# Krust の定理の拡張と極小曲面の変形について
# Extension of Krust theorem and deformations of minimal surfaces

**Shintaro Akamine (Nihon Univ., Japan)**

March 27th, 2023

**部分多様体幾何とリー群作用 2022**
**於　東京理科大学**

This talk is based on the joint work with **Hiroki Fujino** (Nagoya Univ.)

**Aim**: We discuss **geometry of zero mean curvature surfaces** via **harmonic function theory**.

**Concept**: We focus on the benefits and interesting aspects of considering minimal and maximal surfaces at the same time.

**Contents**:

(I) Krust-type theorems for zero mean curvature surfaces (main topic):

– *"Graphness" of surfaces and "univalence" of harmonic functions* –

■ with H. Fujino, *Extension of Krust theorem and deformations of minimal surfaces*, Ann. Mat. Pura Appl. **201** 2583–2601 (2022), DOI: 10.1007/s10231-022-01211-z.

(II) Duality of boundary value problems of minimal and maximal surfaces:

– *"Boundary behaviors" of surfaces and harmonic functions* –

■ with H. Fujino, *Duality of boundary value problems for minimal and maximal surfaces*, to appear in Comm. Anal. Geom., arXiv:1909.00975.

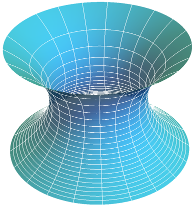
# § Minimal surfaces and maximal surfaces

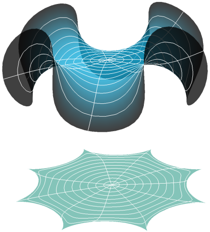
# Minimal surfaces in $\mathbb{E}^3$

- $\mathbb{E}^3 = (\mathbb{R}^3, \langle\ ,\ \rangle_E = dx^2 + dy^2 + dt^2)$: 3-dim. Euclidean space
- A surface in $\mathbb{E}^3$: immersion $X\colon D \subset \mathbb{R}^2 \to \mathbb{E}^3$,
- $D \subset \mathbb{R}^2$: **simply connected**
- A surface $X$ is called **minimal** if its mean curvature $H$ vanishes identically.
- **Weierstrass representation formula**: Any minimal surface $X$ can be represented by

$$X = \mathrm{Re}\int^w \begin{pmatrix} 1 - G^2 \\ -i(1 + G^2) \\ 2G \end{pmatrix} F dw,$$

where $F$ is a holomorphic function and $G$ is a meromorphic function on $D$ such that $FG^2$ is holomorphic. $(F, G)$ is called the **Weierstrass data** (or W-data, for short) of $X$.





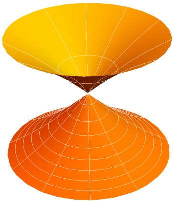
▷ $\mathbb{L}^3 = (\mathbb{R}^3, \langle\ ,\ \rangle_L = dx^2 + dy^2 - dt^2)$: 3-dim. Lorentz-Minkowski space

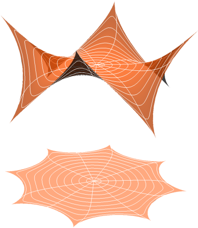
▷ A surface $X\colon D \subset \mathbb{R}^2 \to \mathbb{L}^3$ is called **maximal** if it is spacelike (i.e. $X^*\langle\ ,\ \rangle_L$ is positive definite) and the mean curvature $H$ vanishes identically.

▷ **Weierstrass-type representation formula by O. Kobayashi 1983**:
Any maximal surface $X$ can be represented by

$$X = \mathrm{Re}\int^w \begin{pmatrix} 1+G^2 \\ -i(1-G^2) \\ 2G \end{pmatrix} F dw,$$

where $F$ is a holomorphic function and $G$ is a meromorphic function on $D$ such that $FG^2$ is also holomorphic. $(F, G)$ is called the **Weierstrass data** (W-data, for short) of $X$.





**(1) Associated family/Bonnet transformation**: The transformation of a minimal surface $X = X_0$ with W-data $(F, G)$ to $X_\theta$ with W-data $(e^{i\theta}F, G)$ is called the **Bonnet transformation** of $X$, and $\{X_\theta\}_{\theta \in \mathbb{R}/2\pi\mathbb{Z}}$ is called the **associated family** of $X$.

