-3 for conflations.

$\left(\begin{array}{l} \text{i.e. } \forall\, A \to B \to C \ \ \mathfrak{s}\text{-confl} \\ \text{two of } \{A, B, C\} \in \mathcal{N} \ \Rightarrow \ \text{all} \in \mathcal{N} \end{array} \right)$

(2) $\mathcal{R} := \{ r \mid \overset{\in \mathcal{N}}{N} \to B \xrightarrow{r} C \ \text{confl} \}$

$\mathcal{L} := \{ l \mid A \xrightarrow{l} B \to \overset{\in \mathcal{N}}{N} \ \text{confl} \}$

$\mathcal{S}_{\mathcal{N}} := \{ \text{finite compositions of maps in } \mathcal{R} \cup \mathcal{L} \}$

$= (\cdots \circ \mathcal{R} \circ \mathcal{L} \circ \mathcal{R} \circ \mathcal{L} \circ \cdots)$

(3) $\mathcal{C} \longrightarrow \mathcal{C}/_{[\mathcal{N}]} =: \overline{\mathcal{C}}$ ideal quot.

<u>Thm</u> $\mathcal{C}$ ET cat $\supseteq \mathcal{N}$ thick $\rightsquigarrow \mathcal{S}_{\mathcal{N}}$

If $\mathcal{S}_{\mathcal{N}}$ forms a $\left\{ \begin{array}{l} ① \ \text{multiplicative system} \\ ② \ \text{compatible with extriangulation} \end{array} \right.$ in $\overline{\mathcal{C}}$,

then we have an exact seq of ET cat's

$$\mathcal{N} \longrightarrow \mathcal{C} \xrightarrow{Q} \mathcal{C}/\mathcal{N}$$

① multipicative system $\overline{\mathcal{S}_r}$

(a) 2 - out - of - 3 for composition

(b) closed under finite coproducts

(c)
$$\begin{tikzcd} \bullet \arrow[r, "\bar{s} \in \overline{\mathcal{S}_r}"] \arrow[d, "\bar{f}"'] \arrow[dr, phantom, "\mathcal{Q}"] & \bullet \arrow[d, dashed, "\bar{f}'"] \\ \bullet \arrow[r, dashed, "\bar{s}'"'] & \bullet \end{tikzcd}$$
$\bar{s}' \in \overline{\mathcal{S}_r}$ + (dual)

(d)
$$\begin{tikzcd} \bullet \arrow[r, "\bar{s}"] \arrow[rr, bend right, "0"'] & \bullet \arrow[r, "\bar{f}"] \arrow[rr, dashed, bend right, "0"'] & \bullet \arrow[r, dashed, "\bar{t}"] & \bullet \end{tikzcd}$$
$\overline{\mathcal{S}_r} \ni \bar{s}$, $\bar{t} \in \overline{\mathcal{S}_r}$ + (dual)

② compatible with extriangulation

(a)
$$\begin{tikzcd} A \arrow[r] \arrow[d, "a"'] & B \arrow[r] \arrow[d, "b"'] & C \arrow[d, "c"] \\ A' \arrow[r] & B' \arrow[r] & C' \end{tikzcd}$$
map of $\mathfrak{s}$ - conflation.

If $\bar{a}, \bar{c} \in \overline{\mathcal{S}_r}$, then $\exists (\bar{a}, \bar{b}, \bar{c})$ map of confl. ($\bar{b} \in \overline{\mathcal{S}_r}$)

(b) $\overline{\mathrm{Inf}} := \{ \bar{s} \circ \bar{f} \circ \bar{t} \mid \bar{s}, \bar{t} \in \overline{\mathcal{S}_r}, f \text{ infl} \}$ closed under compositions

+ (dual)

Cor $\mathcal{C}$: ET cat $\supseteq \mathcal{N}$ <u>biresolving</u>

$:\Leftrightarrow$ · $\mathcal{N}$ thick

· For $^\forall C \in \mathcal{C}$, $\exists \begin{cases} C \to N & \text{infl} \\ N' \to C & \text{defl} \end{cases}$ ($N \in \mathcal{N}$, $N' \in \mathcal{N}$)

Then, $\exists\ \mathcal{N} \hookrightarrow \mathcal{C} \to \underset{\text{triang.}}{\mathcal{C}/\mathcal{N}}$ ex. seq of ET cat.

* The above contains §1 (1) self-inj

(1') Gorenstein

(1'') Rump's loc.

(0) der. cat. $\begin{cases} \text{Ver. quot.} \\ \text{homotopy cat of a model cat.} \end{cases}$

Rmk

(1) $C(\mathrm{Mod}A)$: ET cat (abel. ex. cat.)

$\cup$

$C^{ac}(\mathrm{Mod}A)$: biresolv.

