Reeb's sphere theorem for Lipschitz functions

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1 Background

1.1 The original Reeb sphere theorem

Theorem 1.1 (G. Reeb, 1952)
M: m-dim closed C^∞ manifold,
$f : M \to \mathbb{R}$: C^∞ function.
If f has exactly two critical points each of which is nondegenerate, then M is homeomorphic to the sphere $\mathbb{S}^m := \{x \in \mathbb{R}^{m+1} \mid \|x\| = 1\}$.

[Publ. Inst. Math. Univ. Strasbourg]

Definition 1.2 (revision)

- $p \in M$: critical point of f (or critical for f)

$\overset{\text{def}}{\iff}$ for a chart $(U; x_1, \cdots, x_m)$ about p,

$$\frac{\partial f}{\partial x_i}(p) = 0 \quad (i = 1, 2, \cdots m).$$

$\overset{\text{iff}}{\iff}$ $df_p : T_pM \to \mathbb{R} \overset{\text{id}}{=} T_{f(p)}\mathbb{R}$ is not surjective,

i.e., $\mathrm{rk}_p(f) := \mathrm{rank}(df_p) = 0$.

$\overset{\text{iff}}{\iff}$ giving M a Riemannian metric,

$$\nabla f(p) = 0.$$

- Let $p \in M$ be a critical point of f.

  Then p is said to be nondegenerate

  $\overset{\text{def}}{\Longleftrightarrow}$ for a chart $(U; x_1, \cdots, x_m)$ about p,

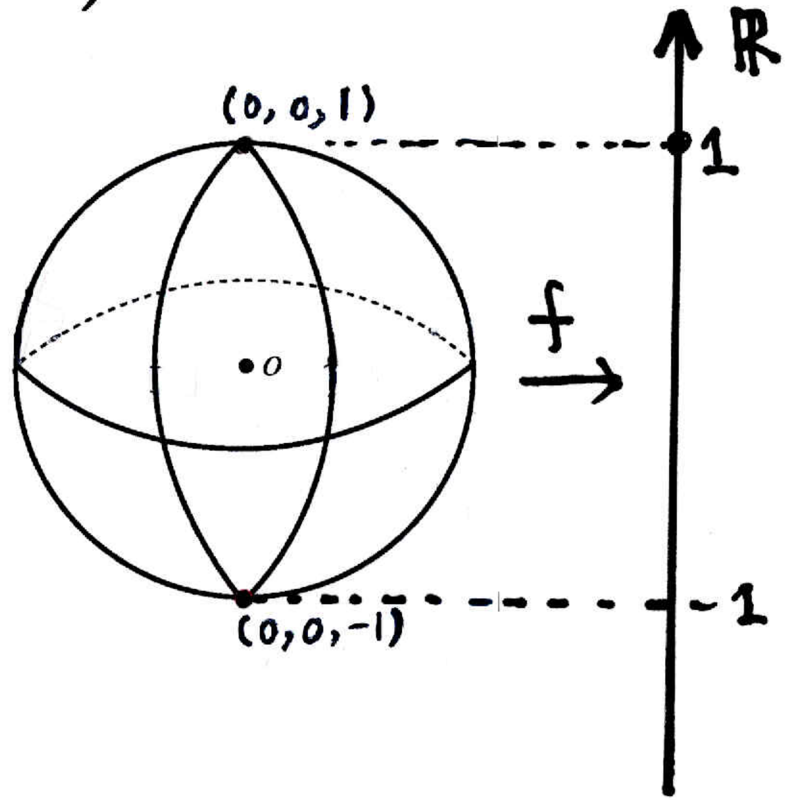
$$\det\left(\frac{\partial^2 f}{\partial x_j \partial x_i}(p)\right) \neq 0.$$

Example 1.3

$\forall (p_1, p_2, p_3) \in \mathbb{S}^2$,

let $f(p_1, p_2, p_3) := p_3$.

Two points $(0, 0, \pm 1)$ are nondegenerate cricical for f.



