HOMOGENEOUS PSEUDO-KAEHLER MANIFOLDS HAVING NON-CONSTANT HOLOMORPHIC FUNCTIONS

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Abstract. In this article we deal with homogeneous pseudo-Kähler manifolds of absolutely simple Lie groups and find out ones having non-constant holomorphic functions from among the pseudo-Kähler manifolds.

Contents

1. Introduction

It is a fundamental problem to judge whether $\mathcal{O}(M) \cong \mathbb{C}$ or $\mathcal{O}(M) \ncong \mathbb{C}$ for a given connected complex manifold $M = (M, J)$, where $\mathcal{O}(M)$ stands for the complex vector space of holomorphic functions on M and $\mathcal{O}(M) \cong \mathbb{C}$ means that all holomorphic functions on M are constant. For example, one knows

(1) $\mathcal{O}(M) \cong \mathbb{C}$ if M is compact;
(2) $\mathcal{O}(M) \ncong \mathbb{C}$ if M is a domain in some $\mathbb{C}^n$.

In this article we solve the problem for a given homogeneous pseudo-Kähler manifold G/L of connected absolutely simple Lie group G and establish

Theorem 1.1 (cf. Borel [B], Matsushima [M], Wolf [W]). *Let G be a connected absolutely simple Lie group, and G/L a homogeneous pseudo-Kähler manifold of G. Suppose that G acts on G/L effectively and* $\dim G/L \neq 0$. *Then, the following conditions* (A) *and* (B) *are equivalent*:

(A) $\mathcal{O}(G/L) \not\cong \mathbb{C}$.
(B) *The Lie group G is non-compact and G/L is a homogeneous Kähler manifold of G.*

Here we refer to Remark 2.1 *for the complex structure J on G/L (see p.*3*).*

Here are comments on Theorem 1.1.

REMARK 1.2. By results of Borel [B] and Matsushima [M] one can show that the (B) is equivalent to the following condition (C):

(C) The Lie group G is of Hermitian type, L is included in a maximal compact subgroup K of G and the fibering of G/L by K/L over G/K is holomorphic.

In addition, Wolf [W] asserts that the (A) is equivalent to the (C) above. These imply that Theorem 1.1 is not new.

At any rate, the main purpose of this article is to prove the part (A) $\Rightarrow$ (B) of Theorem 1.1.

2. A complex structure and homogeneous holomorphic line bundles

Let G be a connected absolutely simple Lie group, and let G/L be a homogeneous pseudo-Kähler manifold of G such that G acts on G/L effectively and $\dim G/L \neq 0$. Here a Lie group is said to be *absolutely simple*, if its Lie algebra is a real form of a complex simple Lie algebra.

2.1. **An invariant complex structure on G/L.** In general, there are several kinds of complex structures on the pseudo-Kähler manifold G/L. In order to study the vector space $\mathcal{O}(G/L)$ one has to fix a complex structure on G/L. We fix a G-invariant complex structure J on G/L by means of the following way. Since G/L is a homogeneous pseudo-Kähler manifold of connected (semi)simple Lie group G and $\dim G/L \neq 0$, there exists a non-zero elliptic element $T \in \mathfrak{g}$ satisfying

$$L = C_G(T)$$

by Dorfmeister-Guan [DG1, DG2].[1] This assures that the center $Z(G)$ is trivial because G acts on G/L effectively, and the Lie group G is isomorphic to the adjoint group of $\mathfrak{g}$. Accordingly there exists a connected complex simple Lie group $G_{\mathbb{C}}$ such that

(1) $Z(G_{\mathbb{C}})$ is trivial,
(2) G is a connected closed subgroup of $G_{\mathbb{C}}$,
(3) $\mathfrak{g}$ is a real form of $\mathfrak{g}_{\mathbb{C}}$

[1] Also see Chapter 10 in [Bm] if necessary.

because G is absolutely simple. Setting

$$\mathfrak{g}^{\lambda} := \{A \in \mathfrak{g}_{\mathbb{C}} \mid \operatorname{ad} T(A) = i\lambda A\} \text{ for } \lambda \in \mathbb{R}, \quad Q^{-} := N_{G_{\mathbb{C}}}(\textstyle\bigoplus_{\mu \geq 0} \mathfrak{g}^{-\mu}),$$

we know that (i) Q^{-} is a connected closed complex parabolic subgroup of $G_{\mathbb{C}}$, (ii) $L = G \cap Q^{-}$ and (iii) the mapping

$$\iota : G/L \to G_{\mathbb{C}}/Q^{-},\, gL \mapsto gQ^{-}$$

is a G-equivariant diffeomorphism of G/L onto a domain in the complex flag manifold $G_{\mathbb{C}}/Q^{-}$. Since $G_{\mathbb{C}}/Q^{-}$ is a complex homogeneous space, one can transfer the complex structure of $\mathbb{C}^m$ to $G_{\mathbb{C}}/Q^{-}$ by means of charts, and fix a $G_{\mathbb{C}}$-invariant complex structure on $G_{\mathbb{C}}/Q^{-}$, where $m := \dim G_{\mathbb{C}}/Q^{-}$. Identifying G/L with $\iota(G/L)$, one may assume that G/L is a domain in $G_{\mathbb{C}}/Q^{-}$. Then, we induce a G-invariant complex structure J on $G/L = \iota(G/L)$ from $G_{\mathbb{C}}/Q^{-}$.

Remark 2.1. In the same way as above, we fix a G-invariant complex structure J on the pseudo-Kähler manifold G/L in Theorem 1.1.

2.2. **Homogeneous holomorphic line bundles and $\mathcal{O}(G/L)$.** We obey the setting of §§2.1. In this subsection we understand the complex vector space $\mathcal{O}(G/L)$ from the viewpoint of homogeneous holomorphic line bundles.

