Equivariant realizations of Hermitian symmetric space of noncompact type

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§1. Introduction

Hermitian symmetric space

$M^{2n} = G/K$: Hermitian symmetric space of noncompact type (HSSNT).

- Riemannian symmetric space $\Leftrightarrow \exists$ geodesic symmetry $\sigma_p \in \mathrm{Isom}(M)$ at each point $p \in M$.
  $\Rightarrow$ $G := \mathrm{Isom}_0(M)$ transitively acts on M, i.e. $M \simeq G/K$.
  - If M is non-cpt type, K becomes a max cpt subgp of G.
- Hermitian symmetric space $\Leftrightarrow$ Riem. sym. sp. admitting a Kähler structure (J, ω) s.t. $\sigma_p \in \mathrm{Aut}(M, \omega, J)$.

HSSs provide nice examples of homogeneous Kähler-Einstein mfds.

$M = G/K$ (noncpt)	$M^* = G^*/K$ (cpt)
$SU(p,q)/S(U(p) \times U(q))$	$SU(p+q)/S(U(p) \times U(q))$
$SO^\circ(2,q)/SO(2) \times SO(q)$	$SO(2+q)/SO(2) \times SO(q)$
$SO^*(2n)/U(n)$	$SO(2n)/U(n)$
$Sp(n,\mathbb{R})/U(n)$	$Sp(n)/U(n)$
$E_6^{-14}/T \cdot Spin(10)$	$E_6/T \cdot Spin(10)$
$E_7^{-25}/T \cdot E_6$	$E_7/T \cdot E_6$

Table: Irreducible Hermitian symmetric spaces

HSSNT as a manifold

Let $M^{2n} = G/K$ be an HSSNT.
Then, M is realized in a vector space of the same dimension by several ways.

Fact 1. M is a simply-connected, complete, Riemannian manifold of non-positive sectional curvature.

$\Longrightarrow$ The Cartan-Hadamard theorem shows that

$$\mathrm{Log}_o : M \to T_o M$$

is a diffeomorphism i.e. M is realized as the vector space $\mathbb{R}^{2n}$ as a manifold.

e.g. $M = \mathbb{C}H^n$.

$$\mathrm{Log}_o : \mathbb{C}H^n \xrightarrow{\sim} T_o M \simeq \mathbb{C}^n, \quad [1 : z] \mapsto \tanh^{-1}|z| \cdot \frac{z}{|z|}$$

Holomorphic realization

Fact 2. There exists a holomorphic diffeomorphism

$$\psi : (M, J) \to D \subset \mathbb{C}^n,$$

where D is a bounded domain in $\mathbb{C}^n$, i.e. (M, J) is realized as a bounded domain $D \subset \mathbb{C}^n$ as a complex manifold.

e.g. $M = \mathbb{C}H^n$.

$$\psi : \mathbb{C}H^n \xrightarrow{\sim} D \subset \mathbb{C}^n, \quad [1 : z] \mapsto z$$

- **É. Cartan** (1935); in his theory of bounded symmetric domain.
- **Harish-Chandra** (1956); first a priori proof based on the representation theory of semi-simple Lie groups.
- The Bergman metric g_B on the bounded symmetric domain D coincides with the Kähler metric of HSSNT M.
  Therefore, HSSNT is often identified with (D, J_0, g_B).

Symplectic realization

Fact 3. There exists a symplectic diffeomorphism

$$\varphi : (M, \omega) \to \mathbb{R}^{2n},$$

where $\mathbb{R}^{2n} = (\mathbb{R}^{2n}, \omega_0)$ is the standard symplectic vector space, i.e. the global version of Darboux thm holds.

e.g. $M = \mathbb{C}H^n$.

$$\varphi : \mathbb{C}H^n \xrightarrow{\sim} \mathbb{C}^n \simeq \mathbb{R}^{2n}, \quad [1 : z] \mapsto \sqrt{\frac{1}{1 - |z|^2}} \cdot z$$

- **McDuff** (1988); the existence theorem when M is a simply-connected, complete Kähler mfd of non-positive sectional curvature.
- **Di Scala-Loi** (2008); first discovery of the explicit formula of the symplectomorphism for general HSSNT M.
  Remark: Di Scala-Loi's formula was independently defined by G. Roos. He proved that the map preserves the volume form.

Di Scala-Loi-Roos realization

Di Scala-Loi-Roos's formula is described by associated Jordan triple system (JTS) (therefore M is regarded as a bounded symmetric domain D).

