Various curvature conditions on weighted Ricci curvature and geometric analysis

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Overview

Weighted Ricci curvature

(M, d, m) : weighted Riemannian manifold, i.e.,

- $M = (M, g)$: n-dim. complete Riemannian manifold
- d : Riemannian distance function
- $m := e^{-f}\,\mathrm{vol}$ for $f \in C^\infty(M)$

N-weighted Ricci curvature

$N \in (-\infty, \infty]$: effective dimension

$$\mathrm{Ric}_f^N := \mathrm{Ric} + \nabla^2 f - \frac{df \otimes df}{N - n}$$

Setting

$N \in (-\infty, 1] \cup [n, \infty]$

ε-range

- $\varepsilon = 0$ for $N = 1$
- $\varepsilon \in \left(-\sqrt{\frac{N-1}{N-n}}, \sqrt{\frac{N-1}{N-n}} \right)$ for $N \neq 1, n$
- $\varepsilon \in \mathbb{R}$ for $N = n$

- $c := \frac{1}{n-1} \left(1 - \varepsilon^2 \frac{N-n}{N-1} \right)$

Setting ([Lu-Minguzzi-Ohta, 22])

$$\mathrm{Ric}_f^N \geq c^{-1} \, \kappa \, e^{-\frac{4(1-\varepsilon)f}{n-1}} g \quad \text{for} \quad \kappa \in \mathbb{R}, \; N \in (-\infty, 1] \cup [n, \infty]$$

Interpolation

Setting ([Lu-Minguzzi-Ohta, 22])

$$\mathrm{Ric}_f^N \geq c^{-1}\,\kappa\, e^{-\frac{4(1-\varepsilon)f}{n-1}} g \quad \text{for} \quad \kappa \in \mathbb{R},\ N \in (-\infty, 1] \cup [n, \infty]$$

Interpolation between...

Curvature-dimension condition

$$\mathrm{Ric}_f^N \geq K g \quad \text{for} \quad K \in \mathbb{R},\ N \in [n, \infty]$$

▷ [Sturm, 06], [Lott-Villani, 09] :

$\mathrm{CD}(K, N)$ for non-smooth metric measure space

Geometric analysis on projectively equivalent affine connection

$$\mathrm{Ric}_f^1 \geq (n-1)\,\kappa\, e^{-\frac{4f}{n-1}} g \quad \text{for} \quad \kappa \in \mathbb{R}$$

▷ [Wylie-Yeroshkin, 16, preprint]

Goal

Setting ([Lu-Minguzzi-Ohta, 22])

$$\mathrm{Ric}_f^N \geq c^{-1}\,\kappa\, e^{-\frac{4(1-\varepsilon)f}{n-1}} g \ \text{ for } \ \kappa \in \mathbb{R},\ N \in (-\infty, 1] \cup [n, \infty]$$

- [Lu-Minguzzi-Ohta, 22], [Kuwae-Sakurai, 21] :

  Comparison geometry of manifolds without boundary

- [Kuwae-Sakurai, 20, preprint] :

  Comparison geometry of manifolds with boundary

- [Kuwae-Sakurai, 20, preprint] : Characterization via optimal transport theory

Background (Curvature-dimension condition)

Curvature-dimension condition

Curvature-dimension condition

$$\mathrm{Ric}_f^N \geq Kg \quad \text{for} \quad K \in \mathbb{R},\ N \in [n, \infty]$$

- [Qian, 97], [Lott, 03], [Wei-Wylie, 09],... : Weighted Riemannian manifold
- [Ohta, 09] : Finsler manifold
- [Sturm, 06], [Lott-Villani, 09] : $\mathrm{CD}(K, N)$
- [Ambrosio-Gigli-Savaré, 14], [Erbar-Kuwada-Sturm, 15] : $\mathrm{RCD}(K, N)$

♣ "$\mathrm{Ric}_f^N \geq Kg \ \fallingdotseq\ \mathrm{Ric} \geq Kg + \dim M \leq N$"

Effective dimension

Curvature-dimension condition

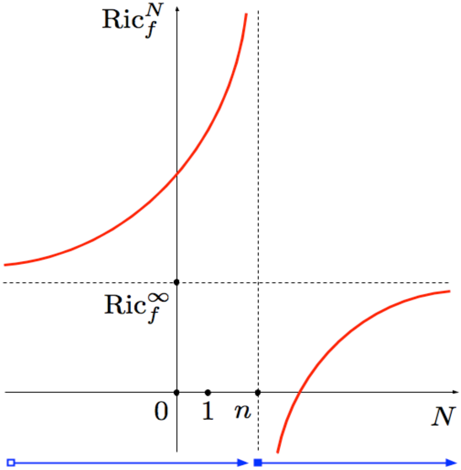
$$\mathrm{Ric}_f^N \geq Kg \text{ for } K \in \mathbb{R},\ N \in [n, \infty]$$

- $N \in [n, \infty]$: traditional case $\Longrightarrow$ "N : upper bound of the dimension"

  $$\text{[Qian, 97]} : N \in [n, \infty),\ \mathrm{Ric}_f^N \geq 0 \Longrightarrow \frac{m\,(B_r(x))}{r^N} \downarrow \ (r \to \infty)$$

- $N \in (-\infty, n)$: complementary case, e.g.,
  - [Ohta, 16] : Curvature-dimension condition $(N \in (-\infty, 0))$
  - [Kolesnikov-Milman, 17] : Poincaré-type inequalities $(N \in (-\infty, 0])$
  - [Klartag, 17] : Needle decomposition $(N \in (-\infty, 1))$
  - [Milman, 17] : Isoperimetric inequalities $(N \in (-\infty, 1))$
  - [Wylie, 17] : Cheeger-Gromoll splitting theorem $(N = 1)$

Monotonicity of Ric_f^N with respect to N



♣ $K \in \mathbb{R},\ N_1 \in [n, \infty],\ N_2 \in (-\infty, n)\ ;\ \mathrm{Ric}_f^{N_1} \geq Kg \implies \mathrm{Ric}_f^{N_2} \geq Kg$

Splitting theorem

Theorem ([Wylie, 17]).

$\sup f < \infty, \ \ \mathrm{Ric}_f^1 \geq 0$

Then

$$\exists\, \gamma : \mathbb{R} \to M : \textit{line} \implies M \equiv \textit{warped product over } \mathbb{R} \times \exists\, \widetilde{M}$$

Moreover, $\exists f_1 : \mathbb{R} \to \mathbb{R}, \ \ \exists f_2 : \widetilde{M} \to \mathbb{R}$ *s.t.*

- $(M, g) \equiv \left(\mathbb{R} \times \widetilde{M},\, dt^2 + \exp\left(2 \frac{f(\gamma_z(t)) - f(z)}{n-1} \right) g_{\widetilde{M}} \right)$
- $f(\gamma_z(t)) = f_1(t) + f_2(z)$

where $\gamma_z : \mathbb{R} \to M$ *is the line with* $\gamma_z(0) = z$ *that is orthogonal to* $\widetilde{M}$

▷ [Wylie, 17] : $N \in (-\infty, 1) \implies$ Direct product splitting

Example (Splitting theorem)

Example ([Wylie, 17]).