For any holomorphic homomorphism $\chi : Q^{-} \to \mathbb{C}^*$ we denote by $G_{\mathbb{C}} \times_{\chi} \mathbb{C}$ the homogeneous holomorphic line bundle over the complex flag manifold $G_{\mathbb{C}}/Q^{-}$ associated with χ, and by $\iota^{\sharp}(G_{\mathbb{C}} \times_{\chi} \mathbb{C})$ the restriction of $G_{\mathbb{C}} \times_{\chi} \mathbb{C}$ to the domain $G/L \subset G_{\mathbb{C}}/Q^{-}$. Note that GQ^{-} is a domain in $G_{\mathbb{C}}$ and set

$$\mathcal{V} := \{h : GQ^{-} \to \mathbb{C}, \text{ holomorphic} \mid h(xq) = \chi(q)^{-1}h(x) \text{ for all } (x,q) \in GQ^{-} \times Q^{-}\}.$$

Then one can regard $\mathcal{V}$ as the complex vector space of holomorphic cross-sections of the bundle $\iota^{\sharp}(G_{\mathbb{C}} \times_{\chi} \mathbb{C})$, where we endow G/L with the complex structure J induced from $\iota : G/L \to G_{\mathbb{C}}/Q^{-}$. In case of $\chi = \mathrm{id}$ (the trivial representation) we have

$$\mathcal{O}(G/L) = \left\{ h : GQ^{-} \to \mathbb{C} \;\middle|\; \begin{array}{l} (1)\ h \text{ is holomorphic}, \\ (2)\ h(xq) = h(x) \text{ for all } (x,q) \in GQ^{-} \times Q^{-} \end{array} \right\}. \tag{2.2}$$

3. Proof of Theorem 1.1

The setting of §2 remains valid here. In this section we will prove Theorem 1.1. Our first aim is to prepare some notation and a lemma. Since the element $T \in \mathfrak{g}$ is elliptic, there exists a Cartan involution θ of $\mathfrak{g}$ satisfying $T = \theta(T)$. Set $\mathfrak{k} := \{X \in \mathfrak{g} \mid \theta(X) = X\}$, $\mathfrak{p} := \{Y \in \mathfrak{g} \mid \theta(Y) = -Y\}$ and denote by $G = K \times P$ the Cartan decomposition of the Lie group G corresponding to $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Here we remark that

$$T \in \mathfrak{k},$$

and that K is a connected compact subgroup of G because G is connected and $Z(G)$ is finite. Let $\mathfrak{k}_{\mathbb{C}}$ and $\mathfrak{p}_{\mathbb{C}}$ be the complex subalgebra and complex vector space of $\mathfrak{g}_{\mathbb{C}}$ generated by $\mathfrak{k}$ and $\mathfrak{p}$, respectively, and let $K_{\mathbb{C}}$ be the connected Lie subgroup of $G_{\mathbb{C}}$ corresponding to the subalgebra $\mathfrak{k}_{\mathbb{C}} \subset \mathfrak{g}_{\mathbb{C}}$. Then, it turns out that

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_{\mathbb{C}}, \quad [\mathfrak{k}_{\mathbb{C}}, \mathfrak{p}_{\mathbb{C}}] \subset \mathfrak{p}_{\mathbb{C}}.$$

Moreover, one can show

Lemma 3.1.

(i) *$K_{\mathbb{C}}$ is a connected closed complex (Lie) subgroup of $G_{\mathbb{C}}$.*

(ii) *The orbit $K_{\mathbb{C}}Q^-/Q^-$ of $K_{\mathbb{C}}$ through the origin $o \in G_{\mathbb{C}}/Q^-$ is a connected compact complex submanifold of the complex flag manifold $G_{\mathbb{C}}/Q^-$ and $K_{\mathbb{C}}Q^-/Q^- \subset \iota(G/L)$. Here $K_{\mathbb{C}}$ acts on $G_{\mathbb{C}}/Q^-$ as follows:*

$$K_{\mathbb{C}} \times G_{\mathbb{C}}/Q^- \ni (k, aQ^-) \mapsto kaQ^- \in G_{\mathbb{C}}/Q^-.$$

(iii) *$K/(K \cap L)$ is a connected compact complex submanifold of G/L, where we identify $K/(K \cap L)$ with KL/L via $K/(K \cap L) \ni y(K \cap L) \mapsto yL \in KL/L$ $(\subset G/L)$.*

Proof. (i) We only confirm that $K_{\mathbb{C}}$ is closed in $G_{\mathbb{C}}$. Extend θ to $\mathfrak{g}_{\mathbb{C}}$ as a $\mathbb{C}$-linear involution. By virtue of $Z(G_{\mathbb{C}}) = \{e\}$ one can lift the θ to $G_{\mathbb{C}}$ as a holomorphic involutive automorphism, and $K_{\mathbb{C}}$ coincides with the identity component of its fixed point set. So, $K_{\mathbb{C}} \subset G_{\mathbb{C}}$ is closed.