Theorem 1.1 (G. Reeb, 1952) (repeat)

M: m-dim closed C^∞ manifold,

$f : M \to \mathbb{R}$: C^∞ function.

If f has exactly two critical points each of which is nondegenerate, then M is homeomorphic to $\mathbb{S}^m$.

♣ The key results in the proof are

Morse Lemma and **Fundamental Theorem.**

Lemma 1.4 (Morse Lemma)

$\forall p_0 \in M$: nondegenerate critical point of f,

$\exists(U, \varphi) = (U; x_1, \cdots, x_m)$: chart about p_0

s.t. $\varphi(p_0) = o := (0, \cdots, 0)(\in \mathbb{R}^m)$, and that

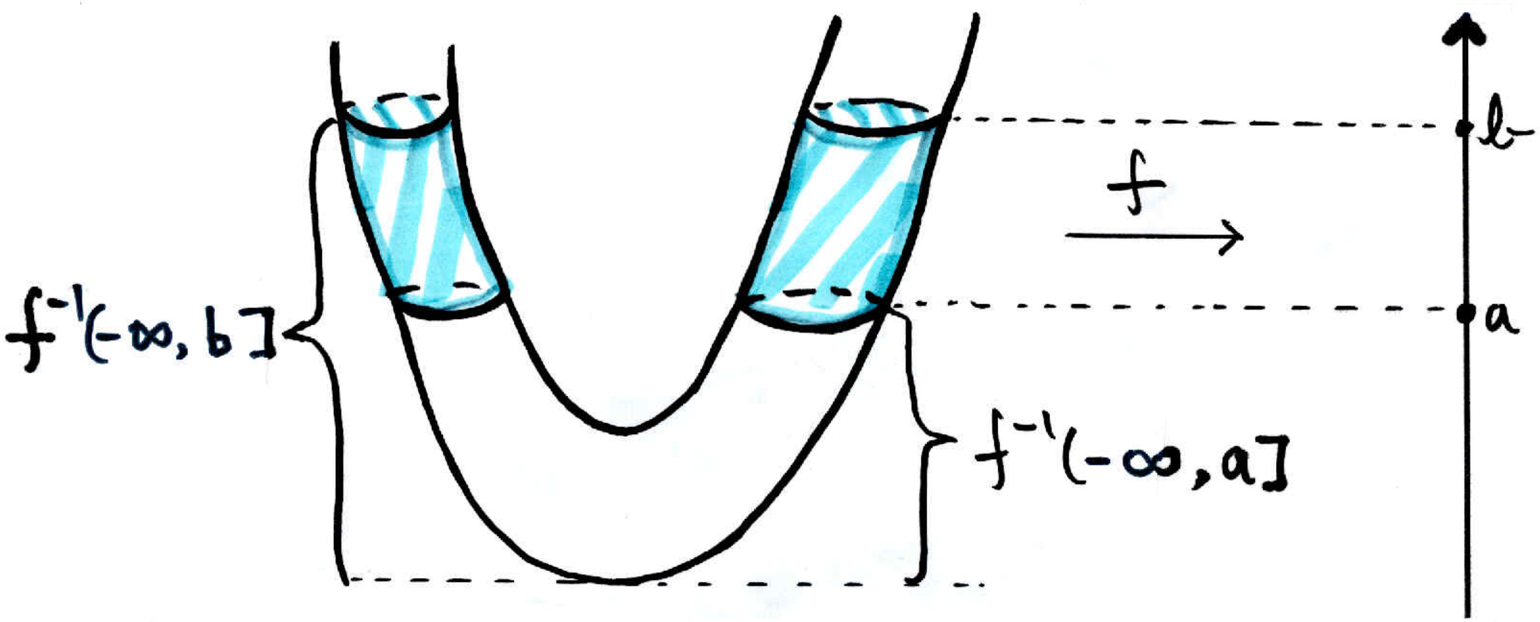
$$f(q) = f(p_0) - x_1^2(q) - \cdots - x_\lambda^2(q) + x_{\lambda+1}^2(q) + \cdots + x_m^2(q), \quad \forall q \in U.$$

Theorem 1.5 (Fundamental Theorem)

If f has no critical point on $f^{-1}[a,b]$ $(a<b)$,

$$f^{-1}(-\infty,a] \overset{\text{diffeo}}{\simeq} f^{-1}(-\infty,b].$$

In particular $f^{-1}[a,b] \overset{\text{diffeo}}{\simeq} f^{-1}(a)\times[a,b]$.