▷ $X^* := X_{\pi/2}$ is called the **conjugate surface** of $X_0$ because

$$X_0 = \mathrm{Re} \int^w \begin{pmatrix} 1 - G^2 \\ -i(1 + G^2) \\ 2G \end{pmatrix} F dw, \quad X_{\pi/2} = -\mathrm{Im} \int^w \begin{pmatrix} 1 - G^2 \\ -i(1 + G^2) \\ 2G \end{pmatrix} F dw.$$

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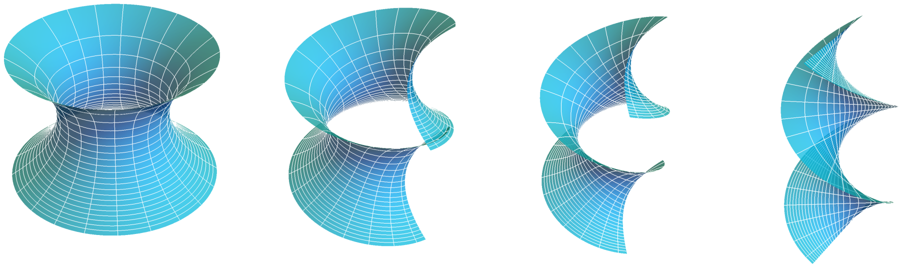

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▷ **Properties of** $\{X_\theta\}_{\theta\in\mathbb{R}/2\pi\mathbb{Z}}$:

- $\{X_\theta\}_{\theta\in\mathbb{R}/2\pi\mathbb{Z}}$ is an isometric deformation i.e. it preserves the first fundamental form $\mathrm{I} = |F|^2\left(1+|G|^2\right)^2 dwd\overline{w}$.
- curvature line of $X_0 \longleftrightarrow$ asymptotic line of $X_\pi/2$
  asymptotic line of $X_0 \longleftrightarrow$ curvature line of $X_\pi/2$
- (H.A. Schwarz 1890): If two simply connected minimal surfaces are isometric, then one of them is congruent in $\mathbb{E}^3$ to an associated surface of the other.

**(2) López-Ros deformation/Goursat transformation**: The deformation of a minimal surface $X = X_0$ with W-data $(F, G)$ to $X_\lambda$ with W-data $(\frac{1}{\lambda}F, \lambda G)$, $\lambda > 0$ is called the **López-Ros deformation** of $X$:

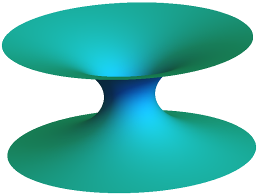
$$X_\lambda = \mathrm{Re}\int^w \begin{pmatrix} 1 - \lambda^2 G^2 \\ -i(1 + \lambda^2 G^2) \\ 2\lambda G \end{pmatrix} \frac{F}{\lambda} dw.$$

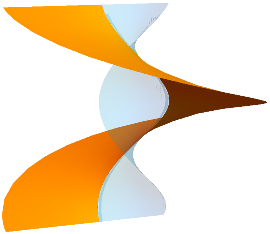
▷ **Properties of** $\{X_\lambda\}_{\lambda>0}$:

- $\{X_\lambda\}_{\lambda>0}$ preserves the second fundamental form $\mathrm{II} = -\mathrm{Re}(FG')dw^2$.
- $\{X_\lambda\}_{\lambda>0}$ preserves the "height function" (the third coordinate of $X_\lambda$).

**(3) Classical duality correspondence**: The transformation of a minimal surface $X$ with W-data $(F, G)$ to the following maximal surface $X^d$ is called the **classical duality correspondence** (**Calabi correspondence**, or **fluid mechanical duality**) of $X$:

$$X = \mathrm{Re}\int^w \begin{pmatrix} 1 - G^2 \\ -i(1 + G^2) \\ 2G \end{pmatrix} F dw \quad \longleftrightarrow \quad X^d = \mathrm{Re}\int^w \begin{pmatrix} 1 - G^2 \\ -i(1 + G^2) \\ 2iG \end{pmatrix} F dw.$$





By using the duality, Calabi 1970 proved the Bernstein problem for maximal surfaces in $\mathbb{L}^3$, that is, any entire maximal graphs are spacelike planes.

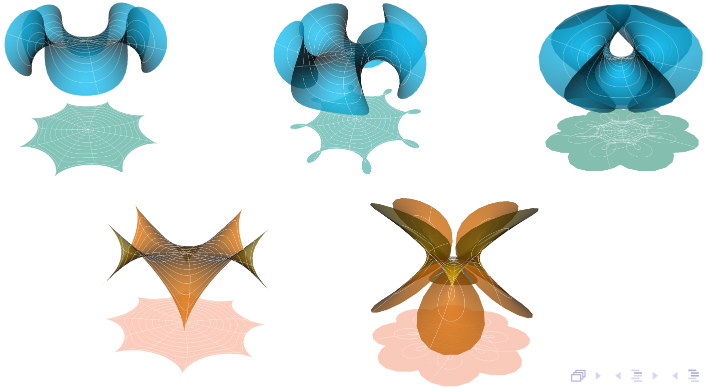
# Topic I: Graphness of zero mean curvature surfaces

Graphness = the property that the surface is whether the graph of a function (defined on the $xy$-plane) or not.