(ii) It follows from $L = G \cap Q^-$ and (i) that the mapping $K/(K \cap L) \ni y(K \cap L) \mapsto y(K_{\mathbb{C}} \cap Q^-) \in K_{\mathbb{C}}/(K_{\mathbb{C}} \cap Q^-)$ is a diffeomorphism. $K_{\mathbb{C}}/(K_{\mathbb{C}} \cap Q^-)$ is homeomorphic to $K_{\mathbb{C}}Q^-/Q^-$. These imply that $K_{\mathbb{C}}Q^-/Q^-$ is a connected compact subset of $G_{\mathbb{C}}/Q^-$. Since $K_{\mathbb{C}}Q^-/Q^-$ is locally closed in $G_{\mathbb{C}}/Q^-$, the orbit $K_{\mathbb{C}}Q^-/Q^-$ is a regular submanifold of $G_{\mathbb{C}}/Q^-$; moreover, it is complex because the tangent space $T_o(K_{\mathbb{C}}Q^-/Q^-)$ is stable under the complex structure on $G_{\mathbb{C}}/Q^-$ and the action of $K_{\mathbb{C}}$ on $G_{\mathbb{C}}/Q^-$ is holomorphic. Now, the rest of proof is to confirm $K_{\mathbb{C}}Q^-/Q^- \subset \iota(G/L)$. From $K(K_{\mathbb{C}} \cap Q^-) = K_{\mathbb{C}}$ we obtain $K_{\mathbb{C}}Q^-/Q^- = KQ^-/Q^- \subset GQ^-/Q^- = \iota(G/L)$.

(iii) is a consequence of (ii) and $\iota(KL/L) = K_{\mathbb{C}}Q^-/Q^-$. □

Our second aim is to introduce a subset $\mathfrak{m}^- \subset \mathfrak{g}_{\mathbb{C}}$ and clarify some properties of $\mathfrak{m}^-$ (see Proposition 3.2). Take $A \in \mathfrak{g}_{\mathbb{C}}$, $h \in \mathcal{O}(G/L)$ and define a mapping $A^*h : GQ^- \to \mathbb{C}$ as follows:

$$(A^*h)(x) := \frac{d}{dt}\Big|_{t=0} f\big(\exp(-tA)x\big) + i\frac{d}{dt}\Big|_{t=0} g\big(\exp(-tA)x\big) \text{ for } x \in GQ^-,$$

where f and g are the real and imaginary parts of h, respectively. Remark that A^*h is holomorphic because h satisfies the Cauchy-Riemann differential equations, and that the mapping $A \mapsto A^*$ gives rise to a representation of the complex Lie algebra $\mathfrak{g}_{\mathbb{C}}$ on the complex vector space $\mathcal{O}(G/L)$. Now, let

$$\mathfrak{m}^- := \{A \in \mathfrak{g}_{\mathbb{C}} \mid (A^*h)(e) = 0 \text{ for all } h \in \mathcal{O}(G/L)\}.$$

Then, we demonstrate

Proposition 3.2.
(i) $\mathfrak{m}^-$ *is a complex subalgebra of* $\mathfrak{g}_{\mathbb{C}}$.
(ii) $\mathfrak{k}_{\mathbb{C}} \subset \mathfrak{m}^-$, $\mathfrak{q}^- \subset \mathfrak{m}^-$.
(iii) $\mathfrak{m}^- = \mathfrak{k}_{\mathbb{C}} \oplus (\mathfrak{p}_{\mathbb{C}} \cap \mathfrak{m}^-)$.
(iv) $\mathfrak{g}_{\mathbb{C}} \subset \mathfrak{m}^-$ *if and only if* $\mathcal{O}(G/L) \cong \mathbb{C}$.

Proof. (i) comes from the mapping $\mathfrak{g}_{\mathbb{C}} \ni A \mapsto A^* \in \mathfrak{gl}(\mathcal{O}(G/L))$ being a homomorphism.

(ii) For an arbitrary holomorphic function on $G/L = \iota(G/L)$, its restriction to $K_{\mathbb{C}}Q^-/Q^-$ is always constant due to Lemma 3.1-(ii); and thus the inclusion $\mathfrak{k}_{\mathbb{C}} \subset \mathfrak{m}^-$ holds, where we remark that $K_{\mathbb{C}}Q^-$ is a closed complex submanifold of $G_{\mathbb{C}}$ and $K_{\mathbb{C}}Q^- \subset GQ^-$. It is immediate from (2.2)-(2) that $\mathfrak{q}^- \subset \mathfrak{m}^-$.

(iii) Denote the lift in the proof of Lemma 3.1-(i) by the same notation θ. Then it turns out that $\theta(Q^-) \subset Q^-$, $\theta(GQ^-) \subset GQ^-$, so that $h \circ \theta \in \mathcal{O}(G/L)$ for all $h \in \mathcal{O}(G/L)$. This assures $\theta(\mathfrak{m}^-) \subset \mathfrak{m}^-$, and hence $\mathfrak{m}^- = \mathfrak{k}_{\mathbb{C}} \oplus (\mathfrak{p}_{\mathbb{C}} \cap \mathfrak{m}^-)$ follows by $\mathfrak{k}_{\mathbb{C}} \subset \mathfrak{m}^-$.

(iv) One has (iv), since every holomorphic function on the domain GQ^- is determined by its derivatives at a point. □

From now on, let us investigate the following three cases individually:
(c1) G is compact;
(c2) G is non-compact
 (c2.1) G is not of Hermitian type;
 (c2.2) G is of Hermitian type.

3.1. **Case (c1).** If G is compact, then G/L is a connected compact complex manifold. This provides us with

Lemma 3.3 (c1). *Suppose that the Lie group G is compact. Then $\mathcal{O}(G/L) \cong \mathbb{C}$.*

3.2. **Case (c2.1).** Let us construct a compact real form $\mathfrak{g}_u \subset \mathfrak{g}_{\mathbb{C}}$ from $\mathfrak{g}_u := \mathfrak{k} \oplus i\mathfrak{p}$, and denote by $\overline{\tau}$ the conjugation of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{g}_u$. By use of this $\overline{\tau}$ we prove

Lemma 3.4 (c2.1). *Suppose that the Lie group G is non-compact and is not of Hermitian type. Then $\mathcal{O}(G/L) \cong \mathbb{C}$.*