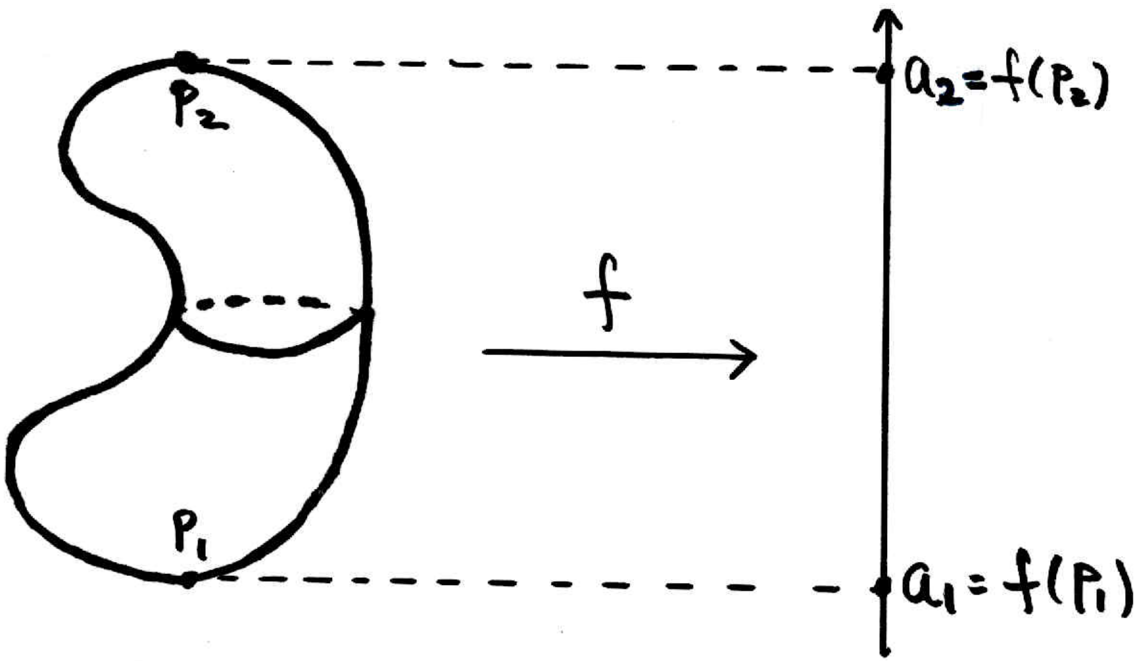
1.2 Proof of Reeb's sphere theorem

