3 次元ハイゼンベルグ群の時間的極小曲面に対する Sym の公式

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Contents

1 Sym-Bobenko formula in Euclidean 3-space

2 Application to timelike surfaces in Heisenberg group

3 Recent research

Sym formula in Euclidean 3-space

Sym-Bobenko formula $\cdots$

immersion formula of non-zero constant mean curvature (CMC in short) surfaces

A. Sym(1985) constant negative Gaussian curvature surfaces in Euclidean space $\mathbb{E}^3$

A. I. Bobenko(1994) CMC surfaces in space forms

T. Taniguchi(1997) spacelike CMC surfaces in Minkowski space $\mathbb{L}^3$

Dorfmeister-Inoguchi-Toda(2003), K-Kobayashi(2022) timelike CMC surfaces in $\mathbb{L}^3$

Sym formula in Euclidean 3-space

$\mathbb{D} \subset \mathbb{C}$; simply connected domain
$f = (f_1, f_2, f_3) : \mathbb{D} \to \mathbb{E}^3$; immersion

f is (locally) conformal on a Riemann surface. First fundamental form of f is

$$I^f = e^u dz d\bar{z}.$$

Notation

- $\partial = \frac{1}{2}\frac{\partial}{\partial x} - i\frac{1}{2}\frac{\partial}{\partial y}$; *the differentiation w.r.t.* z
- $\partial f = (\phi_1, \phi_2, \phi_3)$

$$\textbf{conformality} \Longleftrightarrow (\phi_1)^2 + (\phi_2)^2 + (\phi_3)^2 = 0$$

$$\textbf{non}-\textbf{degeneracy} \Longleftrightarrow |\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2 = e^u/2 > 0$$

Sym formula in Euclidean 3-space

Proposition

There exists a pair of functions $\psi = (\psi_1, \psi_2)$ on $\mathbb{D}$ s.t.

$$\phi_1 = i\left(\overline{\psi_2}^2 + {\psi_1}^2\right), \quad \phi_2 = \overline{\psi_2}^2 - {\psi_1}^2, \quad \phi_3 = 2\psi_1\overline{\psi_2}. \qquad (1)$$

- ψ is called the **generating spinors**.
- ψ describes the amounts of surfaces, e.g.

$$e^u = 4\left(|\psi_1|^2 + |\psi_2|^2\right)^2,$$
$$g = \pi \circ N = \psi_1/\overline{\psi_2},$$
$$Q = 2\left(\overline{\psi_2}\partial\psi_1 - \psi_1\partial\overline{\psi_2}\right).$$

Sym formula in Euclidean 3-space

Proposition

ψ *satisfies the non-linear Dirac equation*;

$$\left\{\begin{pmatrix} 0 & \partial \\ -\overline{\partial} & 0 \end{pmatrix} + \begin{pmatrix} \mathcal{U} & 0 \\ 0 & \mathcal{U} \end{pmatrix}\right\}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0 \quad \textit{where} \quad \mathcal{U} = \frac{1}{2}e^{u/2}H \quad (2)$$

- Non-linear Dirac equation (2) is equivalent to the Kenmotsu formula,

$$H\left(\partial\overline{\partial}g - \frac{\partial\overline{g}}{1+|g|^2}\overline{\partial}g\partial g\right) = \partial H\overline{\partial}g.$$

- $\psi^* = (-\overline{\psi_2}, \overline{\psi_1})$ also satisfies (2).
- $H = 0 \Longrightarrow \psi_1$; holomorphic, ψ_2; anti-holomorphic

Sym formula in Euclidean 3-space

Theorem

Let ψ *be a solution of the non-linear Dirac equation* (2):

$$\left\{ \begin{pmatrix} 0 & \partial \\ -\overline{\partial} & 0 \end{pmatrix} + \begin{pmatrix} \mathcal{U} & 0 \\ 0 & \mathcal{U} \end{pmatrix} \right\} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0.$$

Set ϕ_i *as* (1):

$$\phi_1 = i\left(\overline{\psi_2}^2 + {\psi_1}^2\right), \quad \phi_2 = \overline{\psi_2}^2 - {\psi_1}^2, \quad \phi_3 = 2\psi_1\overline{\psi_2}.$$

Then

$$f = \left(\int \mathrm{Re}(\phi_1 dz), \quad \int \mathrm{Re}(\phi_2 dz), \quad \int \mathrm{Re}(\phi_3 dz)\right)$$

is a surface of the first fundamental form $4\left(|\psi_1|^2 + |\psi_2|^2\right)^2 dz d\bar{z}$, *the mean curvature* $\mathcal{U}\left(|\psi_1|^2 + |\psi_2|^2\right)^{-1}$.