Proof. Since $G_{\mathbb{C}}$ is simple, the supposition allows us to assert that the linear isotropy representation $\mathfrak{k}_{\mathbb{C}} \to \mathfrak{gl}(\mathfrak{p}_{\mathbb{C}})$, $A \mapsto \operatorname{ad} A|_{\mathfrak{p}_{\mathbb{C}}}$, is irreducible. Therefore Proposition 3.2-(i), (iii) tells us that either $\mathfrak{p}_{\mathbb{C}} \cap \mathfrak{m}^- = \{0\}$ or $\mathfrak{p}_{\mathbb{C}} \cap \mathfrak{m}^- = \mathfrak{p}_{\mathbb{C}}$ holds. If $\mathfrak{p}_{\mathbb{C}} \cap \mathfrak{m}^- = \{0\}$, then one deduces $\mathfrak{q}^- \subset \mathfrak{k}_{\mathbb{C}}$ by Proposition 3.2-(ii), (iii). From $\mathfrak{q}^- \subset \mathfrak{k}_{\mathbb{C}}$ we obtain

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{q}^- + \overline{\tau}(\mathfrak{q}^-) \subset \mathfrak{k}_{\mathbb{C}} + \overline{\tau}(\mathfrak{k}_{\mathbb{C}}) \subset \mathfrak{k}_{\mathbb{C}},$$

which is a contradiction since G is non-compact. For this reason $\mathfrak{p}_{\mathbb{C}} \cap \mathfrak{m}^- = \mathfrak{p}_{\mathbb{C}}$ holds. Then one concludes $\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_{\mathbb{C}} \subset \mathfrak{m}^-$, $\mathcal{O}(G/L) \cong \mathbb{C}$ by Proposition 3.2. □

3.3. **Case (c2.2).** Suppose that the Lie group G is of Hermitian type. Then, G/K is an effective, irreducible Hermitian symmetric space of non-compact type, and there exists a non-zero elliptic element $W \in \mathfrak{g}$ such that

(1) the eigenvalue of $\operatorname{ad} W$ is $\pm i$ or zero,
(2) $K = C_G(W)$.

Here the existence of W is unique up to sign $\pm$. Setting $\mathfrak{p}^+ := \{A \in \mathfrak{g}_{\mathbb{C}} \mid \operatorname{ad} W(A) = iA\}$, $\mathfrak{p}^- := \{B \in \mathfrak{g}_{\mathbb{C}} \mid \operatorname{ad} W(B) = -iB\}$ one has

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-, \quad [\mathfrak{k}_{\mathbb{C}}, \mathfrak{p}^\pm] \subset \mathfrak{p}^\pm, \quad \mathfrak{p}_{\mathbb{C}} = \mathfrak{p}^+ \oplus \mathfrak{p}^-.$$

Remark 3.5 (c2.2). The Hermitian symmetric space G/K admits a unique G-invariant complex structure up to sign $\pm$, which is induced by either $\operatorname{ad} W$ or $-\operatorname{ad} W$. Moreover, G/K admits a unique G-invariant Kähler metric up to positive multiplicative constant.

In the setting above, let us give two lemmas and deduce Proposition 3.8 from them.

Lemma 3.6 (c2.2). *Suppose that there exists a maximal compact subgroup $\tilde{K}$ of G such that* (s1) $L \subset \tilde{K}$ *and* (s2) *the fibering of G/L by $\tilde{K}/L$ over $G/\tilde{K}$ is holomorphic. Then, G/L is biholomorphic to $G/\tilde{K} \times \tilde{K}/L$ and $\mathcal{O}(G/L) \ncong \mathbb{C}$.*

Proof. The supposition enables us to construct a Cartan decomposition $G = \tilde{P} \times \tilde{K}$ from $\tilde{K}$ and to define a biholomorphism $\Psi : G/L \to G/\tilde{K} \times \tilde{K}/L$ by

$$\Psi(gL) := (g\tilde{K}, kL) \text{ for } gL \in G/L,$$

where $g = pk \in G = \tilde{P} \times \tilde{K}$, and we remark that $\Psi^{-1}(pk\tilde{K}, \tilde{k}L) = p\tilde{k}L$. Now, we are going to conclude $\mathcal{O}(G/L) \ncong \mathbb{C}$. Since the Hermitian symmetric space $G/\tilde{K}$ is biholomorphic to a domain in some $\mathbb{C}^n$, we have $\mathcal{O}(G/\tilde{K}) \ncong \mathbb{C}$; therefore $\mathcal{O}(G/L) \ncong \mathbb{C}$ because $\Psi : G/L \to G/\tilde{K} \times \tilde{K}/L$ is biholomorphic. □

Lemma 3.7 (c2.2). *Let $\mathfrak{u}^+ := \bigoplus_{\lambda>0} \mathfrak{g}^\lambda$, $\mathfrak{u}^- := \bigoplus_{\lambda>0} \mathfrak{g}^{-\lambda}$. Then, the following items* (i) *and* (ii) *hold*:

(i) $\mathfrak{p}^+ \subset \mathfrak{u}^+$ *if and only if* $\mathfrak{p}^- \subset \mathfrak{u}^-$.
(ii) $\mathfrak{p}^+ \subset \mathfrak{u}^-$ *if and only if* $\mathfrak{p}^- \subset \mathfrak{u}^+$.

Proof. From $\overline{\tau}(\mathfrak{p}^\pm) \subset \mathfrak{p}^\mp$ and $\overline{\tau}(\mathfrak{u}^\pm) \subset \mathfrak{u}^\mp$ we get the conclusion. □

Proposition 3.8 (c2.2). *The following three items hold for $\mathfrak{u}^\pm = \bigoplus_{\lambda>0} \mathfrak{g}^{\pm\lambda}$:*

(i) *If $\mathfrak{p}^+ \subset \mathfrak{u}^+$, then* (i.1) $\mathcal{O}(G/L) \ncong \mathbb{C}$, (i.2) $L \subset K$ *and* (i.3) *G/L is a homogeneous Kähler manifold of G such that the mapping*

$$p : G/L \to G/K,\ gL \mapsto gK$$

is a G-equivariant Kählerian submersion, where the complex structure on G/K is induced by $\operatorname{ad} W$.