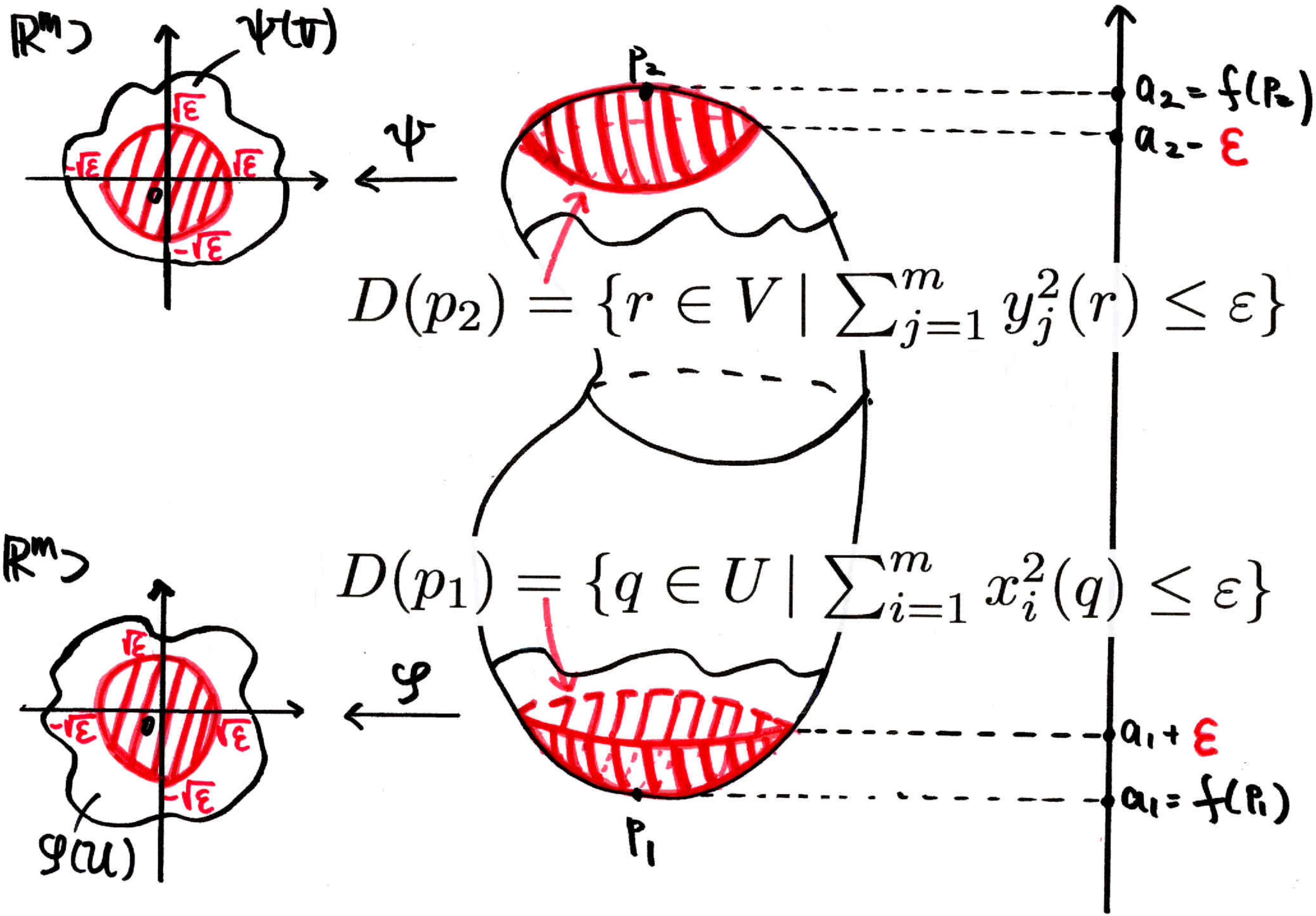
$p_1, p_2 \in M$: exactly two nondeg. critical pts of f.

