# §1 Introduction



$(M^n, g(t))$ : Ricci flow $\quad \partial_t g = -2\mathrm{Ric}(g)$

cpt $\quad\quad t \in [0, T)$

- $|\mathrm{Ric}| \le K$ on $M \times [0, T)$

$$\Rightarrow e^{-2Kt} g(0) \le g(t) \le e^{2Kt} g(0)$$

とくに $\boxed{e^{-Kt} d_0(x,y) \le d_t(x,y) \le e^{Kt} d_0(x,y)}$

distance distortion estimate (DDE)

$dm = dm_t = dm_{g(t)}$ : Riem. vol. meas w.r.t. $g(t)$

$$\partial_t dm = -R\, dm, \quad R : \text{scalar curv.}$$

よって

- $|\mathrm{Ric}| \le K \Rightarrow |R| \le nK$

$$\Rightarrow e^{-nKt} \mathrm{Vol}_{g(0)}(M) \le \mathrm{Vol}_{g(t)}(M) \le e^{nKt} \mathrm{Vol}_{g(0)}(M)$$

- $|\mathrm{Ric}| \le K$ on $M \times [0, T) \Rightarrow T = +\infty$

Long time existence

つまり $|\mathrm{Ric}| \le K$ だと、いろいろできる。

(DDE, Vol. est., long time existence …)

## Question

$|R| \leq R_0 < \infty$ で何が言える?

$|Ric| \leq K$ より弱い条件

$|R| \leq R_0 \longrightarrow$ Volume estimate $\boxed{\text{ok}}$

$\longrightarrow$ distance distortion

$\longrightarrow$ long time existence

?

**Bamler - Zhang '17** along Ricci flow.

$|R| \leq R_0 \Rightarrow$ distance distortion estimate (DDE)

$\Rightarrow$ Gaussian type heat kernel estimate (GHKE)

**K. - Sakurai '22**

Bamler - Zhang を super RF に拡張

## §2 Super Ricci flow & Examples

$(M, g(t))_{t\in[0,T)}$ : super Ricci flow

$$\boxed{\partial_t g \geq -2\mathrm{Ric}(g)}$$

| | |
|---|---|
| Riem mfd (static) | $\mathrm{Ric}=0 \leftrightarrow \mathrm{Ric}\geq 0$ |
| Geom. flow | Ricci flow $\leftrightarrow$ super RF |

- super RF の例

(1) 通常の RF : $\partial_t g = -2\mathrm{Ric}$

(2) $g(t) = g_0$ : static & $\mathrm{Ric}(g_0) \geq 0$

$$\Rightarrow \underbrace{\partial_t g_0}_{=0} \geq -2\mathrm{Ric}(g_0) = 0$$

$\therefore$ super RF

(3) List's flow

$$\begin{cases} \partial_t g = -2\mathrm{Ric}(g) + 2d\varphi\otimes d\varphi \\ \partial_t \varphi = \Delta_{g(t)}\varphi, \quad \varphi : M\times[0,T) \to \mathbb{R} \end{cases}$$

(4) Müller's flow ← List の map ver.

harmonic map heat flow

$\boxed{\text{記号}}$ $h := -\frac{1}{2}\partial_t g$, $H := \mathrm{tr}_g h$ ④
(Ricci curv.) (Scalar curv.)

<u>Müller quantity</u> for general geometric flows

$$\mathcal{D}(V) := \partial_t H - \Delta_{g(t)} H - 2|h|^2 + 4\,\mathrm{div}\,h(V) - 2\langle \nabla H, V\rangle + 2\mathrm{Ric}(V,V) - 2h(V,V)$$

- $\boxed{\mathcal{D}(V) \geq 0}$ $\Rightarrow$ $\begin{cases} \circ \text{ monotonicity of } \mathcal{W}\text{-func., } \mathcal{F}\text{-func.} \\ \circ \text{ reduced geometry} \end{cases}$

(1) ~ (4) の例は
super RF かつ $\mathcal{D}(V) \geq 0$

$\Downarrow$
geometric analysis が
うまく機能する

例の続き

(5) mean curvature flow for <u>spacelike</u> hypersurfaces
in <u>Lorentzian mfd with nonnegative</u> sect. curv.

$\rightsquigarrow$ super RF かつ $\mathcal{D}(V) \geq 0$ をみたす

## §3 Main results

- 以下のような設定を考える

Heat eq. along super RF

$$\begin{cases} (M, g(t)) : \text{s-RF}, \quad t \in [0,T) \\ \partial_t u = \Delta_{g(t)} u, \quad u : M \times [0,T) \to \mathbb{R} \end{cases}$$

$G(x,t;y,s)$ : heat kernel centered at $(y,s) \in M \times [0,T)$

i.e., $$(\partial_t - \Delta_x) G(x,t;y,s) = 0$$

&

$$\lim_{t \searrow s} G(x,t;y,s) = \delta_y$$

<u>Rem</u> $(x,t) \in M \times [0,T)$ を固定すると、

$G(x,t;\cdot,\cdot)$ は conjugate heat eq. の熱核

## Main theorem (K. - Sakurai)

$(M^n, g(t))_{t\in[0,T)}$ : super RF with $\mathcal{D}(V)\geq 0$

$T < +\infty$

For $\forall A>0$ $\exists$ $C = C(n, g(0), T, A) > 0$ s.t.

$H \leq K$ $(K>0)$, $t-s \leq AK^{-1}$, $s \geq (t-s)A^{-1}$

$$\Rightarrow \quad |G(x,t;y,s)| \leq \frac{C}{(t-s)^{\frac{n}{2}}} e^{-\frac{C d_s(x,y)^2}{t-s}}$$

証明は Bamler - Zhang に従う. (詳細は略)

- ポイント

$H \leq K < \infty \rightsquigarrow$ distance distortion est.

(DDE)

### DDE を導くための key lemma

$0 < u \leq A_0$ : sol. to heat eq. on $M\times[t_1, t_2]$

$$\Rightarrow \left(|\Delta u| + \frac{|\nabla u|^2}{u} - A_0 H\right)(x,t) \leq \frac{C_n A_0}{t-t_1} \quad \text{for } t\in[t_1,t_2]$$

※ super RF かつ $\mathcal{D}(V)\geq 0$ だと, この証明がうまくいく

(これが重要)