- $\mathbb{S}^{n-1}_L : (n-1)$-dim. standard sphere of constant curvature L
- $\phi : \mathbb{R} \to \mathbb{R} : C^2$, bounded, $\phi'' :$ bounded

$$M_{L,\phi} := \left(\mathbb{R} \times \mathbb{S}^{n-1}_L, dt^2 + e^{\frac{2\phi(t)}{n-1}} g_{\mathbb{S}^{n-1}_L}\right), \quad f_{L,\phi} := \phi \circ \pi$$

Then

- $\mathrm{Ric}^1_{f_{L,\phi}}(\partial_t) = 0$
- L : sufficiently large $\implies \mathrm{Ric}^1_{f_{L,\phi}}(v) \geq 0, \quad \forall v \perp \partial_t$

Background (Geometric analysis on projectively equivalent affine connection)

1-weighted Ricci curvature bound

1-weighted Ricci curvature bound ([Wylie-Yeroshkin, 16, preprint])

$$\mathrm{Ric}_f^1 \geq (n-1)\,\kappa\, e^{-\frac{4f}{n-1}} g \quad \text{for} \quad \kappa \in \mathbb{R}$$

♣ $f = 0$: unweighted case $\implies$ $\mathrm{Ric} \geq (n-1)\kappa g$

▷ [Wylie, 17]

- Cheeger-Gromoll splitting theorem $(\kappa = 0)$

▷ [Wylie-Yeroshkin, 16, preprint]

- Bonnet-Myers theorem $(\kappa > 0)$
- Cheng maximal diameter theorem $(\kappa > 0)$
- Bishop-Gromov volume comparison theorem $(\kappa \in \mathbb{R})$

Interpretation

- ∇ : Levi-Civita connection
- $\alpha \in \Lambda^1(M)$

Weighted affine connection

$$\nabla^\alpha_X Y := \nabla_X Y - \alpha(X)\, Y - \alpha(Y)\, X$$

♣ ∇^α : torsion free, affine, projectively equivalent to ∇

Interpretation

$$\mathrm{Ric}^1_f \geq (n-1)\,\kappa\, e^{-\frac{4f}{n-1}}\, g \iff \text{“}\mathrm{Ric}^{\nabla^\alpha} \geq (n-1)\kappa g\text{”}$$

Curvature tensor for weighted affine connection

Weighted affine connection

$$\nabla^{\alpha}_X Y := \nabla_X Y - \alpha(X)\, Y - \alpha(Y)\, X$$

- $R^{\nabla^{\alpha}}(X,Y)Z := \nabla^{\alpha}_X \nabla^{\alpha}_Y Z - \nabla^{\alpha}_Y \nabla^{\alpha}_X Z - \nabla^{\alpha}_{[X,Y]} Z$
- $\mathrm{Ric}^{\nabla^{\alpha}}(X,Y) := \mathrm{trace}_g \left[Z \mapsto R^{\nabla^{\alpha}}(Z,X)Y \right]$

Curvature tensor for weighted affine connection

$$\alpha_f := \frac{df}{n-1} \implies \mathrm{Ric}^{\nabla^{\alpha_f}} = \mathrm{Ric}^1_f$$

Geodesic for weighted affine connection

Weighted affine connection

$$\nabla^{\alpha}_{X} Y := \nabla_X Y - \alpha(X)\, Y - \alpha(Y)\, X$$

$x \in M,\ v \in U_x M$

- $\gamma_v : [0, \infty) \to M$: geodesic with $\gamma_v(0) = x,\ \gamma_v'(0) = v$
- $s_{f,v} : [0, \infty] \to [0, s_{f,v}(\infty)]$; $s_{f,v}(t) := \int_0^t e^{-\frac{2f(\gamma_v(\xi))}{n-1}}\, d\xi$
- $\widehat{\gamma}_v : [0, s_{f,v}(\infty)) \to M$; $\widehat{\gamma}_v := \gamma_v \circ s_{f,v}^{-1}$

Geodesic for weighted affine connection

$$\alpha_f := \frac{df}{n-1} \implies \widehat{\gamma}_v : \nabla^{\alpha_f}\text{-geodesic}$$

Interpretation

Weighted affine connection

$$\nabla^{\alpha}_X Y := \nabla_X Y - \alpha(X)\, Y - \alpha(Y)\, X$$

- $\gamma_v : [0,\infty) \to M$: geodesic with $\gamma_v(0) = x,\ \gamma_v'(0) = v$
- $s_{f,v} : [0,\infty] \to [0, s_{f,v}(\infty)]$; $s_{f,v}(t) := \int_0^t e^{-\frac{2f(\gamma_v(\xi))}{n-1}}\, d\xi$
- $\widehat{\gamma}_v : [0, s_{f,v}(\infty)) \to M$; $\widehat{\gamma}_v := \gamma_v \circ s_{f,v}^{-1}$

♣ $\alpha_f := \dfrac{df}{n-1} \implies \begin{cases} \mathrm{Ric}^{\nabla^{\alpha_f}} = \mathrm{Ric}^1_f \\ \widehat{\gamma}_v : \nabla^{\alpha_f}\text{-geodesic} \end{cases}$

Interpretation

$$\mathrm{Ric}^1_f(\gamma_v'(t)) \geq (n-1)\,\kappa\, e^{-\frac{4f(\gamma_v(t))}{n-1}} \iff \mathrm{Ric}^{\nabla^{\alpha_f}}(\widehat{\gamma}_v'(s)) \geq (n-1)\,\kappa$$

Related works (weighted sectional curvature)

$\Phi \in C^\infty(M),\ X, Y \in T_xM,\ X \perp Y,\ \|X\| = \|Y\| = 1$

Weighted sectional curvature ([Wylie, 15])

- $\sec_\Phi^X(Y) := \sec(X,Y) + \nabla^2\Phi(X,X)$
- $\overline{\sec}_\Phi^X(Y) := \sec(X,Y) + \nabla^2\Phi(X,X) + (d\Phi \otimes d\Phi)(X,X)$

♣ $\overline{\sec}_\Phi^X(Y) = g\big(R^{\nabla^{d\Phi}}(Y,X)X, Y\big)$

▷ [Wylie, 15] : Radial curvature equation, Second variation formula,...