By the max-min value theorem, we can assume

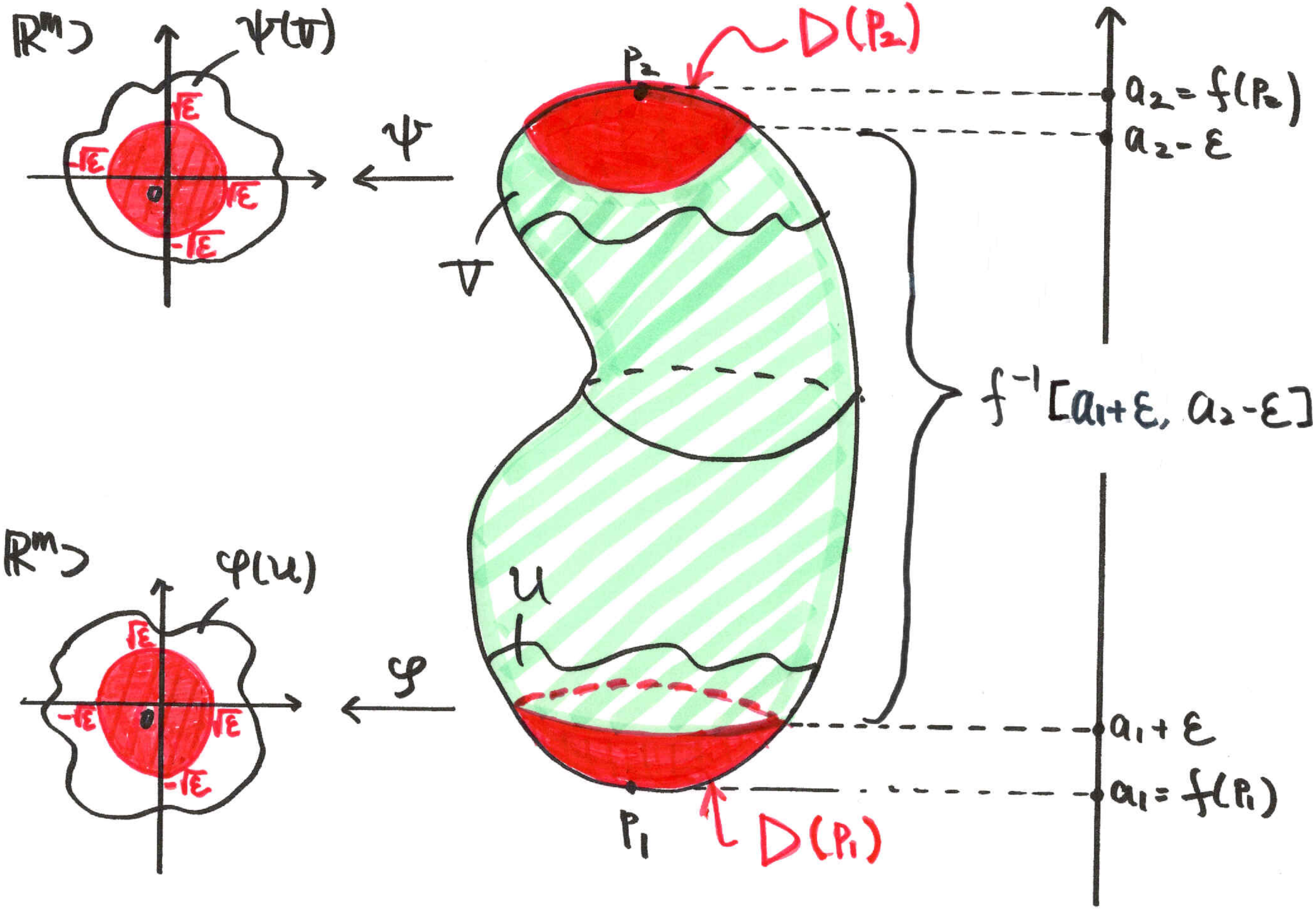
$$a_1 := f(p_1) = \min_{x \in M} f(x),$$

$$a_2 := f(p_2) = \max_{y \in M} f(y).$$





$\mathbb{R}^m \supset$
$\psi(V)$
$\sqrt{\varepsilon}$
$-\sqrt{\varepsilon}$
$\sqrt{\varepsilon}$
$-\sqrt{\varepsilon}$
0
ψ
p_2
$a_2 = f(p_2)$
$a_2 - \varepsilon$
$D(p_2) = \{r \in V \mid \sum_{j=1}^m y_j^2(r) \leq \varepsilon\}$
$D(p_1) = \{q \in U \mid \sum_{i=1}^m x_i^2(q) \leq \varepsilon\}$
$\mathbb{R}^m \supset$
$\sqrt{\varepsilon}$
$-\sqrt{\varepsilon}$
$\sqrt{\varepsilon}$
$-\sqrt{\varepsilon}$
0
$\varphi(U)$
φ
$a_1 + \varepsilon$
$a_1 = f(p_1)$
p_1