Sym formula in Euclidean 3-space

$$F := \frac{1}{\sqrt{|\psi_1|^2 + |\psi_2|^2}} \begin{pmatrix} \psi_1 & \psi_2 \\ -\overline{\psi_2} & \overline{\psi_1} \end{pmatrix} : \mathbb{D} \to \mathrm{SU}(2)$$

$$\text{(2)} \iff \partial F = FU, \quad \overline{\partial} F = FV$$

where

$$U = \begin{pmatrix} \partial u/4 & e^{u/2}H/2 \\ -Qe^{-u/2} & -\partial u/4 \end{pmatrix}, \quad V = \begin{pmatrix} -\overline{\partial} u/4 & \overline{Q}e^{-u/2} \\ -e^{u/2}H/2 & \overline{\partial} u/4 \end{pmatrix}$$

Compatibility condition $\begin{cases} \partial\overline{\partial} u/2 + \mathcal{U}^2 - |Q|^2 e^{-u} = 0 \\ \overline{\partial} Q = e^u \partial H/2 \end{cases}$

Fact

CMC $\iff Q$; *holomorphic*

Sym formula in Euclidean 3-space

$Q \longmapsto \lambda^{-1} Q$ $(\lambda \in \mathbb{S}^1)$ does not change the Gauss equation.

$$\partial F = FU, \quad \overline{\partial} F = FV$$

$$U = \begin{pmatrix} \partial u/4 & e^{u/2}H/2 \\ -Qe^{-u/2} & -\partial u/4 \end{pmatrix}, \quad V = \begin{pmatrix} -\overline{\partial} u/4 & \overline{Q}e^{-u/2} \\ -e^{u/2}H/2 & \overline{\partial} u/4 \end{pmatrix}$$

$$\downarrow$$

$$\partial F^\lambda = F^\lambda U^\lambda, \quad \overline{\partial} F^\lambda = F^\lambda V^\lambda$$

$$U^\lambda = \begin{pmatrix} \partial u/4 & \lambda^{-1} e^{u/2}H/2 \\ -\lambda^{-1} Qe^{-u/2} & -\partial u/4 \end{pmatrix}, \quad V^\lambda = \begin{pmatrix} -\overline{\partial} u/4 & \lambda \overline{Q}e^{-u/2} \\ -\lambda e^{u/2}H/2 & \overline{\partial} u/4 \end{pmatrix}$$

- $F^\lambda : \mathbb{D} \times \mathbb{S}^1 \to \mathrm{SU}(2)$ is called **extended frame** of CMC surface.

Sym formula in Euclidean 3-space

α^λ: Maurer-Cartan form of F^λ

Theorem

Following statements are equivalent;

- H *is constant*
- Q *is holomorphic*
- $d + \alpha^\lambda$ *defines flat connections on* $\mathbb{D} \times \mathrm{SU}(2)$
- $g : \mathbb{D} \to \mathbb{S}^2$ *is harmonic map*

Sym formula in Euclidean 3-space

Let $\mathfrak{su}(2)$ be the Lie algebra of $\mathrm{SU}(2)$.

$$\mathfrak{su}(2) = \left\{ \begin{pmatrix} ia & \bar{b} \\ b & -ia \end{pmatrix} \middle| a \in \mathbb{R}, b \in \mathbb{C} \right\}$$

Define an inner product on $\mathfrak{su}(2)$ as $\langle X, Y \rangle = -2\mathrm{tr}(XY)$. Then $\mathbb{E}^3$ is identified with $\mathfrak{su}(2)$ isometrically.

$$\frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \leftrightarrow (1,0,0),\ \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \leftrightarrow (0,1,0),\ \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \leftrightarrow (0,0,1)$$

Theorem (Sym-Bobenko)

Let H be a constant. Then

$$f^\lambda = -\frac{1}{2H}\left\{ i\lambda \left(\frac{\partial}{\partial \lambda} F^\lambda \right) (F^\lambda)^{-1} + \frac{i}{2} F^\lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} (F^\lambda)^{-1} \right\}$$

describe an isometric family of CMC H surfaces.