▷ [Kennard-Wylie, 17] : $\sec_\Phi > 0,\ \overline{\sec}_\Phi > 0$

▷ [Kennard-Wylie-Yeroshkin, 19] : $\overline{\sec}_\Phi \geq \kappa\, e^{-4\Phi},\ \overline{\sec}_\Phi \leq \kappa\, e^{-4\Phi}$

Related works (generalized weighted affine connection)

$a, b \in \mathbb{R},\ \Phi \in C^\infty(M)$

Generalized weighted affine connection ([Li-Xia, 17])

$$\nabla^{a,b,\Phi}_X Y := \nabla_X Y - a\, d\Phi(X)\, Y - a\, d\Phi(Y)\, X - b\, g(X,Y)\, \nabla\Phi$$

- $\Phi = 0 \implies \nabla^{a,b,\Phi} = \nabla$
- $a = -b \implies \nabla^{a,b,\Phi} = \nabla^{\, e^{-2a\Phi} g}$
- $a = 1,\ b = 0 \implies \nabla^{a,b,\Phi} = \nabla^{d\Phi}$
- $a = 0,\ b = 1 \implies \mathrm{Ric}^{\nabla^{a,b,\Phi}} = \mathrm{Ric} - \dfrac{\nabla^2 e^{-\Phi}}{e^{-\Phi}} - \dfrac{\Delta e^{-\Phi}}{e^{-\Phi}}\, g$

▷ [Li-Xia, 17] : Bochner formula, Reilly formula for $\mathrm{Ric}^{\nabla^{a,b,\Phi}}$

Background (Interpolation)

Question

Curvature-dimension condition

$$\mathrm{Ric}_f^N \geq Kg \text{ for } K \in \mathbb{R},\ N \in [n, \infty]$$

- [Sturm, 06], [Lott-Villani, 09] :

  $\mathrm{CD}(K, N)$ for non-smooth metric measure space

Geometric analysis on projectively equivalent affine connection

$$\mathrm{Ric}_f^1 \geq (n-1)\,\kappa\, e^{-\frac{4f}{n-1}} g \text{ for } \kappa \in \mathbb{R}$$

- [Wylie-Yeroshkin, 16, preprint]

Question

Can we study two curvature bounds in a unified way???

First step

Curvature-dimension condition

$$\mathrm{Ric}_f^N \geq Kg \text{ for } K \in \mathbb{R},\ N \in [n, \infty]$$

- [Sturm, 06], [Lott-Villani, 09] :

  $\mathrm{CD}(K, N)$ for non-smooth metric measure space

Geometric analysis on projectively equivalent affine connection

$$\mathrm{Ric}_f^1 \geq (n-1)\,\kappa\, e^{-\frac{4f}{n-1}} g \text{ for } \kappa \in \mathbb{R}$$

- [Wylie-Yeroshkin, 16, preprint]

First step

$$\mathrm{Ric}_f^N \geq (n-N)\,\kappa\, e^{-\frac{4f}{n-N}} g \text{ for } \kappa \in \mathbb{R},\ N \in (-\infty, 1]$$

- [Kuwae-Li, 21]

Setting

$N \in (-\infty, 1] \cup [n, \infty]$

ε-range

- $\varepsilon = 0$ for $N = 1$
- $\varepsilon \in \left(-\sqrt{\frac{N-1}{N-n}}, \sqrt{\frac{N-1}{N-n}} \right)$ for $N \neq 1, n$
- $\varepsilon \in \mathbb{R}$ for $N = n$

- $c := \frac{1}{n-1} \left(1 - \varepsilon^2 \frac{N-n}{N-1} \right)$

Setting ([Lu-Minguzzi-Ohta, 22])

$$\mathrm{Ric}_f^N \geq c^{-1} \kappa \, e^{-\frac{4(1-\varepsilon)f}{n-1}} g \quad \text{for} \quad \kappa \in \mathbb{R}, \; N \in (-\infty, 1] \cup [n, \infty]$$

Special cases

Setting ([Lu-Minguzzi-Ohta, 22])

$$\mathrm{Ric}_f^N \geq c^{-1}\,\kappa\, e^{-\frac{4(1-\varepsilon)f}{n-1}} g \ \text{ for } \ \kappa \in \mathbb{R},\ N \in (-\infty, 1] \cup [n, \infty]$$

Special cases

- $N \in [n, \infty)$ and $\varepsilon = 1$ ($c = (N-1)^{-1}$) $\implies$ $\mathrm{Ric}_f^N \geq (N-1)\kappa g$
- $N = 1$ and $\varepsilon = 0$ ($c = (n-1)^{-1}$)

  $\implies$ $\mathrm{Ric}_f^1 \geq (n-1)\kappa e^{-\frac{4f}{n-1}} g$ ([Wylie-Yeroshkin, 16, preprint])
- $N \in (-\infty, 1)$ and $\varepsilon = \sqrt{\dfrac{N-1}{N-n}}$ ($c = (n-N)^{-1}$)

  $\implies$ $\mathrm{Ric}_f^N \geq (n-N)\kappa e^{-\frac{4f}{n-N}} g$ ([Kuwae-Li, 21])

♣ $\mathrm{CD}(K, \infty)$ is not included

Comparison geometry of manifolds without boundary

Laplacian comparison

- $\Delta_f := \Delta + g(\nabla f, \nabla \cdot)$: weighted Laplacian
- $\rho_x : M \to \mathbb{R}$; $\rho_x := d(\cdot, x)$: distance function
- $s_{f,v} : [0,\infty] \to [0, s_{f,v}(\infty)]$; $s_{f,v}(t) := \int_0^t e^{-\frac{2(1-\varepsilon)f(\gamma_v(\xi))}{n-1}} \, d\xi$
- $\tau(v) := \sup\{t > 0 \mid \rho_x(\gamma_v(t)) = t\}$
- $\mathfrak{s}_\kappa(s)$: sol. to $\psi''(s) + \kappa\,\psi(s) = 0$ with $\psi(0) = 0$ and $\psi'(0) = 1$

Theorem.

For $\kappa \in \mathbb{R}$, we assume

$$\mathrm{Ric}_f^N \geq c^{-1}\,\kappa\, e^{-\frac{4(1-\varepsilon)f}{n-1}} g$$

Then

$$\Delta_f \rho_x(\gamma_v(t)) \geq -\frac{\mathfrak{s}'_\kappa(s_{f,v}(t))}{\mathfrak{s}_\kappa(s_{f,v}(t))}\, e^{-\frac{2(1-\varepsilon)f(\gamma_v(t))}{n-1}} \quad \textit{for } \forall t \in (0, \tau(v))$$

▷ [Lu-Minguzzi-Ohta, 22]

Rigidity of Laplacian comparison

Theorem ([Kuwae-Sakurai, 21]).

Assume that the equality in Laplacian comparison holds at $t_0 \in (0, \tau(v))$

- $\{Y_{v,i}\}_{i=1}^{n-1}$: *Jacobi fields along* γ_v *with* $Y_{v,i}(0) = 0_x$, $Y'_{v,i}(0) = e_{v,i}$
- $\{E_{v,i}\}_{i=1}^{n-1}$: *parallel vector fields along* γ_v *with* $E_{v,i}(0) = e_{v,i}$

Then the following properties hold on $[0, t_0]$*:*

1. *If* $N = n$*, then* f *is constant, and*

$$Y_{v,i}(t) = \mathfrak{s}_{\kappa\, e^{-\frac{4(1-\varepsilon)f}{n-1}}}(t) E_{v,i}(t)$$

2. *if* $N \neq 1, n$*, then*

$$\varepsilon = 0, \quad f(\gamma_v(t)) \equiv f(x), \quad Y_{v,i}(t) = \mathfrak{s}_{\kappa\, e^{-\frac{4(1-\varepsilon)f(x)}{n-1}}}(t) E_{v,i}(t)$$

3. *if* $N = 1$*, then*

$$\varepsilon = 0, \quad Y_{v,i}(t) = \exp\left(\frac{f(\gamma_v(t)) + f(x)}{n-1}\right) \mathfrak{s}_\kappa(s_{f,v}(t))\, E_{v,i}(t)$$

Re-parametrized distance comparison

- $d_f : M \times M \to \mathbb{R}$; $d_f(x,y) := \inf_{\gamma} \int_0^{d(x,y)} e^{-\frac{2(1-\varepsilon)f(\gamma(\xi))}{n-1}}\, d\xi$

  Here inf. is taken over all min. geod. $\gamma : [0, d(x,y)] \to M$ from x to y

Theorem ([Kuwae-Sakurai, 21]).