## Fact 1 (Bernstein 1915, Calabi 1970)

- *Any entire minimal graphs in $\mathbb{E}^3$ are planes.*
- *Any entire maximal graphs in $\mathbb{L}^3$ are spacelike planes.*

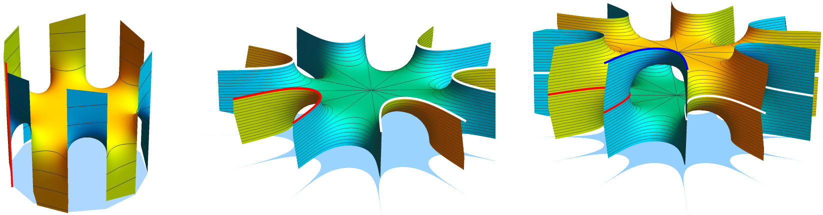
Therefore, in general, non-planar minimal or maximal surface cannot be globally represented by the graph of a function of the form $t = f(x, y)$, $(x, y) \in \mathbb{R}^2$.
How large can a surface be represented as a graph of a function?



▷ To construct (complete) embedded minimal surfaces, the following **Krust theorem** played an important role:

**Fact 2 (Karcher, 1989 (known as the Krust Theorem))**

*If a minimal surface is a graph over a convex domain, then each surface of its associated family is also a graph.*



Minimal graph by Jenkins-Serrin 1966 and Saddle tower by Karcher 1988.

**Remark 1**

- *In this talk, "graph" means the graph $t = \varphi(x, y)$ of a function $\varphi = \varphi(x, y)$ defined on a domain in the xy-plane.* *Graph $\Rightarrow$ embedded.*

**Fact (Krust Theorem)**

*If a minimal surface is a graph over a convex domain, then each surface of its associated family is also a graph.*

$\boxed{\textbf{Question}}$ What is this assumption? Do we need it?

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cf. **Rado (1930-1932)**: *If an (embedded rectifiable) closed curve $\Gamma$ in $\mathbb{E}^3$ can be projected onto a convex curve in a plane, then $\Gamma$ bounds a unique embedded minimal disk which is a graph over the plane.*

**FACT (KRUST THEOREM)**

*If a minimal surface is a graph over a convex domain, then each surface of its associated family is also a graph.*

$\boxed{\textbf{Question}}$ What is this assumption? Do we need it?

▷ The convexity assumption is only a sufficient condition for the surface to be a graph.
$\because$ If we consider the Enneper-type surface $X$, its associated surface $X_\theta$ is nothing but the rotation of $X$ in $\mathbb{E}^3$.
Why is the graphness of the above Enneper-type example preserved?

**Fact (Krust Theorem)**

*If a minimal surface is a graph over a convex domain, then each surface of its associated family is also a graph.*

**Question** Can we consider other deformations preserving the “graphness”?

# § Weierstrass-type representation in different ambient spaces

Based on Weierstrass-type representations

$$\text{in } \mathbb{E}^3 : \mathrm{Re} \int^w \begin{pmatrix} 1 - G^2 \\ -i(1 + G^2) \\ 2G \end{pmatrix} F dw, \qquad \text{in } \mathbb{L}^3 : \mathrm{Re} \int^w \begin{pmatrix} 1 + G^2 \\ -i(1 - G^2) \\ 2G \end{pmatrix} F dw,$$

we first introduce the following $c$-**deformation**

$$X_c = \mathrm{Re} \int^w \begin{pmatrix} 1 - cG^2 \\ -i(1 + cG^2) \\ 2G \end{pmatrix} F dw, \quad c \in \mathbb{R}$$

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- When $c = 1$, $X_1$ is a minimal surface in $\mathbb{E}^3$.
- When $c = -1$, $X_{-1}$ is a maximal surface in $\mathbb{L}^3$.
- When $c = 0$, $X_0 = \operatorname{Re}\int^w {}^t(1, -i, 2G)\, F dw$ is a zero mean curvature surface in the **isotropic 3-space** $\mathbb{I}^3 = (\mathbb{R}^3, dx^2 + dy^2)$.

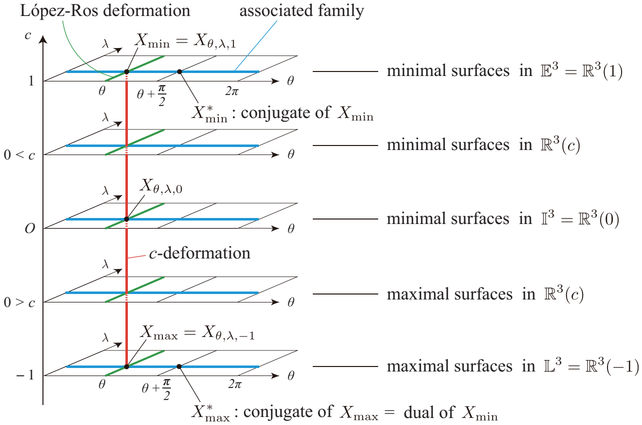
### Proposition 3

*If $FG^2$ is holomorphic and $c|G|^2 \not\equiv -1$, then $X_c$ is a zero mean curvature surface with positive definite induced metric (except singularities) in $\mathbb{R}^3(c) := (\mathbb{R}^3, dx^2 + dy^2 + c\,dt^2)$.*

