部分多様体幾何とリー群作用2021　　2022年3月20日

(無限時間かけて非完備になる非コンパクト山辺フローの具体例)

"An Examples of the Noncompact Yamabe flow having the Infinite-time Incompleteness" (Hikaru Yamamoto)

(This is joint work with Jin Takahashi at Tokyo Institute of Technology)

§1. What is the Yamabe flow?

Notation

- M : n-dim mfd (without boundary)
- $\mathrm{scal}(g)$: the scalar curvature of Riem. met. g on M.

We consider the following parabolic PDE:

$$\frac{\partial}{\partial t} g_t = -\mathrm{scal}(g_t) \cdot g_t \qquad \text{(uYF)}$$

"unnormalized Yamabe flow"

(In this talk, Yamabe flow := uYF)

§1.1. When M is compact ....

- long time ex. and uniqueness are OK. ← Hamilton (1989)
- Convergence $g_t \xrightarrow{t\to\infty} g_\infty$ to a const. scal. met. is OK. (in many cases)

Ye (1994), Schwetlich - Struwe (2003), Brendle (2007)

However, if M is noncompact, these do not hold in general.

§1.2 Noncompact case

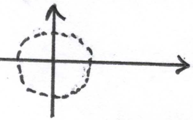
→ Assume M is noncompact.

$\boxed{1}$ Short time ex. is OK? ⟶ No! in general.

But .... [Ma - An (1999): If (M, g_0) is complete, LCF and $Ric \geq -C$, then the short time ex. is OK.]

Many other sufficient condition for the short time ex. are also known.

$\boxed{2}$ Uniqueness is OK? $\longrightarrow$ No! in general.

For example.... [There exist infinitely many Yamabe flows starting from $g_{\mathbb{R}^2}$ on $B^2(1)$ $\longrightarrow$ ]

(See Giesen - Topping (2011) for more interesting phenomena.)

$\boxed{3}$ Long time sol.? $\longrightarrow$ No! in general.

Daskalopoulos - King - Sesum (2013) gave an example.

§2. Motivation and Main result

Completeness is important when we study noncpt. Riem. mfd.

So, we checked the existence of YF having the following properties.

Assume that $\exists\, T \in (0, \infty]$ s.t. g_t is complete for $\forall\, t \in [0, T)$.

Then, the situation is devided into ....

Exist?

① $T < \infty$ ($\rightsquigarrow g_T$ is incomp.)

- (i) flow blows up at some $T' (\geq T)$.
- (ii) flow exists for long time.

② $T = \infty$ ($\rightsquigarrow g_t$ is comp. for $\forall t \in [0, \infty)$)

- (i) $\lim_{t \to \infty} g_t$ does not exist.
- (ii) $g_\infty := \lim_{t \to \infty} g_t$ exists and it's comp.
- (iii) $g_\infty := \lim_{t \to \infty} g_t$ exists and it's incomp.

Main result

$M := \mathbb{R}^n \setminus \{0\}$ $(n \geq 3)$, fix $\frac{n+2}{2} < \lambda < n+2$

$u_0(x) := \left(1 + |x|^{-m\lambda}\right)^{\frac{1}{m}}$ where $m := \frac{n-2}{n+2} \in (0,1)$.

$g_0 := u_0^{\frac{4}{n+2}} g_{\mathbb{R}^n}$ on M.

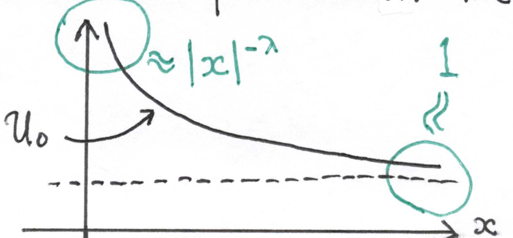
each g_t is complete for $^\forall t \in [0,\infty)$

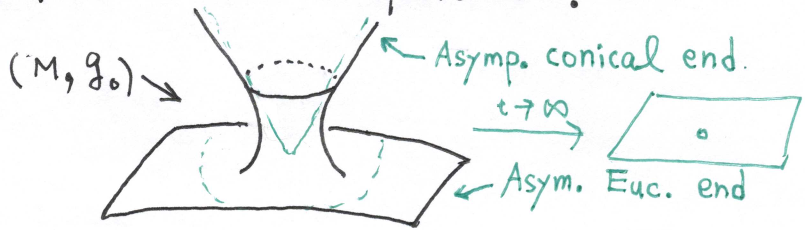
Thm (Takahashi – Y. 2021)

There exists a long time solution $\{g_t\}_{t\in[0,\infty)}$ of Yamabe flow starting from g_0 on M. And the sol. is unique.

Moreover, $g_t \xrightarrow{t\to\infty} g_{\mathbb{R}^n}$, so the limit is incomplete.