- Any HSSNT $M \simeq D$ is associated with JTS $(T_oD, \{\cdot,\cdot,\cdot\})$;

$$\{u, v, w\} := -\frac{1}{2}\{R_o(u, v)w + J_oR_o(u, J_ov)w\},$$

  where R_o is the curvature tensor at o.
- The Bergman operator B is defined by

$$B(u, v) := Id - D(u, v) + Q(u)Q(v),$$

  where $D(u, v)(w) := \{u, v, w\}, \quad Q(u)v := \{u, v, u\}$.

Theorem (Di Scala-Loi '08)

The map

$$\varphi : D \to T_oD, \quad \varphi(z) := B(z, z)^{-1/4}z$$

is a symplectomorphism.

Remark

- The symplectomorphism from M to $\mathbb{R}^{2n}$ is not unique.
  In fact, Di Scala-Loi-Roos (2008) constructed many symplectomorphisms from M to $\mathbb{R}^{2n}$ by determining a special type of map so-called the *symplectic duality map*.
  Nevertheless, we call the canonical symplectomorphism $\varphi(z) = B(z,z)^{-1/4}z$ *Di Scala-Loi-Roos realization*.

We shall reconstruct the Di Scala-Loi-Roos realization by a different method, and show the Di Scala-Loi-Roos realization is characterized as a "canonical" (K-equivariant) symplectic realization of HSSNT as well as Harish-Chandra realization.

§2. Equivariant realizations of HSSNT

Recall that any HSSNT $M = G/K$ is realized as $\mathbb{R}^{2n}$, (D, J_0) and $(\mathbb{R}^{2n}, \omega_0)$ as a mfd, a complex mfd and a symplectic mfd, respectively.
Each realization is regarded as a chart of M.

e.g. $M = \mathbb{C}H^n$.

- Logarithm map (diffeomorphism):

$$\mathrm{Log}_o : \mathbb{C}H^n \xrightarrow{\sim} \mathbb{C}^n \simeq T_oM, \quad [1:z] \mapsto \frac{\tanh^{-1}|z|}{|z|} \cdot z.$$

- Harish-Chandra realization (holomorphic diffeomorphism):

$$\psi : \mathbb{C}H^n \xrightarrow{\sim} D \subset \mathbb{C}^n, \quad [1:z] \mapsto z.$$

- Di Scala-Loi-Roos realization (symplectic diffeomorphism):

$$\varphi : \mathbb{C}H^n \xrightarrow{\sim} \mathbb{C}^n \simeq \mathbb{R}^{2n}, \quad [1:z] \mapsto \sqrt{\frac{1}{1-|z|^2}} \cdot z.$$

Summary of results:

1. **Unified description**; The above realizations are described by a unified and geometric framework for any HSSNT by using the *polarity of K-action* and the *polydisk theorem*.
   In particular, they have some common properties.

2. **Characterization of the Harish-Chandra/Di Scala-Loi-Roos realization;** by means of the polarity of K-action.

3. **Duality;** an analogous map for compact dual M^* is naturally defined. In particular, we define a notion of *duality* of the maps.

4. **Realizations of totally geodesic submfds;** the maps are "hereditary" for a special type of totally geodesic submfd such as
   - totally geodsic submfd of maximal rank
   - complex totally geodesic submfd
   - real forms (totally geodesic Lagrangian submfds)

Polarity and Polydisk theorem

Our stating point is that the above realizations are obtained by K-equivariant embeddings.
Recall that K is a maximal cpt subgp of $G = \mathrm{Isom}_0(M)$.

Let

- $M = G/K$: HSSNT.
- $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$: the Cartan decomposition. We use the identification $\mathfrak{p} \simeq T_o M$.
- K acts on $\mathfrak{p}$ via the isotropy representation.

Theorem (Polarity of K-action)

The K-action on $\mathfrak{p}$ is a polar action, and any maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{p}$ becomes its section, i.e.

- Any $\mathrm{Ad}(K)$-orbit in $\mathfrak{p}$ intersects to $\mathfrak{a}$ orthogonally.
- $\mathrm{Ad}(K)\mathfrak{a} = \mathfrak{p}$ and $K \cdot A = M$, where $A = \mathrm{Exp}_o \mathfrak{a}$.

We say $\mathfrak{a}$ (resp. A) a section of $\mathfrak{p}$ (resp. M) of the K-action.