$\underset{\text{biresolv.}}{C^{ac}(A)} \to \underset{\text{abel.}}{C(A)} \to \underset{\text{tri.}}{D(A)}$ ex. seq. of ET cat

☺

$C^{ac}(A) =: \mathcal{N}$ biresolv.

$(C(A),$ termwise-sp. ex$)$: Frob. ex. cat.

$\forall C \to I_C$ inj. preenvelope in ⓪.
$\cap$
$\mathcal{N}$

$\mathcal{S}_{\mathcal{N}} = \{$ q.i.s. $\}$

$\mathcal{R} = \{ r \mid N \to A \xrightarrow{r} B$ s.e.s $\}$, $N \in \mathcal{N}$

$\mathcal{L} =$ ______________

$\rightsquigarrow$

$$\begin{array}{ccccc} C^{ac}(A) & \longrightarrow & C(A) & \longrightarrow & D(A) \\ | & & | & & \| \\ K^{ac}(A) & \longrightarrow & K(A) & \longrightarrow & D(A) \end{array}$$

ex. seq. of ET cat

ex. seq. of Tri. cat.

(WIC)

(2) $\mathcal{C}$: ET cat equipped with an admissible model str.

$\cup$:$\Leftrightarrow$ (tCof, Fib), (Cof, tFib) } Hovey twin cotorsion pair

$\mathcal{N}$:= Core (tFib, tCof) biresolv

$$\underset{\text{ET}}{\mathcal{N}} \longrightarrow \underset{\text{ET}}{\mathcal{C}} \longrightarrow \underset{\text{tri.}}{\mathrm{Ho}(\mathcal{C})}$$

ex. seq of ET cat.

(3) [Stovicek] $\mathrm{Mod}\,A \supseteq (\mathcal{U}, \mathcal{V})$ resolv. CP.

$$\Downarrow$$

$$C^{ac}(\mathrm{Mod}\,A) \supseteq (C^{ac}(\mathcal{U}), C^{ac}(\mathcal{V}))\quad \text{CP}$$

$$\Downarrow$$

$$\left((C^{ac}(\mathcal{U}), C^{ac}(\mathcal{U})^{\perp_1}), ({}^{\perp_1}C^{ac}(\mathcal{V}), C^{ac}(\mathcal{V})) \right) : \text{HTCP in } C(\mathrm{Mod}\,A)$$

$$\mathcal{N} := \mathrm{Core}(C^{ac}(\mathcal{V}), C^{ac}(\mathcal{W})) = C^{ac}(\mathrm{Mod}\,A)$$

$$\Rightarrow \mathrm{Ho}(C(A)) = D(A)$$

(In particular, $(\mathcal{U}, \mathcal{V}) = (\mathrm{Mod}\,A, \mathrm{Inj}\,A)$

$\Rightarrow$ the above adm. model. cat

= injective model. str.)

§3 Cohomolog. func.

Obs

$$\mathcal{N} \longrightarrow \mathcal{C} \longrightarrow \mathcal{C}/\mathcal{N}$$

tri abel.

$$\left(\begin{array}{ccc} A & \xrightarrow{f} & B \\ \cong & \circlearrowleft & \cong \\ QA' & \xrightarrow[Qf']{} & QB' \end{array} \quad \text{monic in } \mathcal{C}/\mathcal{N} \right)$$

epi

$$\Uparrow$$

$$\left(A' \xrightarrow[f']{} B' \quad \mathfrak{s}\text{-infl in } \mathcal{C} \right)$$

$\mathfrak{s}$ - defl

$$\Rightarrow \mathcal{C}/\mathcal{N} \simeq 0$$

To capture cohom func's,

we tweek the tri. str of $\mathcal{C}$.

<u>Def</u> $\mathcal{C} = (\mathcal{C}, \mathbb{E}, \mathfrak{s})$: tri. cat $\supseteq \mathcal{N}$ extension-closed.

$$\begin{tikzcd}
 & & & \forall N \arrow[d] \arrow[r, dashed] \arrow[dr, phantom, "2"] & N'[1] \arrow[d, dashed] \\
\mathbb{E}_{\mathcal{N}} := \{ C[-1] \arrow[r] \arrow[d, dashed] \arrow[dr, phantom, "2"] & A \arrow[r] \arrow[d] & B \arrow[r] & C \arrow[r] & A[1] \} \\
N'[-1] \arrow[r, dashed] & N & & & \mathbb{E}^{L}_{\mathcal{N}}
\end{tikzcd}$$

$\mathbb{E}^{R}_{\mathcal{N}}$

$\rightsquigarrow (\mathcal{C}, \mathbb{E}_{\mathcal{N}}, \mathfrak{s}_{\mathcal{N}})$: ET cat

$\cup\!\mid$

$\mathcal{N}$: thick

$\overline{\mathcal{S}}_{\mathcal{N}}$ mult. sys. comp. w. extri.