# Deformation family

We give a unified form of the above deformations as follows:

$$X_{\theta,\lambda,c} = \mathrm{Re}\int^{w} \begin{pmatrix} 1 - c\lambda^2 G^2 \\ -i(1 + c\lambda^2 G^2) \\ 2\lambda G \end{pmatrix} \frac{e^{i\theta}}{\lambda} F dw.$$



# § Results

Let us recall the questions of this talk.

**Fact (Krust Theorem)**

*If a minimal surface is a graph over a convex domain, then each surface of its associated family is also a graph.*

**Question** Do we need the convexity assumption?

**Question** Can we consider other deformations preserving the graphness?

**Theorem 4 (Fujino-A., 2021)**

*For the Weierstrass data $(F, G)$ of $X_{\theta,\lambda,c}$, suppose that $G$ is not constant and $|G| < 1$. If there exists $(\theta_0, \lambda_0, c_0)$ such that $|c_0\lambda_0^2| \le 1/\|G\|_\infty^2$ and $S_{\theta_0,\lambda_0,c_0} := X_{\theta_0,\lambda_0,c_0}(D)$ is a graph over a convex domain, then $S_{\theta,\lambda,c} := X_{\theta,\lambda,c}(D)$ is a graph over a close-to-convex domain whenever $|c\lambda^2| \le |c_0\lambda_0^2|$.*

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▷ Here, a domain $\Omega \subset \mathbb{C}$ is called **close-to-convex** if $\mathbb{C} \setminus \Omega$ is a union of half lines that are disjoint except possibly for initial points.

$$\Omega \subset \mathbb{C} \text{ is } \ldots \quad \text{convex} \quad \Rightarrow \quad \text{starlike} \quad \Rightarrow \quad \text{close-to-convex}$$

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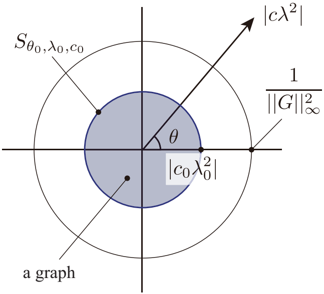


Figure: The highlighted part indicates the region on which the ZMC surfaces $S_{\theta,\lambda,c}$ are graphs.

Putting $(\theta_0, \lambda_0, c_0) = (0, 1, \pm 1)$, we obtain the following.

## Corollary 5

*If $|G| < 1$ and $S_{\mathrm{min}} := S_{0,1,1}$ (or $S_{\mathrm{max}} := S_{0,1,-1}$ ) is a graph over a convex domain, then*

*$S_{\theta,\lambda,c}$ is a graph over a close-to-convex domain whenever $|c\lambda^2| \leq 1$.*

This corollary includes the following known results:

- the original Krust's theorem for minimal surfaces [Karcher 1989],
- Krust-type theorem for maximal surfaces [López 2021],
- Krust-type theorem for López-Ros deform. of minimal surfaces [Dorff 2004].

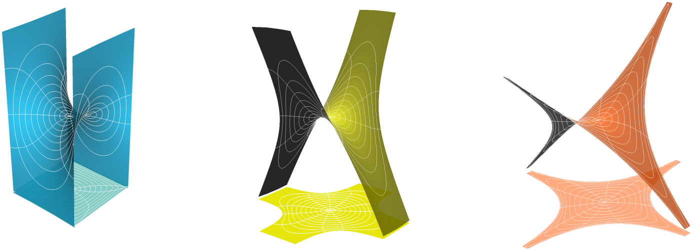

FIGURE: The Scherk's minimal graph $\boxed{S_{0,1,1}}$ and its deformations $\boxed{S_{0,1,0}}$ and $\boxed{S_{0,1,-1}}$.

$$\text{Put} \qquad h := \int^w F dw, \quad g := -\int^w G^2 F dw, \quad T := \int^w 2GF dw.$$

Then, under the identification that $xy$-plane $\cong \mathbb{C}$, $(x, y) \mapsto x + iy$,

$$X_{\theta,\lambda,c} = \mathrm{Re} \int^w \begin{pmatrix} 1 - c\lambda^2 G^2 \\ -i(1 + c\lambda^2 G^2) \\ 2\lambda G \end{pmatrix} \frac{e^{i\theta}}{\lambda} F dw = \begin{pmatrix} \frac{e^{i\theta}}{\lambda}(h + c\lambda^2 e^{-2i\theta}\overline{g}) \\ \mathrm{Re}(e^{i\theta} T) \end{pmatrix}.$$