Polarity and Polydisk theorem

A is a flat totally geodesic submanifold in M, and has a complexification in the following sense:

Theorem (Polydisk theorem)

$A^{\mathbb{C}} := \mathrm{Exp}_o(\mathfrak{a} \oplus J_o\mathfrak{a})$ is a complex totally geodesic submanifold in M, and have a canonical splitting

$$A^{\mathbb{C}} \simeq A_1^{\mathbb{C}} \times \cdots \times A_r^{\mathbb{C}} \simeq \mathbb{C}H^1(-C) \times \cdots \times \mathbb{C}H^1(-C)$$

as a Kähler manifold, where r is the rank of M, i.e. $r = \dim_{\mathbb{R}} \mathfrak{a}$.

Remark M: HSSNT.

- $\Rightarrow$ The associated restricted root system Σ is either type C_r or BC_r.
- $\Rightarrow$ The long roots $\{2e_1, \ldots, 2e_r\}$ consists of *strongly orthogonal roots* (i.e., $\alpha, \beta \in \Sigma \Rightarrow \alpha \pm \beta \notin \Sigma$).
  Moreover, $\{H_i := 2e_i^*\}_{i=1}^r$ is an orthogonal basis of $\mathfrak{a}$.
- $\Rightarrow$ $A_i^{\mathbb{C}} = \mathrm{Exp}_o(\mathfrak{a}_i \oplus J_o\mathfrak{a}_i)$, where $\mathfrak{a}_i = \mathrm{span}_{\mathbb{R}}\{H_i\}$.

A natural construction of K-equivariant emb. (1/5)

Recall

- $\mathfrak{a} \subset \mathfrak{p}$: maximal abelian subsp $\Longrightarrow \mathrm{Ad}(K)\mathfrak{a} = \mathfrak{p}$.
- $A := \mathrm{Exp}_o\mathfrak{a} \subset M \Longrightarrow K \cdot A = M$.

Take

$$\Omega_A : A \to \mathfrak{a}$$

and put

$$\Omega : M \to \mathfrak{p}, \quad \Omega(k\mathrm{Exp}_o v) := \mathrm{Ad}(k)\Omega_A(\mathrm{Exp}_o v)$$

for $k \in K$ and $v \in \mathfrak{a}$.

Then, Ω is K-equivariant map satisfying that $\Omega(A) \subseteq \mathfrak{a}$ if Ω is well-defined. Conversely, any K-equiv. map satisfying $\Omega(A) \subseteq \mathfrak{a}$ is obtained in this way.

A natural construction of K-equivariant emb. (2/5)

Lemma

Ω is well-defined $\Longleftrightarrow$ Ω_A is $\mathcal{W}$-equivariant, where $\mathcal{W}$ is the Weyl group associated with the restricted root system Σ.

Example: $M = \mathbb{C}H^1$.

A natural construction of K-equivariant emb. (3/5)

Recall that the associated restricted root system Σ of HSSNT is either type C_r or BC_r. Using the explicit description of $\mathcal{W}$, we can determine the K-equivariant map in an algebraic way.

Proposition

Let M be an irreducible HSSNT, and $h : \mathfrak{a} \to \mathbb{R}$ be a function satisfying that

(a) h is an odd function with respect to the first variable x_1.

(b) h is an even function with respect to the variable x_i for $i > 1$ and symmetric with respect to x_i and x_j for $i, j > 1$, i.e. $h = h \circ p_{ij}$ for any $i, j > 1$.

Then, the function h yields a well-defined K-equivariant map $\Omega_h : M \to \mathfrak{p}$ such that $\Omega_h(A) \subseteq \mathfrak{a}$, which is defined by

$$\Omega_h(k\mathrm{Exp}_o v) := \mathrm{Ad}(k) \circ \Omega_{h,A}(\mathrm{Exp}_o v), \quad \text{where}$$

$$\Omega_{h,A}\Big(\mathrm{Exp}_o\Big(\sum_{i=1}^{r} x_i \widetilde{H}_i\Big)\Big) = \sum_{i=1}^{r} h \circ p_{1i}(x_1, \ldots, x_r)\widetilde{H}_i.$$

for $k \in K$ and $v = \sum_{i=1}^{r} x_i \widetilde{H}_i \in \mathfrak{a}$. Conversely, any K-equivariant map $\Omega : M \to \mathfrak{p}$ satisfying $\Omega(A) \subseteq \mathfrak{a}$ is obtained in this way.