For $\kappa > 0$, we assume

$$\mathrm{Ric}_f^N \geq c^{-1}\, \kappa\, e^{-\frac{4(1-\varepsilon)f}{n-1}} g$$

Then we have

$$\sup_{x,y \in M} d_f(x,y) \leq \frac{\pi}{\sqrt{\kappa}}$$

♣ $g_f := e^{-\frac{4(1-\varepsilon)f}{n-1}} g \implies d_{g_f}(x,y) \leq d_f(x,y)$

Maximal diameter theorem

Theorem ([Kuwae-Sakurai, 21]).

For $\kappa > 0$, we assume

$$\mathrm{Ric}_f^N \geq c^{-1}\,\kappa\, e^{-\frac{4(1-\varepsilon)f}{n-1}}\, g$$

Then we have

$$\mathrm{diam}_{g_f} M \leq \frac{\pi}{\sqrt{\kappa}}$$

Moreover, if the equality holds for some $x, y \in M$, then

1. *If $N = n$, then f is constant, and M is isometric to a sphere with constant curvature $\kappa\, e^{-\frac{4(1-\varepsilon)f}{n-1}}$*
2. *if $N \neq 1, n$, then $\varepsilon = 0$, f is constant, and M is isometric to a sphere with constant curvature $\kappa\, e^{-\frac{4f}{n-1}}$*
3. *if $N = 1$, then $\varepsilon = 0$, f is radial, and M is homeomorphic to a sphere, and*

$$g = dt^2 + \exp\left(2\frac{f(\gamma_v(t)) + f(x)}{n-1}\right) \mathfrak{s}_\kappa^2\left(s_{f,v}(t)\right) g_{\mathbb{S}^{n-1}}$$

Maximal diameter theorem with bounded density

Theorem ([Kuwae-Sakurai, 21]).

For $\kappa > 0$ and $\delta \in \mathbb{R}$, we assume

$$\mathrm{Ric}_f^N \geq c^{-1}\,\kappa\, e^{-\frac{4(1-\varepsilon)f}{n-1}} g, \quad (1-\varepsilon)f \leq (n-1)\delta$$

Then

$$\operatorname{diam} M \leq \frac{\pi}{\sqrt{\kappa e^{-4\delta}}}$$

Moreover, if the equality holds, then

1. *If $N = n$, then $(1-\varepsilon)f \equiv (n-1)\delta$, and M is isometric to a sphere of constant curvature $\kappa e^{-4\delta}$*
2. *if $N \neq n$, then $\varepsilon = 0$, $f \equiv (n-1)\delta$, and M is isometric to a sphere of constant curvature $\kappa e^{-4\delta}$*

Absolute volume comparison

- $B_{f,r}(x) := \{\, y \in M \mid d_f(x,y) < r \,\}$
- $\mathfrak{m} := e^{-\frac{2(1-\varepsilon)f}{n-1}} m$
- $\mathcal{S}_\kappa(r) := \int_0^{\min\{r,\pi/\sqrt{\kappa}\}} \mathfrak{s}_\kappa^{c^{-1}}(\xi)\, d\xi$

Theorem ([Kuwae-Sakurai, 21]).

For $\kappa \in \mathbb{R}$, we assume

$$\mathrm{Ric}_f^N \geq c^{-1}\, \kappa\, e^{-\frac{4(1-\varepsilon)f}{n-1}} g, \quad c = (n-1)^{-1}$$

Then

$$\mathfrak{m}(B_{f,r}(x)) \leq \omega_{n-1}\, \mathcal{S}_\kappa(r) \quad \text{for} \quad \forall r > 0$$

Here ω_{n-1} is the volume of the $(n-1)$-dimensional unit sphere

Relative volume comparison

Theorem ([Kuwae-Sakurai, 21]).

For $\kappa \in \mathbb{R}$, *we assume*

$$\mathrm{Ric}_f^N \geq c^{-1}\,\kappa\,e^{-\frac{4(1-\varepsilon)f}{n-1}}g$$

Then

$$\frac{\mathfrak{m}(B_{f,R}(x))}{\mathfrak{m}(B_{f,r}(x))} \leq \frac{\mathcal{S}_\kappa(R)}{\mathcal{S}_\kappa(r)} \quad \textit{for} \;\; \forall r \leq \forall R$$

Rigidity of volume comparison

Theorem ([Kuwae-Sakurai, 21]).

For $\kappa \leq 0$, we assume

$$\mathrm{Ric}_f^N \geq c^{-1}\,\kappa\, e^{-\frac{4(1-\varepsilon)f}{n-1}} g, \quad c = (n-1)^{-1}$$

If

$$\varliminf_{r \to +\infty} \frac{\mathfrak{m}(B_{f,r}(x))}{\mathcal{S}_\kappa(r)} \geq \omega_{n-1},$$

then M is diffeomorphic to $\mathbb{R}^n$, and

1. *If $N = n$, then f is constant, and $g = dt^2 + \mathfrak{s}_{\kappa\, e^{-\frac{4(1-\varepsilon)f}{n-1}}}(t)\, g_{\mathbb{S}^{n-1}}$*
2. *if $N \neq 1, n$, then $\varepsilon = 0$, f is constant, and $g = dt^2 + \mathfrak{s}_{\kappa\, e^{-\frac{4f}{n-1}}}(t)\, g_{\mathbb{S}^{n-1}}$*
3. *if $N = 1$, then $\varepsilon = 0$, and*

$$g = dt^2 + \exp\left(2\frac{f(\gamma_v(t)) + f(x)}{n-1}\right) \mathfrak{s}^2_\kappa(s_{f,v}(t)) g_{\mathbb{S}^{n-1}}$$

Comparison geometry of manifolds with boundary

Setting

- $z \in \partial M,\ u_z$: unit inner normal vector
- $H_{f,z} := H_z + g(\nabla f, u_z)$: weighted mean curvature

Setting ([Kuwae-Sakurai, 20, preprint])

$\kappa, \lambda \in \mathbb{R},\ N \in (-\infty, 1] \cup [n, \infty]$

$$\mathrm{Ric}_f^N \geq c^{-1}\,\kappa\, e^{-\frac{4(1-\varepsilon)f}{n-1}} g, \quad H_{f,\partial M} \geq c^{-1}\lambda e^{-\frac{2(1-\varepsilon)f}{n-1}}$$

Special cases

▷ [Heintze-Karcher, 78], [Kasue, 83, 84] : unweighted

- $N \in [n, \infty)$ and $\varepsilon = 1$ $(c = (N-1)^{-1})$

  $\Longrightarrow \quad \mathrm{Ric}_f^N \geq (N-1)\kappa g, \quad H_{f,\partial M} \geq (N-1)\lambda$ ([Sakurai, 19])