## Proposition 6

*Under the above notations, let us define the planar harmonic mapping*

$$f_{\theta,\lambda,c} := h + c\lambda^2 e^{-2i\theta}\overline{g}.$$

*Then, $S_{\theta,\lambda,c} := X_{\theta,\lambda,c}(D)$ is a graph if and only if $f_{\theta,\lambda,c}$ is univalent (i.e. injective).*

Any planar harmonic mapping $f \colon D \to \mathbb{C}$ has unique (up to constants) decomposition

$$f = h + \overline{g}, \quad h, g \in \mathcal{O}(D).$$

By the above proposition, the graphness of each deformation belonging to $\{X_{\theta,\lambda,c}\}$ reduced to the univalence of the planar harmonic mapping of the form

$$f_\varepsilon = h + \varepsilon\overline{g}, \quad \varepsilon \in \mathbb{C}.$$

**Theorem 7 (Fujino-A., 2021)**

*For the Weierstrass data $(F, G)$ of $X_{\theta,\lambda,c}$, suppose that $G$ is not constant and $|G| < 1$. If there exists $(\theta_0, \lambda_0, c_0)$ such that $|c_0\lambda_0^2| \leq 1/\|G\|_\infty^2$ and $S_{\theta_0,\lambda_0,c_0}$ is a graph over a convex domain, then $S_{\theta,\lambda,c}$ is a graph over a close-to-convex domain whenever $|c\lambda^2| \leq |c_0\lambda_0^2|$.*

To prove the result, we use recent developments of planar harmonic mapping theory:

**Theorem 8 (Kalaj 2010)**

*Let $f = h + \overline{g}\colon D \to \mathbb{C}$ be a univalent sense-preserving harmonic mapping. If $f$ is convex, then $f_\varepsilon := h + \varepsilon\overline{g}$ is close-to-convex and sense-preserving for all $\varepsilon$ with $|\varepsilon| \leq 1$.*

This is related to the following theorem:

**Fact 9 (Clunie and Sheil-Small, 1984)**

*Let $f = h + \overline{g}$ be locally univalent. If there exists $\varepsilon$ such that $|\varepsilon| \leq 1$ and $h + \varepsilon g$ is convex, then $f$ is univalent close-to-convex.*

$\boxed{\textbf{Question}}$ Can we obtain/recover the "graphness" of minimal (or maximal) surfaces from ZMC surfaces in the isotropic 3-space $\mathbb{I}^3$?

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**Theorem 10 (Fujino-A., 2021)**

*If $|G| < 1$ and $S_{\theta_0,\lambda_0,0}$ in $\mathbb{I}^3$ is a graph over a convex domain for some $(\theta_0, \lambda_0)$, then $S_{\theta,\lambda,c}$ is a graph over a close-to-convex domain whenever $|c\lambda^2| \leq 1/\|G\|_\infty^2$.*
*In particular, the minimal and the maximal surfaces $S_{\theta,1,\pm 1}$ are graphs.*

## THEOREM 10 (FUJINO-A., 2021)

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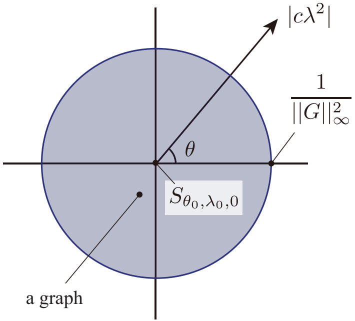


FIGURE: The region on which $S_{\theta,\lambda,c}$ is a graph.

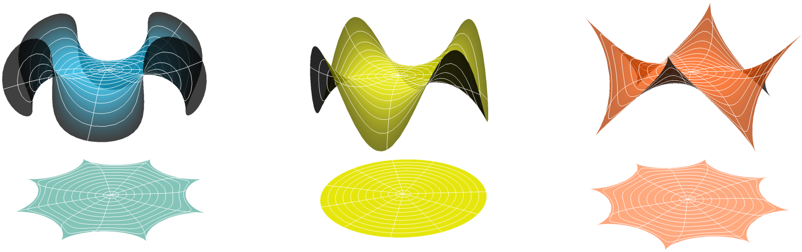

FIGURE: The Enneper-type minimal graph $\boxed{S_{0,1,1}}$ with $(F, G) = (1, w^3)$ on $\mathbb{D}$ and its deformations $\boxed{S_{0,1,0}}$ and $\boxed{S_{0,1,-1}}$.