(ii) *If* $\mathfrak{p}^+ \subset \mathfrak{u}^-$, *then* (ii.1) $\mathcal{O}(G/L) \ncong \mathbb{C}$, (ii.2) $L \subset K$ *and* (ii.3) G/L *is a homogeneous Kähler manifold of G such that $p : G/L \to G/K$, $gL \mapsto gK$, is a G-equivariant Kählerian submersion, where the complex structure on G/K is induced by* $-\operatorname{ad} W$.

(iii) *If* $\mathfrak{p}^+ \not\subset \mathfrak{u}^+$ *and* $\mathfrak{p}^+ \not\subset \mathfrak{u}^-$, *then* $\mathcal{O}(G/L) \cong \mathbb{C}$. *Moreover, there never exists any maximal compact subgroup $\tilde{K}$ of G such that* (s1) $L \subset \tilde{K}$ *and* (s2) *the fibering of G/L by $\tilde{K}/L$ over $G/\tilde{K}$ is holomorphic.*

Proof. (i) Suppose that $\mathfrak{p}^+ \subset \mathfrak{u}^+$. Then, Lemma 3.7-(i) yields $\mathfrak{p}^- \subset \mathfrak{u}^-$. In view of $\mathfrak{p}^\pm \subset \mathfrak{u}^\pm$ we see that $L \subset K$, and that the mapping

$$p : G/L \to G/K, \, gL \mapsto gK$$

is holomorphic when the complex structure on G/K is induced by $\operatorname{ad} W$. Since G/K is biholomorphic to a domain in $\mathbb{C}^n$, one has $\mathcal{O}(G/K) \ncong \mathbb{C}$. Consequently $\mathcal{O}(G/L) \ncong \mathbb{C}$ follows by $p : G/L \to G/K$ being surjective holomorphic. Now, let $\mathfrak{u} := [T, \mathfrak{g}]$. Then we obtain $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{u}$ and $\operatorname{Ad} L(\mathfrak{u}) \subset \mathfrak{u}$, so the homogeneous space G/L is reductive. Defining G-invariant Riemann metrics g on G/L, and h on G/K by

$$\mathsf{g}_o(X_1, X_2) := -B_{\mathfrak{g}}(X_1, \theta(X_2)) \text{ for } X_1, X_2 \in T_o(G/L) = \mathfrak{u}, \text{ and}$$
$$\mathsf{h}_o(Y_1, Y_2) := B_{\mathfrak{g}}(Y_1, Y_2) \text{ for } Y_1, Y_2 \in T_o(G/K) = \mathfrak{p},$$

respectively, we conclude that g is a G-invariant Kähler metric on G/L and $p : G/L \to G/K$, $gL \mapsto gK$, is a G-equivariant Kählerian submersion. Here $B_{\mathfrak{g}}$ stands for the Killing form of $\mathfrak{g}$.

(ii) Similarly, one can conclude (ii).

(iii) On the one hand; it follows from $\mathfrak{p}^+ \not\subset \mathfrak{u}^+$ (resp. $\mathfrak{p}^+ \not\subset \mathfrak{u}^-$) that $\{0\} \neq \mathfrak{p}^+ \cap \mathfrak{q}^- \subset \mathfrak{p}^+ \cap \mathfrak{m}^-$ (resp. $\{0\} \neq \mathfrak{p}^- \cap \mathfrak{m}^-$). On the other hand; the representations $\mathfrak{k}_{\mathbb{C}} \to \mathfrak{gl}(\mathfrak{p}^\pm)$, $A \mapsto \operatorname{ad} A|_{\mathfrak{p}^\pm}$, are irreducible. Consequently, Proposition 3.2 enables us to see that $\mathfrak{p}^+ = \mathfrak{p}^+ \cap \mathfrak{m}^-$ and $\mathfrak{p}^- = \mathfrak{p}^- \cap \mathfrak{m}^-$; and furthermore, $\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^- \subset \mathfrak{m}^-$, and $\mathcal{O}(G/L) \cong \mathbb{C}$. Then we complete the proof, by the contraposition of Lemma 3.6. □

Lemma 3.1-(iii) and Proposition 3.8 lead to

Corollary 3.9 (c2.2). *Suppose that the Lie group G is of Hermitian type.*

(c2.2.1) *If* $\mathfrak{p}^+ \subset \mathfrak{u}^+$ *or* $\mathfrak{p}^+ \subset \mathfrak{u}^-$, *then* (1) $\mathcal{O}(G/L) \ncong \mathbb{C}$, (2) $L \subset K$ *and* (3) G/L *is a homogeneous Kähler manifold of G such that the fibering of G/L by K/L over G/K is holomorphic.*

(c2.2.2) *If* $\mathfrak{p}^+ \not\subset \mathfrak{u}^+$ *and* $\mathfrak{p}^+ \not\subset \mathfrak{u}^-$, *then* $\mathcal{O}(G/L) \cong \mathbb{C}$. *Moreover, there never exists any maximal compact subgroup $\tilde{K}$ of G such that* (s1) $L \subset \tilde{K}$ *and* (s2) *the fibering of G/L by $\tilde{K}/L$ over $G/\tilde{K}$ is holomorphic.*

3.4. **Summary.** Let us prove Theorem 1.1.

Proof of Theorem 1.1. There are four cases where

(c1) G is compact;

(c2) G is non-compact

(c2.1) G is not of Hermitian type;

(c2.2) G is of Hermitian type

(c2.2.1) $\mathfrak{p}^+ \subset \mathfrak{u}^+$ or $\mathfrak{p}^+ \subset \mathfrak{u}^-$;

(c2.2.2) $\mathfrak{p}^+ \not\subset \mathfrak{u}^+$ and $\mathfrak{p}^+ \not\subset \mathfrak{u}^-$.