- $N \in (-\infty, 1]$ and $\varepsilon = 0$ $(c = (n-1)^{-1})$

  $\Longrightarrow \quad \mathrm{Ric}_f^N \geq (n-1)\kappa e^{-\frac{4f}{n-1}} g, \quad H_{f,\partial M} \geq (n-1)\lambda e^{-\frac{2f}{n-1}}$ ([Sakurai, 20])

Laplacian comparison

- $\rho_{\partial M} : M \to \mathbb{R}$; $\rho_{\partial M} := d(\cdot, \partial M)$: distance function
- $\gamma_z : [0,T] \to M$: geodesic with $\gamma_z(0) = z,\ \gamma_z'(0) = u_z$
- $s_{f,z} : [0,T] \to [0, s_{f,z}(T)]$; $s_{f,z}(t) := \int_0^t e^{-\frac{2(1-\varepsilon)f(\gamma_z(\xi))}{n-1}}\, d\xi$
- $\tau(z) := \sup\{t > 0 \mid \rho_{\partial M}(\gamma_z(t)) = t\}$
- $\mathfrak{s}_{\kappa,\lambda}(s)$: sol. to $\psi''(s) + \kappa\,\psi(s) = 0$ with $\psi(0) = 1$ and $\psi'(0) = -\lambda$

Theorem ([Kuwae-Sakurai, 20, preprint]).

For $\kappa, \lambda \in \mathbb{R}$*, we assume*

$$\mathrm{Ric}_f^N \geq c^{-1}\,\kappa\, e^{-\frac{4(1-\varepsilon)f}{n-1}} g, \quad H_{f,\partial M} \geq c^{-1}\lambda e^{-\frac{2(1-\varepsilon)f}{n-1}}$$

Then

$$\Delta_f \rho_{\partial M}(\gamma_z(t)) \geq -\frac{\mathfrak{s}'_{\kappa,\lambda}(s_{f,z}(t))}{\mathfrak{s}_{\kappa,\lambda}(s_{f,z}(t))}\, e^{-\frac{2(1-\varepsilon)f(\gamma_z(t))}{n-1}} \quad \textit{for}\ \ \forall t \in (0, \tau(z))$$

Rigidity of Laplacian comparison

Theorem ([Kuwae-Sakurai, 20, preprint]).

Assume that the equality in Laplacian comparison holds at $t_0 \in (0, \tau(z))$

- $\{Y_{z,i}\}_{i=1}^{n-1}$: *Jacobi fields along* γ_z *with* $Y_{z,i}(0) = e_{z,i}$, $Y'_{z,i}(0) = -A_{u_z} e_{z,i}$
- $\{E_{z,i}\}_{i=1}^{n-1}$: *parallel vector fields along* γ_z *with* $E_{z,i}(0) = e_{z,i}$

Then the following properties hold on $[0, t_0]$*:*

1. *If* $N = n$, *then* f *is constant, and*

$$Y_{z,i}(t) = \mathfrak{s}_{\kappa\, e^{-\frac{4(1-\varepsilon)f}{n-1}},\lambda\, e^{-\frac{2(1-\varepsilon)f}{n-1}}}(t) E_{z,i}(t)$$

Rigidity of Laplacian comparison

Theorem ([Kuwae-Sakurai, 20, preprint]).

Assume that the equality in Laplacian comparison holds at $t_0 \in (0, \tau(z))$

- $\{Y_{z,i}\}_{i=1}^{n-1}$: *Jacobi fields along* γ_z *with* $Y_{z,i}(0) = e_{z,i}$, $Y'_{z,i}(0) = -A_{u_z} e_{z,i}$
- $\{E_{z,i}\}_{i=1}^{n-1}$: *parallel vector fields along* γ_z *with* $E_{z,i}(0) = e_{z,i}$

Then the following properties hold on $[0, t_0]$*:*

1. *If* $N \neq 1, n$, *then*

$$f(\gamma_z(t)) = f(z) - \varepsilon \frac{N-n}{N-1} c^{-1} \log \mathfrak{s}_{\kappa,\lambda}(s_{f,z}(t)),$$

$$Y_{z,i}(t) = \mathfrak{s}_{\kappa,\lambda}^{\frac{c^{-1}}{n-1}\left(1-\varepsilon \frac{N-n}{N-1}\right)}(s_{f,z}(t)) E_{z,i}(t)$$

2. *if* $N = 1$, *then*

$$\varepsilon = 0, \quad Y_{z,i}(t) = \exp\left(\frac{f(\gamma_z(t)) - f(z)}{n-1}\right) \mathfrak{s}_{\kappa,\lambda}(s_{f,z}(t))\, E_{z,i}(t)$$

Splitting theorem

Theorem ([Kuwae-Sakurai, 20, preprint]).

Let $\kappa \leq 0$ *and* $\lambda := \sqrt{|\kappa|}$. *We assume*

$$\mathrm{Ric}_f^N \geq c^{-1}\,\kappa\, e^{-\frac{4(1-\varepsilon)f}{n-1}} g, \quad H_{f,\partial M} \geq c^{-1}\lambda e^{-\frac{2(1-\varepsilon)f}{n-1}}$$

Suppose that $(1-\varepsilon)f$ *is bounded from above. If* $\tau(z_0)=\infty$ *for some* $z_0 \in \partial M$, *then* M *is diffeomorphic to* $[0,\infty)\times \partial M$, *and*

1. *If* $N \neq 1, n$, *then for any* $z \in \partial M$

$$f(\gamma_z(t)) = f(z) - \varepsilon \frac{N-n}{N-1} c^{-1} \log \mathfrak{s}_{\kappa,\lambda}(s_{f,z}(t)),$$

$$g = dt^2 + \mathfrak{s}_{\kappa,\lambda}^{2\frac{c^{-1}}{n-1}\left(1-\varepsilon\frac{N-n}{N-1}\right)}(s_{f,z}(t)) g_{\partial M};$$

2. *if* $N = 1$, *then*

$$\varepsilon = 0, \quad g = dt^2 + \exp\left(2\frac{f(\gamma_z(t)) - f(z)}{n-1}\right) \mathfrak{s}_{\kappa,\lambda}^2(s_{f,z}(t))\, g_{\partial M}.$$

Model spaces

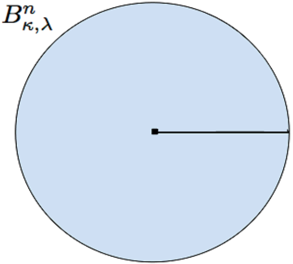
Model spaces

$\kappa, \lambda \in \mathbb{R}$, M^n_κ : n-dim. space form with constant curvature κ

$\kappa, \lambda \in \mathbb{R}$: ball-condition $:\Longleftrightarrow$ $\exists B^n_{\kappa,\lambda} \subset M^n_\kappa$: closed ball with $\mathcal{H}_{\partial B^n_{\kappa,\lambda}} \equiv (n-1)\lambda$

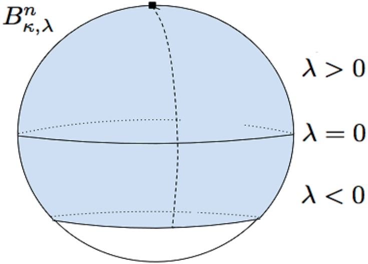
- $\kappa \leq 0$ $(\lambda > \sqrt{|\kappa|})$

$\mathbb{R}^n$ or $\mathbb{H}^n(\kappa)$



- $\kappa > 0$ $(\lambda \in \mathbb{R})$

$\mathbb{S}^n(1/\sqrt{\kappa})$



Maximal inscribed radius theorem

Theorem ([Kuwae-Sakurai, 20, preprint]).