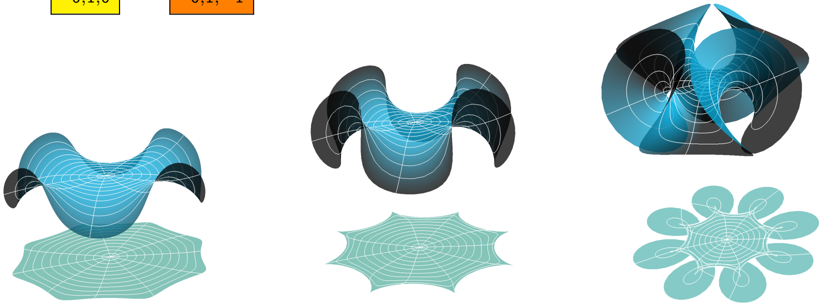

FIGURE: The López-Ros deformation $\boxed{S_{0,\lambda,1}}$: From left $\lambda = 0.6, \lambda = 1, \lambda = 2$.

**Theorem 11 (Fujino-A., 2021)**

*If* $|G| < 1$ *and* $S_{\theta_0,\lambda_0,0}$ *in* $\mathbb{I}^3$ *is a graph over a convex domain for some* $(\theta_0, \lambda_0)$, *then* $S_{\theta,\lambda,c}$ *is a graph over a close-to-convex domain whenever* $|c\lambda^2| \leq 1/\|G\|_\infty^2$.
*In particular, the minimal and the maximal surfaces* $S_{\theta,1,\pm 1}$ *are graphs.*

We use recent developments of planar harmonic mapping theory again:

**Fact 12 (Partyka-Sakan, 2022)**

*Let* $h, g \colon D \to \mathbb{C}$ *be holomorphic functions such that* $|h'| > |g'|$ *and* $\omega = g'/h'$ *is not constant. If there exists* $\varepsilon_0 \in \mathbb{C}$ *such that* $|\varepsilon_0|\|\omega\|_\infty \leq 1$ *and* $h + \varepsilon_0 g$ *is a conformal mapping and its image is convex, then* $f_\varepsilon := h + \varepsilon\overline{g}$ *is a close-to-convex univalent harmonic mapping for any* $\varepsilon \in \mathbb{C}$ *with* $|\varepsilon|\|\omega\|_\infty \leq 1$.

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**Outline of the proof**: Put $\varepsilon_0 = 0$ and consider

$$f_{\theta,\lambda,c} = h + c\lambda^2 e^{-2i\theta}\overline{g} =: h + \varepsilon g, \quad -G^2 = \omega = \frac{g'}{h'} \text{ (analytic dilatation of } f_{0,1,1}).$$

Then the condition $|\varepsilon|\|\omega\|_\infty = |c\lambda^2|\|G^2\|_\infty \leq 1$ is a sufficient condition for $f_{\theta,\lambda,c}$ to be univalent, which is equivalent to the condition that $S_{\theta,\lambda,c}$ is a graph. □

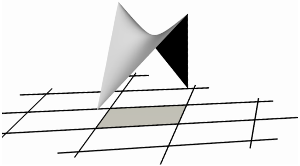
# Summary of Topic I: Graphness

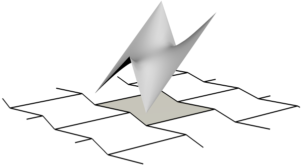
By using recent developments of univalent **harmonic function theory**, we obtain the following results of **geometry of ZMC surfaces**:

- We can extend Krust's theorem for minimal surfaces to various deformations of ZMC surfaces in different ambient spaces.
- We can recover the graphness of minimal and maximal surfaces from ZMC surfaces in the isotropic 3-space $\mathbb{I}^3 = (\mathbb{R}^3, dx^2 + dy^2)$.
- As a corollary, we can relax the convexity assumption of Krust's theorem.
  This gives a reason why the graphness is preserved under the isometric deformations even for non-convex minimal graphs like as Enpper-type minimal surfaces.

# Topic II: Duality of boundary value problems for minimal and maximal surfaces

The following boundary value problem for maximal surfaces was geometrically unknown.
How can we solve the Lightlike line boundary value problem for the maximal surfaces?





Lightlike line segment: a straight line segment $L = \mathrm{span}\{\vec{v}\}$ satisfying $\langle \vec{v}, \vec{v} \rangle_L = 0$.
By definition, the induced metric on surface degenerates along Lightlike line segment.
This has made solving this boundary value problem difficult until now.

Minimal surface equation: $(1+\varphi_y^2)\varphi_{xx} - 2\varphi_x\varphi_y\varphi_{xy} + (1+\varphi_x^2)\varphi_{yy} = 0$,
Maximal surface equation: $(1-\psi_y^2)\psi_{xx} + 2\psi_x\psi_y\psi_{xy} + (1-\psi_x^2)\psi_{yy} = 0$, $|\nabla\psi(p)|^2 < 1$.