Needless to say, the cases (c1), (c2.1), (c2.2.1) and (c2.2.2) are mutually exclusive and one of them necessarily occurs. Lemma 3.3, Lemma 3.4 and Corollary 3.9 imply that

(c1) $\mathcal{O}(G/L) \cong \mathbb{C}$;

(c2.1) $\mathcal{O}(G/L) \cong \mathbb{C}$;

(c2.2.1) $\mathcal{O}(G/L) \not\cong \mathbb{C}$, $L \subset K$ and G/L is a homogeneous Kähler manifold of G such that the fibering of G/L by K/L over G/K is holomorphic;

(c2.2.2) $\mathcal{O}(G/L) \cong \mathbb{C}$. Moreover, there never exists any maximal compact subgroup $\tilde{K}$ of G such that (s1) $L \subset \tilde{K}$ and (s2) the fibering of G/L by $\tilde{K}/L$ over $G/\tilde{K}$ is holomorphic.

(A) $\Rightarrow$ (B): Suppose that $\mathcal{O}(G/L) \not\cong \mathbb{C}$. Then (c2.2.1) only occurs, and so G is non-compact and G/L is a homogeneous Kähler manifold of G.

(B) $\Rightarrow$ (A): Suppose that G is non-compact and G/L is a homogeneous Kähler manifold of G. Then, Borel [B] and Matsushima [M] show that G is of Hermitian type, L is included in a maximal compact subgroup K' of G and the fibering of G/L by K'/L over G/K' is holomorphic. Hence (c2.2.1) only occurs, and so $\mathcal{O}(G/L) \not\cong \mathbb{C}$. □

Remark 3.10. We have proven Theorem 1.1 by slightly modifying the proof of Zierau [Z, Proposition 3.16].

3.5. **How to determine homogeneous pseudo-Kähler manifolds having non-constant holomorphic functions.** First, we recall the setting of §§3.3 and give a necessary and sufficient condition for the pseudo-Kähler manifold G/L to satisfy

$$\text{(c2.2.1) } \mathfrak{p}^+ \subset \mathfrak{u}^+ \text{ or } \mathfrak{p}^+ \subset \mathfrak{u}^-,$$

cf. Lemma 3.12. Fix a maximal torus $\mathfrak{t}$ of $\mathfrak{k}$ containing the element T. Here the element W automatically belongs to $\mathfrak{t}$ due to $K = C_G(W)$. Let $\mathfrak{t}_{\mathbb{C}}$ be the complex subalgebra of $\mathfrak{g}_{\mathbb{C}}$ generated by $\mathfrak{t}$, and let $\triangle = \triangle(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ be the root system of $\mathfrak{g}_{\mathbb{C}}$ relative to $\mathfrak{t}_{\mathbb{C}}$. Take a fundamental root system $\Pi = \{\alpha_j\}_{j=1}^{\ell} \subset \triangle$ such that

$$\alpha_j(-iW) \geq 0 \text{ for all } 1 \leq j \leq \ell.$$

In addition, let $\tilde{\alpha} = \sum_{j=1}^{\ell} m_j \alpha_j$ $(\in \triangle)$ and $\{Z_j\}_{j=1}^{\ell}$ $(\subset i\mathfrak{t})$ be the highest root with respect to $\Pi = \{\alpha_j\}_{j=1}^{\ell}$ and the dual basis of $\{\alpha_j\}_{j=1}^{\ell}$, respectively. Then, there exists a unique index $1 \leq p \leq \ell$ such that

$$m_p = 1, \quad W = iZ_p,$$

because the eigenvalue of $\operatorname{ad} W$ is $\pm i$ or zero. Taking this p into account, we set

$$\Pi_p := \Pi - \{\alpha_p\}, \quad \mathcal{W}_{\mathfrak{k}} := \{H \in \mathfrak{t} \,|\, \beta(-iH) \geq 0 \text{ for all } \beta \in \Pi_p\}.$$

It follows from $T \in \mathfrak{t}$ that there exists a $k \in K$ satisfying $\operatorname{Ad} k(T) \in \mathcal{W}_{\mathfrak{k}}$. The invariant complex structure on G/K is induced by either $\operatorname{ad} W$ or $-\operatorname{ad} W$. These, together with $K = C_G(W)$, allow us to assume that

$$T \in \mathcal{W}_{\mathfrak{k}}. \tag{3.11}$$

On the assumption (3.11) we establish

Lemma 3.12. *$\alpha_p(-iT) > 0$ if and only if $\mathfrak{p}^+ \subset \mathfrak{u}^+$.*

Proof. Let $\mathfrak{g}_\alpha$ denote the root subspace of $\mathfrak{g}_{\mathbb{C}}$ for $\alpha \in \triangle$, and let $\triangle_1 := \{\gamma = \sum_{j=1}^{\ell} n_j\alpha_j \in \triangle \,|\, n_p = 1\}$. First, let us suppose that $\alpha_p(-iT) > 0$. Then, it follows from (3.11) that $\gamma(-iT) > 0$ for all $\gamma \in \triangle_1$, so that $\mathfrak{p}^+ = \bigoplus_{\gamma \in \triangle_1} \mathfrak{g}_\gamma \subset \mathfrak{u}^+$. Now, we suppose that $\mathfrak{p}^+ \subset \mathfrak{u}^+$. Then $\mathfrak{g}_{\alpha_p} \subset \mathfrak{p}^+ \subset \mathfrak{u}^+$ holds, and hence there exist a non-zero $X \in \mathfrak{g}_{\alpha_p}$ and a $\lambda > 0$ satisfying $\operatorname{ad} T(X) = i\lambda X$; besides, $i\lambda X = [T, X] = \alpha_p(T)X = i\alpha_p(-iT)X$. This assures $\alpha_p(-iT) > 0$. □