We called this phenomenon the "infinite-time incompleteness".





§3. The proof

We have the following correspondence (if g_0 is conformal to a scalar flat metric $\bar{g}$).

The solution of $\qquad\qquad\qquad\qquad$ The solution of

$$\begin{cases} \dfrac{\partial}{\partial t} g = -\mathrm{scal}(g)\cdot g \\ g(\cdot, 0) = g_0 \end{cases} \quad \underset{g := u^{\frac{4}{n+2}} \bar{g}}{\longleftrightarrow} \quad \begin{cases} \dfrac{\partial}{\partial t} u = \Delta(u^m) \\ u(\cdot, 0) = u_0 \end{cases} \qquad m := \frac{n-2}{n+2}$$

"fast diffusion equation"

So, it's enough to solve the right hand side. $\rightsquigarrow$ Purely PDE!

Rem Proving the existence and uniqueness is not sufficient.
We should say that $g_t := u_t^{\frac{4}{n+2}} g_{\mathbb{R}^n}$ is complete.
A sufficient condition is $u_t \approx \begin{cases} |x|^{-\lambda} & (|x| \to 0) \\ 1 & (|x| \to \infty). \end{cases}$

The strategy

① Prove the existence in two ways.

② Prove the uniqueness

③ Prove the convergence to a const.

Then, we get two solutions
$\tilde{u}$ having some property A and
$\tilde{\tilde{u}}$ having some property B.

Then, we can say $\tilde{u} = \tilde{\tilde{u}}$.
So, $u := \tilde{u} \, (= \tilde{\tilde{u}})$ has
properties A and B.

①-(i) Existence by approximation method.

Deform the initial $u_0(x) = (1 + |x|^{-m\lambda})^{1/m}$ by

This is unbounded around the origin.

$$u_\varepsilon(x) := \left((1+\varepsilon) + (|x|^2 + \varepsilon)^{-\frac{m\lambda}{2}} \right)^{1/m}.$$

This is bounded.

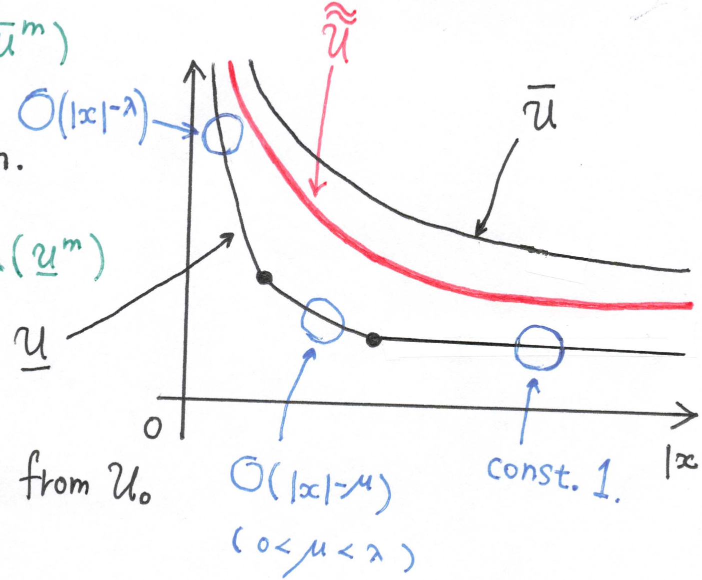
Then, there exists a sol. $u_\varepsilon(x,t)$ of the fast diff. eq. starting from u_ε.

Letting $\varepsilon \to 0$, we get a sol. $\tilde{u}(x,t) := \lim_{\varepsilon \to 0} u_\varepsilon(x,t)$.

Since $u_\varepsilon(x,t)$ is radialy sym. and decreasing in $|x|$ and t, $\tilde{u}$ is also so.

①-(ii) Existence by super-sub method.

- super sol. is $\overline{u}(x,t) := (1+|x|^{-m\lambda})^{\frac{1}{m}}$. This is indep. of t.
  - $\overline{u} \geq u_0$ at $t=0$ and $\frac{\partial}{\partial t}\overline{u} \leq \Delta(\overline{u}^m)$
- sub sol. is as the following graph.
  - $\underline{u} \leq u_0$ at $t=0$ and $\frac{\partial}{\partial t}\underline{u} \geq \Delta(\underline{u}^m)$



⇝ Then, by a general theory,

$\exists$ sol. $\tilde{\tilde{u}}$ of the fast diff. eq. starting from u_0

s.t. $\underline{u} \leq \tilde{\tilde{u}} \leq \overline{u}$.

By this property, we can easily see that

$$\tilde{\tilde{u}}_t(x) \approx \begin{cases} |x|^{-\lambda} & (|x|\to 0) \\ 1 & (|x|\to\infty) \end{cases}.$$

This is a sufficient condition so that g_t is complete.