A natural construction of K-equivariant emb. (4/5)

Recall that the polydisk

$$A^{\mathbb{C}} \simeq A_1^{\mathbb{C}} \times \cdots \times A_r^{\mathbb{C}} \simeq \mathbb{C}H^1 \times \cdots \times \mathbb{C}H^1$$

lies in M.

Proposition

Let $\Omega : M \to \mathfrak{p}$ be a K-equivariant map s.t. $\Omega(A) \subseteq \mathfrak{a}$.
$\Longrightarrow \exists$ a single odd function $\eta : \mathbb{R} \to \mathbb{R}$ s.t.

$$\Omega|_{A_i^{\mathbb{C}}} = \Omega_{\eta,0} \quad \forall i = 1, \ldots, r$$

under the identification $A_i^{\mathbb{C}} \simeq \mathbb{C}H^1$.

Conversely, we obtain the following simple way of construction of K-equivariant embedding of M into $T_oM \simeq \mathfrak{p}$:

A natural construction of K-equivariant emb. (5/5)

① First, we take a $\widehat{K}$-equivariant embedding $\widehat{\Omega} : \mathbb{C}H^1(-C) = \widehat{G}/\widehat{K} \to \widehat{\mathfrak{p}} \simeq T_o\mathbb{C}H^1$ such that $\widehat{\Omega}(\widehat{A}) \subset \widehat{\mathfrak{a}}$. Such an embedding is obtained by a "radial map" associated with any injective odd function $\eta : \mathbb{R} \to \mathbb{R}$:

$$\widehat{\Omega}(\mathrm{Exp}_o z) = \Omega_{\eta,0}(\mathrm{Exp}_o z) := \eta(|z|)\frac{z}{|z|}.$$

② Extend $\Omega_{\eta,0}$ to the polydisk $A^{\mathbb{C}} \simeq \mathbb{C}H^1(-C) \times \cdots \times \mathbb{C}H^1(-C)$ by the direct product:

$$\Omega_{\eta,A^{\mathbb{C}}} := \Omega_{\eta,0} \times \cdots \times \Omega_{\eta,0} : A^{\mathbb{C}} \to \mathfrak{a}^{\mathbb{C}}.$$

We put $\Omega_{\eta,A} := \Omega_{\eta,A^{\mathbb{C}}}|_A$. Then, $\Omega_{\eta,A}(A) \subseteq \mathfrak{a}$.

③ Extending $\Omega_{\eta,A}$ (or $\Omega_{\eta,A^{\mathbb{C}}}$) to M by the K-action, we finally obtain a well-defined K-equivariant embedding

$$\Omega_\eta : M \to \mathfrak{p}, \quad \Omega_\eta(k\mathrm{Exp}_o v) = \mathrm{Ad}(k)\Omega_{\eta,A}(\mathrm{Exp}_o v)$$

such that $\Omega_\eta(A) \subseteq \mathfrak{a}$, where $k \in K$ and $v \in \mathfrak{a}$.
We call the resulting embedding Ω_η strongly diagonal realization.

Remark

- A corresponding map has been appeared in another context in terms of the Jordan triple system. [cf. Loos '77, Di Scala-Loi-Roos '08, Loi-Mossa '11]. Our construction above is regarded as a geometric reconstruction of it.
- $\Omega_\eta(M) = \mathfrak{p}$ if and only if η is surjective. Otherwise,

  $$\Omega_\eta(M) = \mathrm{Ad}(K)\square_{\eta,\mathfrak{a}} \quad \text{(a } K\text{-invariant bounded domain)}$$

  where

  $$\square_{\eta,\mathfrak{a}} := \left\{ \sum_{i=1}^{r} x_i \widetilde{H}_i \mid |x_i| < \sup\eta \right\}$$

  This is a generalization of the picture of Harish-Chandra realization.

Dual maps

The above construction can be applied to the compact dual $M^* = G^*/K$ with a slight modification.

- $U^* \subseteq (M^*)^o := M^* \setminus \mathrm{Cut}_o(M^*)$: K-invariant open nbd of o.
- $\eta : (-R^*, R^*) \to \mathbb{R}$: odd fct.

$\Longrightarrow \Omega^*_\eta : U^* \to \mathfrak{p}^*, \quad \Omega^*_\eta(k\mathrm{Exp}_o v^*) = \mathrm{Ad}(k)\Omega_{\eta,A^*}(\mathrm{Exp}_o v^*)$.