### Fact 13 (cf. Bers 1958, Calabi 1970: **Duality**)

*Let $\Omega$ be a simply connected domain. For any solution $\varphi\colon\Omega\to\mathbb{R}$ to the minimal surface equation, there exists a unique (up to an additive constant) solution $\psi$ to the maximal surface equation satisfying the relation:*

$$\begin{pmatrix}\psi_x\\ \psi_y\end{pmatrix} = \frac{1}{\sqrt{1+\varphi_x^2+\varphi_y^2}}\begin{pmatrix}-\varphi_y\\ \varphi_x\end{pmatrix} \quad (\psi_x^2+\psi_y^2<1) \qquad \cdots(\bigstar).$$

*Conversely, for any solution $\psi\colon\Omega\to\mathbb{R}$ to the maximal surface equation, there exists unique solution $\varphi$ to the minimal surface equation satisfying $(\bigstar)$.*
*We call such $\psi$ (resp. $\varphi$) as the dual of $\varphi$ (resp. $\psi$).*

### Example 14

The dual of $\varphi(x,y) = \log(\cos x/\cos y)$ (Scherk surface) is $\psi(x,y) = -\arcsin(\sin x\sin y)$.

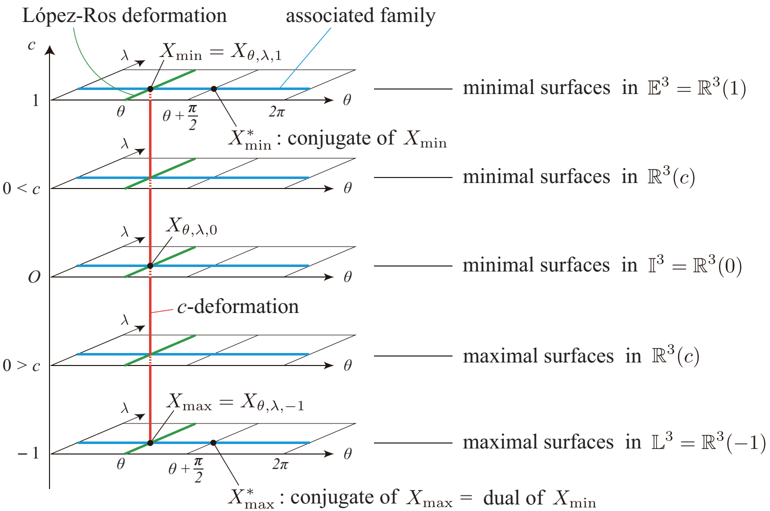
H. Lee (2011) (cf. Fujino-A., arXiv:1909.00975): The duality $(\bigstar)$ corresponds to

$$X_{\mathrm{min}} = (x,y,t) \longmapsto X_{\mathrm{max}} = (x,y,t^*),$$

where $X_{\mathrm{min}} = (x,y,t)$ is a minimal surface, whose coordinates functions are harmonic, and $t^*$ is the conjugate of $t$.

# Where is dual in the deformation family?

$$X_{\theta,\lambda,c} = \mathrm{Re}\int^{w} \begin{pmatrix} 1 - c\lambda^2 G^2 \\ -i(1 + c\lambda^2 G^2) \\ 2\lambda G \end{pmatrix} \frac{e^{i\theta}}{\lambda} F dw.$$

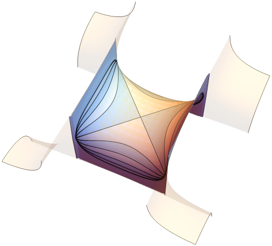


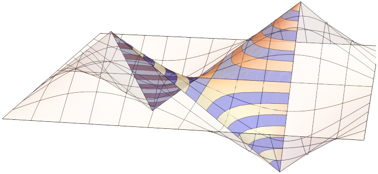
**Theorem 15 (A.-Fujino, arXiv:1909.00975)**

*Let $\Omega$ be a bounded simply connected Jordan domain whose boundary contains a line segment $I$. We let $\varphi$ be a solution of the minimal surface equation over $\Omega$, $\psi$ its dual solution of the maximal surface equation. Then the following statements (I) and (II) are equivalent.*

(I) *$\varphi$ tends to plus (resp. minus) infinity on $I$, and*

(II) *$\psi$ tamely degenerates to a future-directed (resp. past-directed) lightlike line segment on $I$.*





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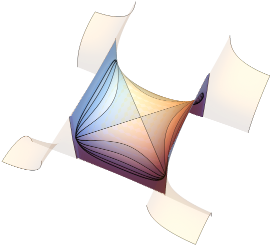

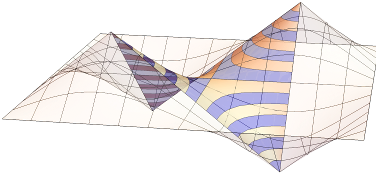

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(I) *$\varphi$ tends to plus (resp. minus) infinity on $I$, and*

(II) *$\psi$ tamely degenerates to a future-directed (resp. past-directed) lightlike line segment on $I$.*

**Remarks**:

- Tamely degenerateness defines a degeneration of $\mathrm{graph}(\psi)$ to a lightlike line segment on the boundary with an asymptotic estimate.