Sym formula in Euclidean 3-space

CMC surfaces		Gauss maps
in Euclidean space	$\Longleftrightarrow$	harmonic into sphere $\mathbb{S}^2$
spacelike in Minkowski space	$\Longleftrightarrow$	harmonic into hyperbolic plane $\mathbb{H}^2$
timelike in Minkowski space	$\Longleftrightarrow$	harmonic into de-Sitter sphere $\mathbb{S}^2_1$

Remark

Sym formula of spacelike/timelike surfaces in Minkowski space can be constructed as in Euclidean case.

Application to timelike surfaces in Heisenberg group

Thurston's conjecture proposed a complete characterization of geometric structures on three-dimensional manifolds.

Fact (Thurston, Perelman)

Every three-dimensional manifold is locally isometric to just one of a family of eight distinct types.

Model spaces for Thurston's geometry are

$$\mathbb{S}^3,\ \mathbb{E}^3,\ \mathbb{H}^3,$$
$$\mathbb{S}^2 \times \mathbb{R},\ \mathbb{H}^2 \times \mathbb{R},$$
$$\mathrm{Nil}_3,\ \widetilde{\mathrm{SL}_2\mathbb{R}},\ \mathrm{Sol}_3.$$

Application to timelike surfaces in Heisenberg group

Heisenberg group Nil_3;

$$\mathrm{Nil}_3 = \left\{ \begin{pmatrix} 1 & x_1 & x_3 + \frac{1}{2}x_1x_2 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix} \middle| \quad x_i \in \mathbb{R} \right\} \cong \left(\mathbb{R}^3(x_1, x_2, x_3), \cdot \ \right),$$

$$(x_1, x_2, x_3) \cdot (y_1, y_2, y_3) = \left(x_1 + y_1, x_2 + y_2, x_3 + y_3 + \frac{1}{2}(x_1y_2 - x_2y_1) \right)$$

Left invariant **Riemannian** metric g;

$$g = dx_1{}^2 + dx_2{}^2 + \left\{ dx_3 + \frac{1}{2}(x_2dx_1 - x_1dx_2) \right\}^2$$

Left invariant non-flat **Lorentzian** metric g;

$$g = \mp dx_1{}^2 + dx_2{}^2 \pm \left\{ dx_3 + \frac{1}{2}(x_2dx_1 - x_1dx_2) \right\}^2$$

Application to timelike surfaces in Heisenberg group

Isometry group $\mathrm{Iso}(\mathrm{Nil}_3)$;

$$\mathrm{Iso}_\circ(\mathrm{Nil}_3) = \mathrm{Nil}_3 \rtimes \mathrm{U}_1$$

Lie algebra $\mathfrak{nil}_3$;

$$\mathfrak{nil}_3 = \left(\mathbb{R}^3, [\ ,\ ]\right), \quad [e_1, e_2] = e_3, \quad [e_2, e_3] = [e_3, e_1] = 0$$

(e_1, e_2, e_3); standard orthonormal basis of $\mathbb{R}^3$

Notation

- E_1, E_2, E_3; *ONB of left-invariant vector fields*
- $e_1 \leftrightarrow E_1 = \frac{\partial}{\partial x_1} - \frac{1}{2}\frac{\partial}{\partial x_3}, \quad e_2 \leftrightarrow E_2 = \frac{\partial}{\partial x_2} + \frac{1}{2}\frac{\partial}{\partial x_3},$
  $e_3 \leftrightarrow E_3 = \frac{\partial}{\partial x_3}$

Application to timelike surfaces in Heisenberg group

(Nil_3, g) with $g = -{dx_1}^2 + {dx_2}^2 + \left\{dx_3 + \frac{1}{2}(x_2 dx_1 - x_1 dx_2)\right\}^2$

Definition

$f : M^2 \to \mathrm{Nil}_3$; *immersion from orientable connected 2-manifold*
f; **timelike surface** $\overset{def}{\Longleftrightarrow}$ f^*g; *Lorentzian metric*

Para-complex number $\mathbb{C}'$;

$$\mathbb{C}' = \mathbb{R} \otimes \mathbb{R} = \langle 1, i' \rangle \quad \text{with} \quad (i')^2 = 1, \quad 1 \cdot i' = i' \cdot 1 = i'$$