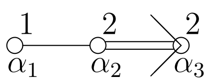
Lemma 3.12 provides a necessary and sufficient condition for the G/L to satisfy (c2.2.1) $\mathfrak{p}^+ \subset \mathfrak{u}^+$ or $\mathfrak{p}^+ \subset \mathfrak{u}^-$, because one can substitute $\mathfrak{p}^+$ for $\mathfrak{p}^-$ by changing the sign of the complex structure on G/K and further $\mathfrak{p}^- \subset \mathfrak{u}^-$ implies $\mathfrak{p}^+ \subset \mathfrak{u}^+$ (cf. Lemma 3.7). Therefore, we can determine homogeneous pseudo-Kähler manifolds satisfying the condition (c2.2.1) by means of the following three steps:

- (S1) Fix an irreducible root system $\triangle(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ and a fundamental root system $\Pi = \{\alpha_j\}_{j=1}^{\ell} \subset \triangle(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$, provided that $\{m_1, m_2, \ldots, m_\ell\}$ contains 1 for the highest root $\tilde{\alpha} = \sum_{j=1}^{\ell} m_j\alpha_j$ with respect to Π.
- (S2) If $m_p = 1$, then we put $\Pi_p := \Pi - \{\alpha_p\}$ and define a compact Lie algebra $\mathfrak{k}$ so that Π_p is a fundamental root system of $\triangle(\mathfrak{k}, \mathfrak{t})$, and define a non-compact real form $\mathfrak{g} \subset \mathfrak{g}_{\mathbb{C}}$ so that $\mathfrak{k}$ is a maximal compact subalgebra of $\mathfrak{g}$.
- (S3) Take any subset $S \subset \Pi_p$ and define a compact Lie algebra $\mathfrak{l}$ so that S is a fundamental root system of $\triangle(\mathfrak{l}, \mathfrak{t})$.

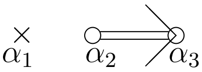
In Example 3.13, let us take the three steps.

Example 3.13. Let $\mathfrak{g}_{\mathbb{C}} = \mathfrak{so}(7, \mathbb{C})$. Assume that the Dynkin diagram of $\triangle(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ is as follows (cf. Plate II in Bourbaki [Br]):

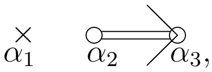
Π

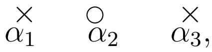


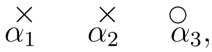
Then $\tilde{\alpha} = \alpha_1 + 2\alpha_2 + 2\alpha_3$ is the highest root with respect to $\Pi = \{\alpha_j\}_{j=1}^{3}$. Therefore we put $\Pi_1 := \Pi - \{\alpha_1\}$, and obtain $\mathfrak{k} = \mathfrak{so}(5) \oplus \mathfrak{so}(2)$, $\mathfrak{g} = \mathfrak{so}(5, 2)$ from Π_1.

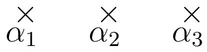
Π_1 

All subdiagrams of $\Pi_1 = \{\alpha_2, \alpha_3\}$ are

(i) 

, (ii) 

, (iii) 

, (iv) 

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These give rise to

(i) $\mathfrak{l}_1 = \mathfrak{so}(5) \oplus \mathfrak{so}(2)$, (ii) $\mathfrak{l}_2 = \mathfrak{u}(2) \oplus \mathfrak{so}(2)$, (iii) $\mathfrak{l}_3 = \mathfrak{u}(2) \oplus \mathfrak{so}(2)$,
(iv) $\mathfrak{l}_4 = \mathfrak{u}(1) \oplus \mathfrak{u}(1) \oplus \mathfrak{so}(2)$.

Note that $\mathfrak{k} = \mathfrak{c}_{\mathfrak{g}}(W)$ for $W = iZ_1$, and that (i) $\mathfrak{l}_1 = \mathfrak{c}_{\mathfrak{g}}(T_1)$ for $T_1 = \lambda i Z_1$, (ii) $\mathfrak{l}_2 = \mathfrak{c}_{\mathfrak{g}}(T_2)$ for $T_2 = \lambda i Z_1 + \nu i Z_3$, (iii) $\mathfrak{l}_3 = \mathfrak{c}_{\mathfrak{g}}(T_3)$ for $T_3 = \lambda i Z_1 + \mu i Z_2$ and (iv) $\mathfrak{l}_4 = \mathfrak{c}_{\mathfrak{g}}(T_4)$ for $T_4 = \lambda i Z_1 + \mu i Z_2 + \nu i Z_3$, where $\{Z_j\}_{j=1}^3$ is the dual basis of $\{\alpha_j\}_{j=1}^3$ and λ, μ, ν are positive real numbers. Consequently,

$$SO_0(5,2)/(SO(5)\cdot SO(2)),\ SO_0(5,2)/(U(2)\cdot SO(2)),\ SO_0(5,2)/(U(1)\cdot U(1)\cdot SO(2))$$

are homogeneous pseudo-Kähler manifolds of $SO_0(5,2)$ satisfying the condition (c2.2.1) $\mathfrak{p}^+ \subset \mathfrak{u}^+$ or $\mathfrak{p}^+ \subset \mathfrak{u}^-$; besides, they exhaust all the homogeneous pseudo-Kähler manifolds G/L of $G = SO_0(5,2)$ such that G acts on G/L effectively, $\dim G/L \neq 0$ and G/L satisfy the condition (c2.2.1). Here $SO_0(5,2)$ is the adjoint group of $\mathfrak{g} = \mathfrak{so}(5,2)$.