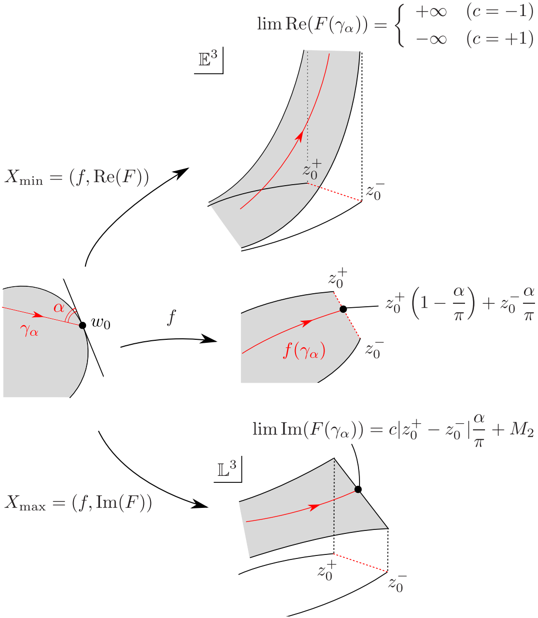
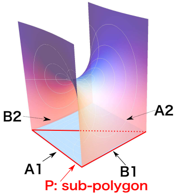
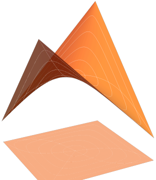


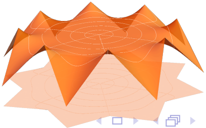

FIGURE: **Key observation in [A.-Fujino, arXiv:1909.00975]**: Boundary behaviors of a minimal surface and its dual maximal surface around a discontinuous point $w_0$ of the harmonic mapping $f = x + iy$.

## Fact 16 (Jenkins-Serrin 1966)

$\Omega \subset \mathbb{R}^2$: **polygonal domain** *whose boundary consists of* $A_i$ *and* $B_i$,
$\mathcal{P}$: *simple closed polygon whose vertices chosen from among the endpoints of* $A_i$ *and* $B_i$.
$\alpha := \sum_i |A_i|$ *for* $A_i \subset \mathcal{P}$, $\beta := \sum_i |B_i|$ *for* $B_i \subset \mathcal{P}$, $\gamma$: *perimeter of* $\mathcal{P}$.
*Assume that no two segments of* $A_i$ *and no two segments of* $B_i$ *have a common endpoint.*
*Then,* ${}^{\exists !}\varphi$: *solution of the min. surf. eq. on* $\Omega$ *satisfying* $\varphi \to \infty$ *on* $A_i$ *and* $\varphi \to -\infty$ *on* $B_i$ $\iff$ $2\alpha < \gamma$ *and* $2\beta < \gamma$ *for each polygon* $\mathcal{P} \neq \overline{\Omega}$ *and* $\alpha = \beta$ *when* $\mathcal{P} = \overline{\Omega}$.





As a direct corollary of previous theorem and Jenkins-Serrin's result, we obtain existence and uniqueness of solutions of maximal surface equation with lightlike line boundary.

**Theorem 17 (A.-Fujino, existence and uniqueness)**

*Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain whose boundary consists of line segments $A_i$ and $B_i$. Then, there exists a solution of maximal surface equation which tamely degenerates to future-directed lightlike line segments on $A_i$ and past-directed lightlike line segments on $B_i$ if and only if*

(I) *the polygon $\Gamma \subset \mathbb{L}^3$ whose edges consist of above lightlike line segments is a closed curve, and*

(II) *each line segment $L$ connecting vertices of $\Gamma$ are spacelike when the projection of $L$ to the $xy$-plane is in $\Omega$.*

*Moreover, such solution is unique (up to a constant vector)*

# Summary of Topic II: Duality of boundary value problems

- Infinite boundary value problem for the minimal surface equation $\underset{1-1}{\longleftrightarrow}$ lightlike line boundary value problem for the maximal surface equation.
- It is closely related to discontinuous boundary behavior of harmonic functions.

  ■ with H. Fujino, *Reflection principle for lightlike line segments on maximal surfaces*, Ann. Global Anal. Geom. **59** (2021), 93–108. DOI: 10.1007/s10455-020-09743-4.

  ■ with H. Fujino, *Reflection principles for zero mean curvature surfaces in the simply isotropic 3-space*, Result Math. **74(4)** (2022), DOI: 10.1007/s00025-022-01718-0.