Remark

- *$\mathbb{C}'$ is NOT a field.*
- *$z\bar{z}$ does not always become positive for $z \in \mathbb{C}'$.*

Application to timelike surfaces in Heisenberg group

$\mathbb{D} \subset \mathbb{C}'$; simply connected domain
$f = (f_1, f_2, f_3) : \mathbb{D} \to \mathrm{Nil}_3$; timelike surface

f is (locally) conformal immersion from Lorentz surface into Nil_3.
First fundamental form of f is

$$e^u dz d\bar{z}.$$

Notation

- $\partial = \frac{1}{2}\frac{\partial}{\partial x} + i'\frac{1}{2}\frac{\partial}{\partial y}$; *differentiation w.r.t. conformal coordinate* z
- $\partial f = \phi^1 E_1 + \phi^2 E_2 + \phi^3 E_3$

$$\textbf{conformality} \iff -(\phi^1)^2 + (\phi^2)^2 + (\phi^3)^2 = 0$$

$$\textbf{non}-\textbf{degeneracy} \iff -\phi^1\overline{\phi^1} + \phi^2\overline{\phi^2} + \phi^3\overline{\phi^3} = e^u/2$$

Application to timelike surfaces in Heisenberg group

Proposition

There exist a pair of $\mathbb{C}'$*-valued functions* $\psi = (\psi_1, \psi_2)$ *on* $\mathbb{D}$ *and* $\epsilon \in \{\pm i'\}$ *s.t.*

$$\phi^1 = \epsilon\left(\overline{\psi_2}^2 + \psi_1{}^2\right), \quad \phi^2 = \epsilon i'\left(\overline{\psi_2}^2 - \psi_1{}^2\right), \quad \phi^3 = 2\psi_1\overline{\psi_2}$$

Proposition

ψ *satisfies the non-linear Dirac equation*:

$$\left\{\begin{pmatrix} 0 & \partial \\ -\overline{\partial} & 0 \end{pmatrix} + \begin{pmatrix} \mathcal{U} & 0 \\ 0 & \mathcal{U} \end{pmatrix}\right\}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0$$

$$\mathcal{U} = \tilde{\epsilon} e^{w/2} = -\frac{1}{2} e^{u/2} H + \frac{i'}{4} h, \quad \tilde{\epsilon} \in \{\pm 1, \pm i'\}$$

$h = -e^{u/2} g(N, E_3)$; support function of f

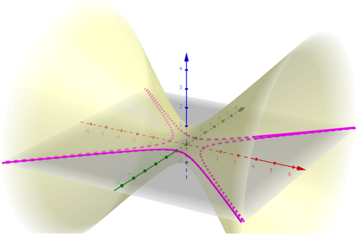
Application to timelike surfaces in Heisenberg group

Definition

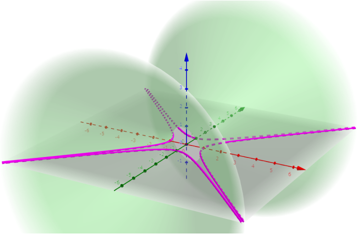
timelike surface is called **non-vertical** *if* $h \neq 0$ *anywhere.*

Fact (c.f. Daniel, Inoguchi)

The normal Gauss maps of (*spacelike/timelike*) *non-vertical surfaces with* $H = 0$ *in the* 3*-dimensional Heisenberg group define harmonic maps into the hyperbolic plane* ($\mathbb{S}^2$*/de-Sitter sphere*)*.*



$\{-x_1^2 + x_2^2 + x_3^2 = 1\} \subset \mathfrak{nil}_3$



$\{x_1^2 - x_2^2 + x_3^2 = 1\} \subset \mathbb{L}^3_{(+,-,+)}$

Application to timelike surfaces in Heisenberg group

For simplicity, $\epsilon = i', h > 0$. The extended frame is obtained from

$$\partial F^{\mu} = F^{\mu} U^{\mu}, \quad \overline{\partial} F^{\mu} = F^{\mu} V^{\mu} \quad \mu \in \{e^{i't} | t \in \mathbb{R}\}$$

$$U^{\mu} = \begin{pmatrix} \partial w/4 + \tilde{\epsilon} e^{-w/2+u/2} \partial H/2 & -\mu^{-1} \tilde{\epsilon} e^{w/2} \\ \mu^{-1} Q \tilde{\epsilon} e^{-w/2} & -\partial w/4 \end{pmatrix}$$
$$V^{\mu} = \begin{pmatrix} -\overline{\partial} w/4 & -\mu \overline{Q} \tilde{\epsilon} e^{-w/2} \\ \mu \tilde{\epsilon} e^{w/2} & \overline{\partial} w/4 + \tilde{\epsilon} e^{-w/2+u/2} \overline{\partial} H/2 \end{pmatrix}$$