Let κ and λ satisfy the ball-condition. We assume

$$\mathrm{Ric}_f^N \geq c^{-1}\,\kappa\, e^{-\frac{4(1-\varepsilon)f}{n-1}} g, \quad H_{f,\partial M} \geq c^{-1}\lambda e^{-\frac{2(1-\varepsilon)f}{n-1}}$$

Then we have

$$\mathrm{InRad}_{g_f} M \leq C_{\kappa,\lambda}$$

If $\rho^{g_f}_{\partial M}(x_0) = C_{\kappa,\lambda}$ for some $x_0 \in M$, then M is diffeomorphic to a closed ball centered at x_0, and

1. *If $N \neq 1, n$, then f is constant, and*

$$\varepsilon = 0, \quad g = dt^2 + \mathfrak{s}^2_{\kappa\, e^{-\frac{4f}{n-1}}}(t) g_{\mathbb{S}^{n-1}}$$

2. *if $N = 1$, then f is radial with respect to x_0, and*

$$\varepsilon = 0, \quad g = dt^2 + \exp\left(2\frac{f(\gamma_v(t)) + f(x_0)}{n-1}\right) \mathfrak{s}^2_{\kappa}(s_{f,v}(t)) g_{\mathbb{S}^{n-1}}$$

Maximal inscribed radius theorem with bounded density

Theorem ([Kuwae-Sakurai, 20, preprint]).

Let κ and λ satisfy the ball-condition. We assume

$$\mathrm{Ric}_f^N \geq c^{-1}\,\kappa\,e^{-\frac{4(1-\varepsilon)f}{n-1}} g, \quad H_{f,\partial M} \geq c^{-1}\lambda e^{-\frac{2(1-\varepsilon)f}{n-1}}, \quad (1-\varepsilon)f \leq (n-1)\delta$$

for $\delta \in \mathbb{R}$. Then we have

$$\mathrm{InRad}\, M \leq C_{\kappa\, e^{-4\delta}, \lambda\, e^{-2\delta}}$$

If $\rho_{\partial M}(x_0) = C_{\kappa\, e^{-4\delta}, \lambda\, e^{-2\delta}}$ for some $x_0 \in M$, then $(1-\varepsilon)f = (n-1)\delta$ and

1. *If $N = n$, then M is isometric to $B^n_{\kappa\, e^{-4\delta}, \lambda\, e^{-2\delta}}$*
2. *if $N \neq n$, then $\varepsilon = 0$, and M is isometric to $B^n_{\kappa\, e^{-4\delta}, \lambda\, e^{-2\delta}}$*

Absolute volume comparison

- $\rho_{\partial M,f} : M \to \mathbb{R}\ ;\ \rho_{\partial M,f}(x) := \inf_{z \in \partial M} \int_0^{\rho_{\partial M}(x)} e^{-\frac{2(1-\varepsilon) f(\gamma_z(\xi))}{n-1}} \, d\xi$

  Here inf. is taken over all foot points $z \in \partial M$ of x (i.e., $\rho_{\partial M}(x) = d(x,z)$)

- $B_{f,r}(\partial M) := \{\, x \in M \mid \rho_{\partial M,f}(x) < r \,\}$

- $\mathcal{S}_{\kappa,\lambda}(r) := \int_0^{\min\{r, C_{\kappa,\lambda}\}} \mathfrak{s}_{\kappa,\lambda}^{c^{-1}}(\xi)\, d\xi$

Theorem ([Kuwae-Sakurai, 20, preprint]).

For $\kappa, \lambda \in \mathbb{R}$, we assume

$$\mathrm{Ric}_f^N \geq c^{-1}\, \kappa\, e^{-\frac{4(1-\varepsilon)f}{n-1}} g, \quad H_{f,\partial M} \geq c^{-1} \lambda e^{-\frac{2(1-\varepsilon)f}{n-1}}$$

Let ∂M be compact. Then

$$\mathfrak{m}\,(B_{f,r}(\partial M)) \leq \mathcal{S}_{\kappa,\lambda}(r)\, m_{\partial M}(\partial M) \quad \textit{for} \quad \forall r > 0$$

Relative volume comparison

Theorem ([Kuwae-Sakurai, 20, preprint]).

For $\kappa, \lambda \in \mathbb{R}$, *we assume*

$$\mathrm{Ric}_f^N \geq c^{-1}\, \kappa\, e^{-\frac{4(1-\varepsilon)f}{n-1}} g, \quad H_{f,\partial M} \geq c^{-1} \lambda e^{-\frac{2(1-\varepsilon)f}{n-1}}$$

Let ∂M *be compact. Then*

$$\frac{\mathfrak{m}(B_{f,R}(\partial M))}{\mathfrak{m}(B_{f,r}(\partial M))} \leq \frac{\mathcal{S}_{\kappa,\lambda}(R)}{\mathcal{S}_{\kappa,\lambda}(r)} \quad \textit{for } \forall r \leq \forall R$$

Rigidity of volume comparison

Theorem ([Kuwae-Sakurai, 20, preprint]).

Assume that κ and λ do not satisfy the ball-condition. Let ∂M be compact. If

$$\liminf_{r\to\infty} \frac{\nu(B_{f,r}(\partial M))}{\mathcal{S}_{\kappa,\lambda}(r)} \geq m_{\partial M}(\partial M),$$

then M is diffeomorphic to $[0,\infty)\times\partial M$, and

1. *if $N \neq 1, n$, then*

$$f(\gamma_z(t)) = f(z) - \varepsilon\frac{N-n}{N-1}c^{-1}\log\mathfrak{s}_{\kappa,\lambda}(s_{f,z}(t)),$$

$$g = dt^2 + \mathfrak{s}_{\kappa,\lambda}^{2\frac{c-1}{n-1}\left(1-\varepsilon\frac{N-n}{N-1}\right)}(s_{f,z}(t))g_{\partial M}$$

2. *if $N = 1$, then*

$$\varepsilon = 0, \quad g = dt^2 + \exp\left(2\frac{f(\gamma_z(t)) - f(z)}{n-1}\right)\mathfrak{s}_{\kappa,\lambda}^2(s_{f,z}(t))\, g_{\partial M}$$

Dirichlet Eigenvalue comparison

- $\nu_{m,p}(M) := \inf_{\psi \in W_0^{1,p}(M,m)\setminus\{0\}} \dfrac{\int_M \|\nabla \psi\|^p \, dm}{\int_M |\psi|^p \, dm}$

Theorem ([Kuwae-Sakurai, 20, preprint]).