<u>Thm</u> [O] $\mathcal{C}$: tri. cat $\supseteq \mathcal{N}$ ext-cl.

$\exists$ $\mathcal{N} \longrightarrow (\mathcal{C}, \mathbb{E}_{\mathcal{N}}, \mathfrak{s}_{\mathcal{N}}) \xrightarrow{Q} \mathcal{C}/\mathcal{N}$ ex. seq. of ET cat.

$\mathcal{N}$	ext-cl.	thick	$\mathrm{Cone}(\mathcal{N}, \mathcal{N}) = \mathcal{C}$	$(\mathcal{C}, \mathbb{E}, \mathfrak{s})$
$\mathcal{N}$	thick	biresolv.	Serre	$(\mathcal{C}, \mathbb{E}_{\mathcal{N}}, \mathfrak{s}_{\mathcal{N}})$
$\mathcal{C}/\mathcal{N}$	ET	tri	abel.	

* The above contains §1 (2) heart of t-str.

(2) heart of CP

(3) rigid object.

Rmk

(1) $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ t-str.

$\mathcal{N} := \operatorname{add}(\mathcal{C}^{\leq -1} * \mathcal{C}^{\geq 1})$: ext-cl. with $\operatorname{Core}(\mathcal{N}, \mathcal{N}) = \mathcal{C}$.

$$\begin{tikzcd}
 & \mathcal{C} & \\
\mathcal{N} \arrow[r] & (\mathcal{C}, \mathbb{E}_{\mathcal{N}}, \mathfrak{s}_{\mathcal{N}}) \arrow[u, hook] \arrow[r] & \mathcal{C}/\mathcal{N} \simeq \mathcal{C}^{\leq 0} \cap \mathcal{C}^{\geq 0}
\end{tikzcd}$$

Serre ET abel

(2) $(\mathcal{U}, \mathcal{V})$ CP $\mathcal{N} := \operatorname{add}(\mathcal{U} * \mathcal{V})$.

(3) $\mathcal{C} \supseteq \mathcal{J}$ contrav. fin. rigid subcat

$\mathcal{N} := \operatorname{Ker}(\mathcal{J}, -)$: ext-cl. with $\operatorname{Core}(\mathcal{N}, \mathcal{N}) = \mathcal{C}$

$$\mathcal{N} \longrightarrow (\mathcal{C}, \mathbb{E}_{\mathcal{N}}, \mathfrak{s}_{\mathcal{N}}) \longrightarrow \mathcal{C}/\mathcal{N} \simeq \operatorname{mod} \mathcal{J}$$

Serre ET abel.

Thm [0] $\mathcal{C}$: tri. cat $\supseteq \mathcal{N}$ ext-cl. with $\operatorname{Core}(\mathcal{N}, \mathcal{N}) = \mathcal{C}$.

$$\begin{tikzcd}
 & (\mathcal{C}, \mathbb{E}^R_{\mathcal{N}}, \mathfrak{s}^R_{\mathcal{N}}) \arrow[dl] \arrow[drr, bend left, "\text{right-ex.}"] & & \\
(\mathcal{C}, \mathbb{E}, \mathfrak{s}) \arrow[rrr, bend right=15, "\text{cohom.}"'] & & (\mathcal{C}, \mathbb{E}_{\mathcal{N}}, \mathfrak{s}_{\mathcal{N}}) \arrow[ul, hook] \arrow[dl] \arrow[r, "\text{exact}"] & \mathcal{C}/\mathcal{N} \\
 & (\mathcal{C}, \mathbb{E}^L_{\mathcal{N}}, \mathfrak{s}^L_{\mathcal{N}}) \arrow[ul, hook] \arrow[urr, bend right, "\text{left-ex.}"'] & &
\end{tikzcd}$$

tri abel.

Reference

Relative theory

[Auslander - Solberg] Relative homology and representation theory I

[Beligiannis] Relative homological algebra and purity in triangulated categories

[Krause] Smashing subcategories and the telescope conjecture

ET cat

[Herschend - Liu - Nakaoka] n-extriangulated categories (I)

[Nakaoka - Palu] Extriangulated categories, Hovey twin cotorsion pairs and model structures

[Nakaoka - O - Sakai] Localization of extriangulated categories

[O] Localization of triangulated categories with respect to extension-closed subcategories

Others

[Abe - Nakaoka] General heart construction on a triangulated category (II)

[Buan - Marsh] From triangulated categories to module categories via localization (II)

[Nakaoka] General heart construction on a triangulated category (I)

[Šťovíček] Exact model categories, approximation theory, and cohomology of quasi-coherent sheaves