We define a notion of dual map of Ω_η as follows if the odd function η is a real analytic function:

- Define a dual function of η by

$$\eta^* : (-R, R) \to \mathbb{R}, \ \eta^*(x) := -\sqrt{-1}\eta(\sqrt{-1}x),$$

  where $R \in (0, \pi/2]$.
- We call the corresponding K-equivariant map $\Omega^*_{\eta^*}$ the *dual map* of Ω_η. Note that $U^*_{\eta^*} = (M^*)^o = M^* \setminus \mathrm{Cut}_o(M^*)$ iff $R = \pi/2$.

Holomorphic/symplectic realization

An advantage of this construction is that the map is obtained from a radial map on $\mathbb{C}H^1$:

$$\Omega_{\eta,0} : \mathbb{C}H^1(-C) \to \widehat{\mathfrak{p}} \simeq \mathbb{C}, \quad \mathrm{Exp}_o z \mapsto \eta(|z|)\frac{z}{|z|}.$$

e.g. By taking $\Omega_{\eta,0} = \mathrm{Log}_o^{\mathbb{C}H^1}$, or equivalently, $\eta = id$, we recover the Logarithm map of M, i.e.

$$\Omega_{id} = \mathrm{Log}_o : M \to \mathfrak{p} \simeq T_oM.$$

The dual function of $\eta = id$ is given by $\eta^* = id$.
The dual map of $\Omega_{id} = \mathrm{Log}_o : M \to \mathfrak{p}$ is the logarithm map of M^*:

$$\Omega_{id}^* = \mathrm{Log}_o^* : (M^*)^o \to \mathfrak{p}^* \simeq T_oM^*.$$

By solving an O.D.E., it is easy to see that

- $\Omega_{\eta,0}$ is holomorphic $\iff \eta(x) = \tanh x$ (up to constant multiple).
- $\Omega_{\eta,0}$ is symplectic $\iff \eta(x) = \sinh x$ (up to sign).

Moreover, the holomorphic/symplectic $\Omega_{\eta,0}$ is extended to a holomorphic/symplectic embedding of M and its compact dual M^*:

Holomorphic/symplectic realization

Theorem (Hashinaga-K.)

Let $M = G/K$ be an irreducible HSSNT. Then,

- $\Psi := \Omega_{\tanh} : M \to D$ is a K-equivariant holomorphic diffeomorphism onto a bounded domain $D \subset \mathfrak{p}$.
- $\Phi := \Omega_{\sinh} : M \to \mathfrak{p}$ is a K-equivariant symplectic diffeomorphism onto $\mathfrak{p} \simeq T_o M$.

Moreover, Ψ (resp. Φ) is the unique K-equivariant holomorphic (resp. symplectic) embedding from M to $\mathfrak{p}$ such that A is mapped into $\mathfrak{a}$ (up to appropriate constant multiple).

The dual function of $\eta = \tanh$ (resp. $\sinh$) is $\eta^* = \tan$ (resp. $\sin$) on $(-\pi/2, \pi/2)$.

Theorem

Let $M^* = G^*/K$ be the compact dual of M. Then,

- The dual map $\Psi^* := \Omega^*_{\tan} : (M^*)^o \to \mathfrak{p}^*$ of Ψ is a K-equivariant holomorphic diffeomorphism onto $\mathfrak{p}^* \simeq T_o M^*$.
- The dual map $\Phi^* := \Omega^*_{\sin} : (M^*)^o \to D^*$ of Φ is a K-equivariant symplectic diffeomorphism onto a bounded domain $D^* \subset \mathfrak{p}^*$.

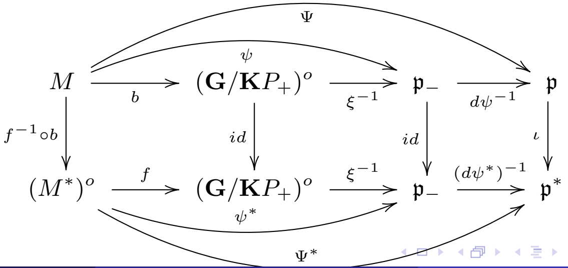
Remark

- We use the restricted root system for the proof. This provides an alternative and unified proof of the results due to É. Cartan, Harish-Chandra and Di Scala-Loi. In fact,

Proposition

Under appropriate identification of spaces, we see

- Ψ coincides with the Harish-Chandra realization.
- Φ coincides with the Di Scala-Loi-Roos realization.
- $(\Psi^*)^{-1} \circ \Psi$ coincides with the Borel embedding.