Let $p \in (1,\infty)$. Let κ and λ satisfy the ball-condition, and $\lambda \geq 0$. Assume

$$\mathrm{Ric}_f^N \geq c^{-1}\,\kappa\, e^{-\frac{4(1-\varepsilon)f}{n-1}} g, \quad H_{f,\partial M} \geq c^{-1}\lambda e^{-\frac{2(1-\varepsilon)f}{n-1}}, \quad (1-\varepsilon)f \leq (n-1)\delta$$

for $\delta \in \mathbb{R}$. Let M be compact. Then

$$\nu_{m,p}(M) \geq \nu_p(B^n_{\kappa e^{-4\delta},\lambda e^{-2\delta}})$$

If the equality holds, then $(1-\varepsilon)f = (n-1)\delta$ on M, and

1. *If $N = n$, then M is isometric to $B^n_{\kappa\, e^{-4\delta},\lambda\, e^{-2\delta}}$*
2. *if $N \neq n$, then $\varepsilon = 0$, and M is isometric to $B^n_{\kappa\, e^{-4\delta},\lambda\, e^{-2\delta}}$*

Spectrum rigidity

Theorem ([Kuwae-Sakurai, 20, preprint]).

Let $p \in (1, \infty)$*. Let* $\kappa < 0$ *and* $\lambda := \sqrt{|\kappa|}$*. Assume*

$$\mathrm{Ric}_f^N \geq c^{-1}\,\kappa\, e^{-\frac{4(1-\varepsilon)f}{n-1}} g, \quad H_{f,\partial M} \geq c^{-1}\lambda e^{-\frac{2(1-\varepsilon)f}{n-1}}, \quad (1-\varepsilon)f \leq (n-1)\delta$$

for $\delta \in \mathbb{R}$*. Let* ∂M *be compact. Then*

$$\nu_{m,p}(M) \geq e^{-2p\delta} \left(\frac{c^{-1}\lambda}{p}\right)^p$$

If the equality holds, then M *is diffeomorphic to* $[0,\infty) \times \partial M$*, and*

1. *If* $N \neq 1, n$*, then*

$$f(\gamma_z(t)) = f(z) - \varepsilon \frac{N-n}{N-1} c^{-1} \log \mathfrak{s}_{\kappa,\lambda}(s_{f,z}(t)),$$

$$g = dt^2 + \mathfrak{s}_{\kappa,\lambda}^{2\frac{c^{-1}}{n-1}\left(1-\varepsilon\frac{N-n}{N-1}\right)}(s_{f,z}(t))g_{\partial M}$$

Spectrum rigidity

Theorem ([Kuwae-Sakurai, 20, preprint]).

Let $p \in (1, \infty)$. Let $\kappa < 0$ and $\lambda := \sqrt{|\kappa|}$. Assume

$$\mathrm{Ric}_f^N \geq c^{-1}\,\kappa\, e^{-\frac{4(1-\varepsilon)f}{n-1}} g, \quad H_{f,\partial M} \geq c^{-1}\lambda e^{-\frac{2(1-\varepsilon)f}{n-1}}, \quad (1-\varepsilon)f \leq (n-1)\delta$$

for $\delta \in \mathbb{R}$. Let ∂M be compact. Then

$$\nu_{m,p}(M) \geq e^{-2p\delta} \left(\frac{c^{-1}\lambda}{p} \right)^p$$

If the equality holds, then M is diffeomorphic to $[0, \infty) \times \partial M$, and

1. *If $N = 1$, then*

$$\varepsilon = 0, \quad g = dt^2 + \exp\left(2\frac{f(\gamma_z(t)) - f(z)}{n-1} \right) \mathfrak{s}^2_{\kappa,\lambda}(s_{f,z}(t))\, g_{\partial M}$$

Characterization via optimal transport theory

Wasserstein space

- $P(M)$: the set of all Borel probability measures on M
- $\pi \in P(M \times M)$: coupling of (μ, ν)

  $:\Longleftrightarrow \ \pi(X \times M) = \mu(X),\ \pi(M \times X) = \nu(X), \quad \forall X \subset M$: Borel
- $\Pi(\mu, \nu)$: the set of all coupling of (μ, ν)
- $P_2(M) := \left\{ \mu \in P(M) \,\middle|\, \exists x_0 \in M \ ; \ \int_M d(x, x_0)^2 \, d\mu(x) < \infty \right\}$
- $\mu,\, \nu \in P_2(M)$

$$W_2(\mu, \nu) := \inf_{\pi \in \Pi(\mu,\nu)} \left(\int_{M \times M} d(x, y)^2 \, d\pi(x, y) \right)^{\frac{1}{2}}$$: Wasserstein distance

- $P_2(M) := (P_2(M), W_2)$: Wasserstein space
- $\pi \in \Pi(\mu, \nu)$: optimal coupling of (μ, ν) $\ :\Longleftrightarrow \ \pi$ attains the infimum

Entropy

- $\mathcal{DC}$: the set of all $U : [0,\infty) \to \mathbb{R}$: continuous, convex, $U(0) = 0$ s.t.

$$\varphi_U(r) := r^{\frac{c+1}{c}} U(r^{-\frac{c+1}{c}}) \text{ : convex}$$

♣ $H(r) := \dfrac{c+1}{c}\, r\, \left(1 - r^{-\frac{c}{c+1}}\right) \implies H \in \mathcal{DC}$

- $\mu = \rho\, m \in P_2(M),\ U \in \mathcal{DC}$,

$$U_m(\mu) := \int_M U(\rho)\, dm$$

- $U = H \implies H_m := U_m$: Rényi entropy

Twisted coefficient

- $t \in [0,1], \quad d_{f,t} : M \times M \to \mathbb{R}\ ;\ d_{f,t}(x,y) := \inf_{\gamma} \int_0^{t\,d(x,y)} e^{-\frac{2(1-\varepsilon)f(\gamma(\xi))}{n-1}}\,d\xi$,

  Here inf. is taken over all min. geod. $\gamma : [0, d(x,y)] \to M$ from x to y

♣ $t = 1 \implies d_{f,1}(x,y) = d_f(x,y)$ (Bonnet-Myers type theorem holds for d_f)

♣ $t \neq 0, 1 \implies d_{f,t}(x,y) \neq d_{f,t}(y,x)$ in general

Twisted coefficient ([Kuwae-Sakurai, 20, preprint])

$$\beta_{\kappa,f,t}(x,y) := \left(\frac{\mathfrak{s}_\kappa(d_{f,t}(x,y))}{t\mathfrak{s}_\kappa(d_f(x,y))} \right)^{c^{-1}}$$

Twisted curvature-dimension condition

Definition ([Kuwae-Sakurai, 20, preprint]).