Furthermore, Corollary 3.9 implies that $SO_0(5,2)/(SO(5)\cdot SO(2))$, $SO_0(5,2)/(U(2)\cdot SO(2))$ and $SO_0(5,2)/(U(1)\cdot U(1)\cdot SO(2))$ exhaust all the homogeneous pseudo-Kähler manifolds G/L of $G = SO_0(5,2)$ such that G acts on G/L effectively, $\dim G/L \neq 0$ and $\mathcal{O}(G/L) \ncong \mathbb{C}$.

Remark 3.14. There is a homogeneous pseudo-Kähler manifold G/L of $G = SO_0(5,2)$ such that G/L and $SO_0(5,2)/(U(1)\cdot U(1)\cdot SO(2))$ look the same, but $\mathcal{O}(G/L) \cong \mathbb{C}$.

By taking the three steps (S1), (S2) and (S3) we determine all the homogeneous pseudo-Kähler manifolds G/L of connected absolutely simple Lie groups G of Hermitian type such that G act on G/L effectively, $\dim G/L \neq 0$ and $\mathcal{O}(G/L) \ncong \mathbb{C}$; and then we make List 1. In fact, the manifolds G/L in List 1 exhaust all the homogeneous pseudo-Kähler manifolds G'/L' of connected absolutely simple Lie groups G' such that G' act on G'/L' effectively, $\dim G'/L' \neq 0$ and $\mathcal{O}(G'/L') \ncong \mathbb{C}$.

List 1	
type	G/L, where we assume the center $Z(G)$ to be trivial.
AIII	$SU(k,\ell{+}1-k)/S\big(U(i_1)\times U(i_2-i_1)\times\cdots\times U(i_n-i_{n-1})\times U(k-i_n)$ $\times U(j_1-k)\times U(j_2-j_1)\times\cdots\times U(j_m-j_{m-1})\times U(\ell+1-j_m)\big)$, where $\ell \geq 1$, $1 \leq k \leq \ell$, $0 \leq n \leq k-1$, $0 \leq m \leq \ell-k$, $1 \leq i_1 < i_2 < \cdots < i_n \leq k-1$ and $k+1 \leq j_1 < j_2 < \cdots < j_m \leq \ell$.

BI	$SO_0(2\ell-1,2)/\big(U(i_1-1)\cdot U(i_2-i_1)\cdots U(i_n-i_{n-1})\cdot SO(2\ell+1-2i_n)\cdot SO(2)\big)$, where $\ell \geq 2$, $0 \leq n \leq \ell-1$ and $2 \leq i_1 < i_2 < \cdots < i_n \leq \ell$.
CI	$Sp(\ell,\mathbb{R})/\big(U(i_1)\cdot U(i_2-i_1)\cdots U(i_n-i_{n-1})\cdot U(\ell-i_n)\big)$, where $\ell \geq 3$, $0 \leq n \leq \ell-1$ and $1 \leq i_1 < i_2 < \cdots < i_n \leq \ell-1$.
DI	$SO_0(2\ell-2,2)/\big(U(i_1-1)\cdot U(i_2-i_1)\cdots U(i_n-i_{n-1})\cdot SO(2\ell-2i_n)\cdot SO(2)\big)$, where $\ell \geq 4$, $0 \leq n \leq \ell-1$ and $2 \leq i_1 < i_2 < \cdots < i_n \leq \ell$.
DIII	$SO^*(2\ell)/\big(U(j_1)\cdot U(j_2-j_1)\cdots U(j_m-j_{m-1})\cdot U(\ell-j_m)\big)$, where $\ell \geq 4$, $0 \leq m \leq \ell-1$ and $1 \leq j_1 < j_2 < \cdots < j_m \leq \ell-1$.
EIII	$G = E_{6(-14)}$ and L is one of the following twelve groups: $SO(10)\cdot T^1$, $SO(8)\cdot T^2$, $SU(5)\cdot T^2$, $SU(4)\cdot SU(2)\cdot T^2$, $SU(3)\cdot SU(2)\cdot SU(2)\cdot T^2$, $SU(4)\cdot T^3$, $SU(3)\cdot SU(2)\cdot T^3$, $SU(2)\cdot SU(2)\cdot SU(2)\cdot T^3$, $SU(3)\cdot T^4$, $SU(2)\cdot SU(2)\cdot T^4$, $SU(2)\cdot T^5$, T^6.
EVII	$G = E_{7(-25)}$ and L is one of the following seventeen groups: $E_6\cdot T^1$, $SO(10)\cdot T^2$, $SU(6)\cdot T^2$, $SU(5)\cdot SU(2)\cdot T^2$, $SU(3)\cdot SU(3)\cdot SU(2)\cdot T^2$, $SO(8)\cdot T^3$, $SU(5)\cdot T^3$, $SU(4)\cdot SU(2)\cdot T^3$, $SU(3)\cdot SU(3)\cdot T^3$, $SU(3)\cdot SU(2)\cdot SU(2)\cdot T^3$, $SU(4)\cdot T^4$, $SU(3)\cdot SU(2)\cdot T^4$, $SU(2)\cdot SU(2)\cdot SU(2)\cdot T^4$, $SU(3)\cdot T^5$, $SU(2)\cdot SU(2)\cdot T^5$, $SU(2)\cdot T^6$, T^7.

Here for integers $m, p, i_1, i_2, \ldots, i_m$, the notation $0 \leq m$, $p \leq i_1 < i_2 < \cdots < i_m$ means $i_1 = i_2 = \cdots = i_m = p-1$ whenever $m = 0$.

Remark 3.15. One can also make List 1 from Xu's results [X] that determine all the homogeneous Kähler manifolds G'/L' on which connected, non-compact absolutely simple Lie groups G' act effectively.

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