Remark

- $Q dz^2$; **Abresch-Rosenberg differential**,
$$Q = \frac{1}{4}(2H - i') g(\nabla_{\partial} \partial f, N) - \frac{{\phi^3}^2}{4}$$
- H; *constant* $\Longrightarrow$ Q; *para-holomorphic*

Application to timelike surfaces in Heisenberg group

α^μ; Maurer-Cartan form of F^μ

Theorem (K-Kobayashi, 2022)

The following statements are equivalent;

- $H = 0$
- $d + \alpha^\mu$ *defines flat connections on* $\mathbb{D} \times \mathrm{SU}'_{1,1}$
- $\pi_{\mathbb{L}^3} \circ {\pi_{\mathfrak{nil}}}^{-1} \circ N$ *is a harmonic map into* $\mathbb{S}^2_1 = \mathrm{SU}'_{1,1}/\mathrm{U}'_1$

Application to timelike surfaces in Heisenberg group

Notation

$$\mathrm{SU}'_{1,1} = \left\{ \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \middle| \alpha, \beta \in \mathbb{C}', \alpha\bar{\alpha} - \beta\bar{\beta} = 1 \right\}$$

$$\mathfrak{su}'_{1,1} = \left\{ \begin{pmatrix} i'r & -p - i'q \\ -p + i'q & -i'r \end{pmatrix} \middle| p, q, r \in \mathbb{R} \right\}$$

The Lie alg. $\mathfrak{su}'_{1,1}$ can be identified with $\mathfrak{nil}_3$ as vector spaces.

$$\frac{1}{2}\begin{pmatrix} 0 & -i' \\ i' & 0 \end{pmatrix} \leftrightarrow e_1,\ \frac{1}{2}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \leftrightarrow e_2,\ \frac{1}{2}\begin{pmatrix} i' & 0 \\ 0 & -i' \end{pmatrix} \leftrightarrow e_3$$

Notation

For $X \in \mathfrak{su}'_{1,1}$

- $(X)^o$; *off-diagonal part of* X
- $(X)^d$; *diagonal part of* X

Application to timelike surfaces in Heisenberg group

Theorem (K-Kobayashi, 2022)

$$f^{\mu} = \exp\left\{ (f_{\mathbb{L}^3})^o - \frac{i'}{2}\mu \left(\frac{\partial}{\partial \mu} f_{\mathbb{L}^3} \right)^d \right\}$$

describe timelike minimal surfaces in Nil_3 *where the map* $f_{\mathbb{L}^3} : \mathbb{D} \to \mathfrak{su}'_{1,1}$ *is given by*

$$f_{\mathbb{L}^3} = -\left\{ i'\mu \left(\frac{\partial}{\partial \mu} F^{\mu} \right) (F^{\mu})^{-1} + \frac{i'}{2} F^{\mu} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} (F^{\mu})^{-1} \right\}.$$

Remark

The normalized Gauss map N^{μ} *of* f^{μ} *defines harmonic map into de-Sitter sphere* $S^2_1 \subset \mathbb{L}^3$

$$\frac{i'}{2} F^{\mu} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} (F^{\mu})^{-1}$$

through the stereographic projections.

Recent research

Compatibility conditions for non-vertical surfaces;

$$\frac{1}{2}\partial\overline{\partial}w+e^{w}-Q\overline{Q}e^{-w}+\frac{1}{2}(\partial\overline{\partial}H+p)\tilde{\epsilon}e^{-w/2}e^{u/2}=0$$

$$\partial\overline{Q}=-\frac{1}{2}\overline{Q}\partial H\tilde{\epsilon}e^{-w/2}e^{u/2}-\frac{1}{2}\overline{\partial}H\tilde{\epsilon}e^{w/2}e^{u/2}$$

$$\overline{\partial}Q=-\frac{1}{2}Q\overline{\partial}H\tilde{\epsilon}e^{-w/2}e^{u/2}-\frac{1}{2}\partial H\tilde{\epsilon}e^{w/2}e^{u/2}$$