(M, d, m) : Twisted curvature-dimension condition $\mathrm{TwCD}(\kappa, N, \varepsilon)$ $:\Longleftrightarrow$

$\forall \mu_i = \rho_i\, m \in P_2^{ac}(M)$ $(i = 0, 1)$,

$$U_m(\mu_t) \le (1-t) \int_{M \times M} U\left(\frac{\rho_0(x)}{\beta_{\kappa,f,1-t}(y,x)}\right) \frac{\beta_{\kappa,f,1-t}(y,x)}{\rho_0(x)} d\pi(x,y) + t \int_{M \times M} U\left(\frac{\rho_1(y)}{\beta_{\kappa,f,t}(x,y)}\right) \frac{\beta_{\kappa,f,t}(x,y)}{\rho_1(y)} d\pi(x,y)$$

for $\forall U \in \mathcal{DC}$, $\forall t \in (0,1)$, where

- π : unique optimal coupling of (μ_0, μ_1)
- $(\mu_t)_{t \in [0,1]}$: unique minimal geodesic in $P_2(M)$ from μ_0 to μ_1

▷ [Lott-Villani, 09] : $N \in [n, \infty)$, $\varepsilon = 1$ $\mathrm{CD}((N-1)\kappa, N) = \mathrm{TwCD}(\kappa, N, 1)$

▷ [Sakurai, 20] : $N = 1$, $\varepsilon = 0$

Relaxed twisted curvature-dimension condition

Definition ([Kuwae-Sakurai, 20, preprint]).

(M, d, m) : Relaxed twisted curv.-dim. condition $\mathrm{TwCD}_{\mathrm{rel}}(\kappa, N, \varepsilon)$ $:\Longleftrightarrow$

$\forall \mu_i = \rho_i\, m \in P_2^{ac}(M)$ $(i = 0, 1)$,

$$H_m(\mu_t) \le (1-t) \int_{M\times M} H\left(\frac{\rho_0(x)}{\beta_{\kappa,f,1-t}(y,x)}\right) \frac{\beta_{\kappa,f,1-t}(y,x)}{\rho_0(x)} d\pi(x,y) + t \int_{M\times M} H\left(\frac{\rho_1(y)}{\beta_{\kappa,f,t}(x,y)}\right) \frac{\beta_{\kappa,f,t}(x,y)}{\rho_1(y)} d\pi(x,y)$$

for $\forall t \in (0,1)$, where

- π : unique optimal coupling of (μ_0, μ_1)
- $(\mu_t)_{t\in[0,1]}$: unique minimal geodesic in $P_2(M)$ from μ_0 to μ_1

▷ [Sturm, 06] : $N \in [n, \infty)$, $\varepsilon = 1$ $\mathrm{CD}((N-1)\kappa, N) = \mathrm{TwCD}(\kappa, N, 1)$

▷ [Sakurai, 20] : $N = 1$, $\varepsilon = 0$

Characterization via optimal transport theory

Theorem ([Kuwae-Sakurai, 20, preprint]).

Assume that if $N \neq 1, n$, then $\varepsilon \neq 0$. TFAE:

- $\mathrm{Ric}_f^N \geq c^{-1} \kappa \, e^{-\frac{4(1-\varepsilon)f}{n-1}} g$
- $(M, d, m) : \mathrm{TwCD}(\kappa, N, \varepsilon)$
- $(M, d, m) : \mathrm{TwCD}_{\mathrm{rel}}(\kappa, N, \varepsilon)$

▷ [Sturm, 06], [Lott-Villani, 09] : $N \in [n, \infty),\ \varepsilon = 1$

▷ [Sakurai, 20] : $N = 1,\ \varepsilon = 0$

Prékopa-Leindler inequality

- $t \in [0,1],\ X, Y \subset M$

$$Z_t(X,Y) := \{\gamma(t) \mid \gamma : [0,1] \to M \text{ s.t. } \gamma(0) \in X,\ \gamma(1) \in Y\}$$

Corollary ([Kuwae-Sakurai, 20, preprint]).

- $\psi_i, \psi : M \to [0,\infty)\ (i = 0,1)$
- X, Y : *bounded,* $\operatorname{supp}[\psi_0] \subset X$, $\operatorname{supp}[\psi_1] \subset Y$

For $t \in (0,1)$*, we assume that* $\forall (x,y) \in X \times Y$ *and* $\forall z \in Z_t(\{x\},\{y\})$,

$$\psi(z) \geq \left(\frac{\psi_0(x)}{\beta_{\kappa,f,1-t}(y,x)} \right)^{1-t} \left(\frac{\psi_1(y)}{\beta_{\kappa,f,t}(x,y)} \right)^{t}$$

For $\kappa \in \mathbb{R}$*, if* $\mathrm{Ric}_f^N \geq c^{-1}\,\kappa\, e^{-\frac{4(1-\varepsilon)f}{n-1}} g$*, then*

$$\int_M \psi\, dm \geq \left(\int_M \psi_0 dm \right)^{1-t} \left(\int_M \psi_1\, dm \right)^{t}$$

Borel-Brascamp-Lieb inequality

Corollary ([Kuwae-Sakurai, 20, preprint]).

- $\psi_i, \psi : M \to [0, \infty)$ $(i = 0, 1)$
- X, Y : *bounded*, $\operatorname{supp}[\psi_0] \subset X$, $\operatorname{supp}[\psi_1] \subset Y$

We suppose $\int_M \psi_0 \, dm = \int_M \psi_1 \, dm = 1$

For $t \in (0,1)$, *we assume that* $\forall (x, y) \in X \times Y$ *and* $\forall z \in Z_t(\{x\}, \{y\})$,

$$\psi(z)^{-\frac{c}{c+1}} \leq (1-t) \left(\frac{\psi_0(x)}{\beta_{\kappa,f,1-t}(y,x)} \right)^{-\frac{c}{c+1}} + t \left(\frac{\psi_1(y)}{\beta_{\kappa,f,t}(x,y)} \right)^{-\frac{c}{c+1}}$$

For $\kappa \in \mathbb{R}$, *if* $\operatorname{Ric}_f^N \geq c^{-1} \, \kappa \, e^{-\frac{4(1-\varepsilon) f}{n-1}} g$, *then*

$$\int_M \psi \, dm \geq 1$$

Brunn-Minkowski inequality

Corollary ([Kuwae-Sakurai, 20, preprint]).

Let $X, Y \subset M$ *denote two bounded Borel subsets with* $m(X), m(Y) \in (0, \infty)$

For $\kappa \in \mathbb{R}$, *if* $\mathrm{Ric}_f^N \geq c^{-1} \kappa \, e^{-\frac{4(1-\varepsilon)f}{n-1}} g$, *then* $\forall t \in (0,1)$

$$m(Z_t(x,y))^{\frac{c}{c+1}} \geq (1-t) \left(\inf_{(x,y)\in X\times Y} \beta_{\kappa,f,1-t}(y,x)^{\frac{c}{c+1}} \right) m(X)^{\frac{c}{c+1}} + t \left(\inf_{(x,y)\in X\times Y} \beta_{\kappa,f,1-t}(x,y)^{\frac{c}{c+1}} \right) m(Y)^{\frac{c}{c+1}}$$

Summary

Setting ([Lu-Minguzzi-Ohta, 22])

$$\mathrm{Ric}_f^N \geq c^{-1}\,\kappa\, e^{-\frac{4(1-\varepsilon)f}{n-1}} g \quad \text{for} \quad \kappa \in \mathbb{R},\ N \in (-\infty, 1] \cup [n, \infty]$$

- [Lu-Minguzzi-Ohta, 22], [Kuwae-Sakurai, 21] :

  Comparison geometry of manifolds without boundary

- [Kuwae-Sakurai, 20, preprint] :

  Comparison geometry of manifolds with boundary

- [Kuwae-Sakurai, 20, preprint] : Characterization via optimal transport theory

Thank you for your attention!!!