Remark

- The uniqueness may not hold in general if we drop the assumption that "A is mapped into $\mathfrak{a}$". In fact, Di Scala-Loi-Roos (2008) constructed many K-equivariant symplectic diffeomorphisms from M to $\mathfrak{p}$.

- Ψ and Ψ^* (resp. Φ and Φ^*) may be regarded as a "canonical" K-equivariant holomorphic (resp. symplectic) chart of M and M^*, respectively.
The induced Kähler structure on an open dense subset of the image in $\mathfrak{p}$ (or $\mathfrak{p}^*$) can be described explicitly by using the restricted root system.

- Another example of the pair of strongly diagonal realization and its dual map is given by Loi-Mossa (2011) by using the JTS. They construct "diastatic exponential maps" for M and M^*.

- The fact that Φ and Φ^* are symplectomorphisms of each other is equivalent to the remarkable property of Di Scala-Loi-Roos realization so-called the *symplectic duality*.

Realizations of totally geodesic submfds

Theorem (Ciriza '93)

The symplectomorphism $M \to \mathbb{C}^n$ constructed by McDuff sends any totally geodesic complex submfd N through the origin to a complex linear subspace in $\mathbb{C}^n$.

Theorem (Di Scala-Loi '08)

The same property holds for the Di Scala-Loi-Roos realization φ. Indeed, φ satisfies the following property:

$$\varphi|_N = \varphi_N$$

for any totally geodesic complex submfd N in M through the origin, where $\varphi_N : N \to \mathfrak{p}_N$ is the symplectomorphism assoaciated with N.

This is generalized to any strongly diagonal realization:

Theorem (Hashinaga-K.)

- N: totally geodesic submfd N in M through the origin.
- $\mathfrak{a}_N$: maximal abelian subsp in $\mathfrak{p}_N \simeq T_o N$.

If $\mathrm{Exp}_o(\mathfrak{a}_N \oplus J_o \mathfrak{a}_N) \subset M$ splits into a polydisk ($\Longleftrightarrow$ $\mathfrak{a}_N \oplus J_o \mathfrak{a}_N$ is a complex Lie triple system in $\mathfrak{p}$), then

- K_N-equiv. emb. $\Omega_{\eta,N} : N \to \mathfrak{p}_N$ can be defined similar to the strongly diagonal realization of M.
- $\Omega_\eta|_N = \Omega_{\eta,N}$ for any injective odd function $\eta : \mathbb{R} \to \mathbb{R}$.

In particular, we have the following commutative diagram:

$$\begin{array}{ccc} (M, A) & \xrightarrow[\text{diffeo.}]{\Omega_\eta \times \Omega_\eta|_A} & (D_\eta, \square_{\eta,\mathfrak{a}}) \\ \uparrow & & \uparrow \\ (N, A_N) & \xrightarrow[\text{diffeo.}]{\Omega_{\eta,N} \times \Omega_{\eta,N}|_{A_N}} & (D_{\eta,N}, \square_{\eta,\mathfrak{a}_N}) \end{array}$$

Conversely, if (N, A_N) is mapped onto $(D_{\eta,N}, \square_{\eta,\mathfrak{a}_N})$ by $\Omega_\eta \times \Omega_\eta$ for any injective odd function $\eta : \mathbb{R} \to \mathbb{R}$, then $\mathfrak{a}_N$ has a complexification as LTS in $\mathfrak{p}$.

Examples

Examples of totally geodesic submfds whose maximal abelian subspace $\mathfrak{a}_N$ has a complexification as LTS in $\mathfrak{p}$:

- totally geodesic submfds of maxmal rank. e.g. any totally geodesic submfd in $\mathbb{C}H^n$.
- totally geodesic complex submfds.
- real forms (totally geodesic Lagrangian submfds).

Thus, these submfds are always realized as either linear subspaces of bounded domains in linear subspaces in $\mathfrak{p}$ by any strongly diagonal realizations.

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2. **Characterization of the Harish-Chandra/Di Scala-Loi-Roos realization;** by means of the polarity of K-action.

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   - totally geodsic submfd of maximal rank
   - complex totally geodesic submfd
   - real forms (totally geodesic Lagrangian submfds)

Thank you for your attention!