In the case of non-vertical, H; constant, and $Q\overline{Q}=0$

$$\frac{1}{2}\partial\overline{\partial}w+e^{w}=0 \quad \text{and} \quad \overline{\partial}Q=0$$

Recent research

For $w : \mathbb{D} \to \mathbb{C}'$, denote $w = u + i'v$. Then the Liouville equation

$$\partial\overline{\partial}w = ce^{dw} \tag{3}$$

for real constants $c, d \in \mathbb{R}$ is equivalent to the system

$$\partial\overline{\partial}u = ce^{du}\cosh dv, \quad \partial\overline{\partial}v = ce^{du}\sinh dv$$

The solution of (3) is given by

$$w = \frac{u_1 + u_2}{2} + i'\frac{u_1 - u_2}{2}$$

where u_1, u_2 are the solutions of the Liouville equation for real function.

Recent research

Problem

Specify the minimal surfaces of the Abresch-Rosenberg differential Qdz^2 with $Q\overline{Q} = 0$ and $Q \neq 0$.

Definition

null-scroll *is the surface in* Nil_3 *of the form*

$$\gamma \cdot \exp(tB)$$

where $\gamma : I \to \mathrm{Nil}_3$ *is a null curve and* $B : I \to \mathfrak{nil}_3$ *is a null vector field.*

Remark

- *The null-scroll is a natural generalization of the ruled surface with a null-ruler* $\gamma + tB$*.*
- *Null-scroll is timelike.*

Recent research

Definition (L. Graves, 1979)

*Let $\gamma : I \to \mathbb{L}^3$ be a null Frenet curve with (A, B, C). Then the map $f(s,t) := \gamma(s) + tB(s)$ is called B-**scroll**.*

Null curve $\gamma : I \to \mathbb{L}^3$ is called **null Frenet curve** if ${}^\exists(A\ B\ C)$; frame along γ, ${}^\exists\kappa, {}^\exists\tau : I \to \mathbb{R}$ s.t.

$$A = \gamma', \qquad \langle A, B\rangle = \langle C, C\rangle = 1,$$
$$\langle A, A\rangle = \langle B, B\rangle = \langle A, C\rangle = \langle B, C\rangle = 0,$$
$$(A, B, C)' = (A, B, C)\begin{pmatrix} 0 & 0 & -\tau \\ 0 & 0 & -\kappa \\ \kappa & \tau & 0 \end{pmatrix}$$

Remark

The mean curvature of B-scroll is τ.

Recent research

Theorem

For arbitrary functions $\kappa \neq 0$ and τ, there exists a null Frenet curve γ with (A, B, C) s.t.

$$(A, B, C)' = (A, B, C) \begin{pmatrix} 0 & 0 & -\tau \\ 0 & 0 & -\kappa \\ \kappa & \tau & 0 \end{pmatrix}.$$

Null Frenet curve γ is given by

$$\gamma = \left(\int_0^s A^1 ds, \int_0^s A^2 ds, \int_0^s A^3 ds \right)$$

where $A = (A^1, A^2, A^3)$.

Recent research

Theorem (K)

The non-vertical timelike minimal surfaces with $Q\overline{Q} = 0, Q \neq 0$ are null-scrolls with the mean curvature $H = 0$.

($\Leftarrow$) Denote the conformal coordinate for null-scroll by $z = lx + \bar{l}y$, where $l = \frac{1+i'}{2}$ and the ruler as B. The $\bar{l}$-part of the Abresch-Rosenberg differential can be computed as

$$\bar{l}Q = \bar{l}\frac{1}{2}(t_y)^2 g(B, e_3)^2 H,$$

that means $\bar{l}Q = 0$ if and only if $H = 0$ or $g(B, e_3) = 0$.
($\Rightarrow$) The minimal surfaces induce the B-scrolls with $\tau = 1/2$. By the relation between f^μ and $f^3_{\mathbb{L}}$, we can see that the null curve and null vector field of B-scroll derive a null curve and null vector field of Nil_3.

Issues

- Construction of the null-scroll with the mean curvature 0 by the curve theory in Nil_3.
- Classify the CMC surfaces in Nil_3 by $Q\overline{Q} = 0$.
- Details of "ruled surfaces" $\gamma \cdot \exp(tB)$.

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