② Uniqueness

$m = \frac{n-2}{n+2} \in (0,1)$

Let u_1 and u_2 be solutions of $\frac{\partial}{\partial t} u = \Delta (u|u|^{m-1})$ on $\mathbb{R}^n \setminus \{0\}$ with $u_1(\cdot, 0) = u_2(\cdot, 0)$.

(In this talk, I assume $u_1 \geq 0$ and $u_2 \geq 0$ for notational simplicity. But, the following is OK by replacing u^m with $u|u|^{m-1}$.)

Put $w(x,t) := \int_0^t |u_1^m(x,\tau) - u_2^m(x,\tau)|\, d\tau$

Goal $w(\cdot, t) \equiv 0$ for $\forall t \in [0, \infty)$. ($\rightsquigarrow$ Then, $u_1 \equiv u_2$.)

How to prove it? Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ be a nonnegative test function. Then, we can prove that

Very hard computation! (*)

$$\int_{\mathbb{R}^n} w(x,t) \cdot \Delta\varphi(x)\, dx \geq \int_{\mathbb{R}^n} \varphi(x) \cdot |u_1(x,t) - u_2(x,t)|\, dx \geq 0.$$

$\rightsquigarrow$ This means that $\Delta w \geq 0$ in "weak sense". $\rightsquigarrow$ w is subharmonic.

Then, by the mean value ineq. for subharmonic functions, we have

$$(0 \leq)\ w(x,t) \leq \frac{1}{\omega_n R^n} \int_{B(x,R)} w(\xi,t)\, d\xi .$$

Moreover, we can also prove that

$$\left| \int_{B(x,R)} w(\xi,t)\, d\xi \right| \leq C(t) \cdot R^{n - \frac{2m}{1-m}} .$$

Thus,

$$|w(x,t)| \leq C(t) \cdot \underbrace{R^{-\frac{2m}{1-m}}}_{\text{negative power!}} \qquad \left(m = \frac{n-2}{n+2} \in (0,1) \right).$$

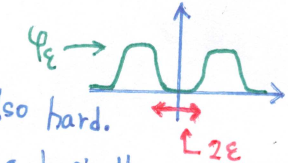
Letting $R \to \infty$ implies $w(\cdot,t) \equiv 0$ for $\forall t \in [0,\infty)$. ▨

So, $\tilde{u}$ in ①-(i) and $\tilde{\tilde{u}}$ in ①-(ii) are same.

Put $u := \tilde{u}\ (= \tilde{\tilde{u}})$. Then, u has prop. of $\tilde{u}$ and $\tilde{\tilde{u}}$.

For (*) ....

First, we prove the ineq. for φ_ε s.t. $\varphi_\varepsilon \equiv 0$ on $B(0,\varepsilon)$.



This is also hard.

The proof is basically integration by part.

Next, let $\varepsilon \to 0$.

③ Convergence to a const.

(More precisely, we need some delicate argument with parabolic regulatity.)

Step 1 Since u (actually $\tilde{u}$) is monotone in t, we have

$$u(\cdot, t) \longrightarrow \exists\, u_\infty \in C^\infty(\mathbb{R}^n \setminus \{0\}) \text{ as } t \to \infty.$$

Since u is a sol. of $\frac{\partial}{\partial t} u_t = \Delta(u_t^m)$, letting $t \to \infty$, we see

$$\Delta(u_\infty^m) \equiv 0 \rightsquigarrow u_\infty^m \text{ is a harmonic func. on } \mathbb{R}^n \setminus \{0\}. \quad (a)$$

Moreover, recall that u (actually $\tilde{\tilde{u}}$) satisfies $1 \le u \le (1+|x|^{-m\lambda})^{\frac{1}{m}}$.

So, $1 \le u_\infty^m \le 1 + |x|^{-m\lambda}$ (b) $\rightsquigarrow u_\infty^m = o(|x|^{-(n-2)})$ as $|x| \to 0$. (b)'

small order

We used $\lambda < n+2$ and $m = \frac{n-2}{n+2}$.

Step 2 Use the "removable sing. thm" for harmonic func.

By (a) and (b)', we can apply it for u_∞^m. Then, ...

Then, we can say that

"u_∞^m is actually a entire harmonic function on $\mathbb{R}^n$."

including the origin!

Then, by ⓑ, we also know that

"u_∞^m is bounded on $\mathbb{R}^n$."

(This implies that u_∞^m is bounded around the origin.)

Then, by the usual Liouville's theorem, we can say that

$u_\infty^m \equiv C$ and, by ⓑ again, this C is actually 1.

Thus, we proved that

$$u_\infty \equiv 1. \quad \blacksquare$$