$\mathbb{R}^m \supset$
$\psi(V)$
$\sqrt{\varepsilon}$
$-\sqrt{\varepsilon}$
$\sqrt{\varepsilon}$
$-\sqrt{\varepsilon}$
0
ψ
$D(p_2)$
p_2
V
$a_2 = f(p_2)$
$a_2 - \varepsilon$
$f^{-1}[a_1+\varepsilon, a_2-\varepsilon]$
$\mathbb{R}^m \supset$
$\varphi(U)$
$\sqrt{\varepsilon}$
$-\sqrt{\varepsilon}$
$\sqrt{\varepsilon}$
$-\sqrt{\varepsilon}$
0
φ
U
$a_1 + \varepsilon$
$a_1 = f(p_1)$
p_1
$D(p_1)$

Remark 1.6

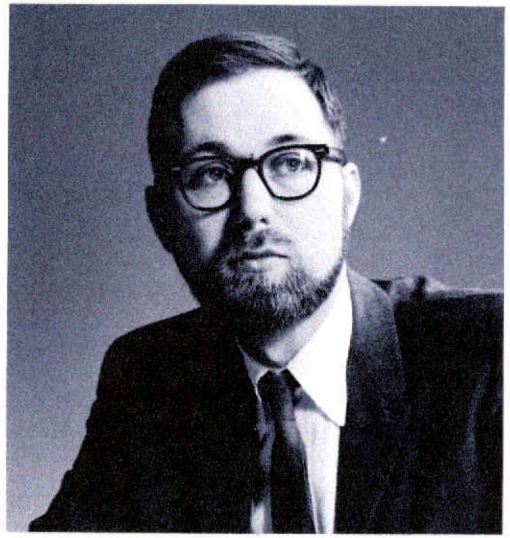
- M is a twisted m-sphere.
- Every exotic m-sphere ($m \geq 7$) is diffeom. to a twisted m-sphere.

  [Smale, Ann. of Math. (1961)]
- If $m \leq 6$, then M is diffeomorphic to $\mathbb{S}^m$.

  [Cerf, Kervaire-Milnor, Palais, Smale ('60s)]

1.3 Reeb-Milmor-Rosen sphere theorem

Theorem 1.7 (J. Milnor (1959), R. Rosen (1960))



M: m-dim closed C^∞ manifold,

$f : M \to \mathbb{R}$: C^∞ function.

If f has exactly two critical points,

then M is homeomorphic to $\mathbb{S}^m$.

[Bull. Soc. Math. France, Notices of AMS]

★ The main theorem of the talk is the extension of Theorem 1.7 to **Lipshitz function**.

1.3 Reeb-Milmor-Rosen sphere theorem

Theorem 1.7 (J. Milnor (1959), R. Rosen (1960))

M: m-dim closed C^∞ manifold,

$f : M \to \mathbb{R}$: C^∞ function.

If f has exactly two critical points,

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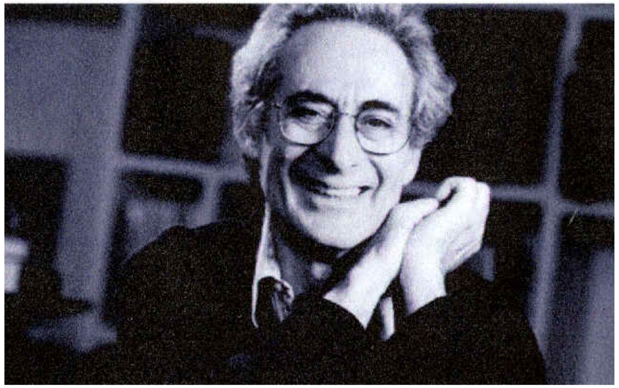


♣ The key result in the proof is

Mazur-Brown's theorem.

Theorem 1.8 (Mazur-Brown (1959, 1961))
X: topological space, Σ: suspension over X.
If Σ is locally m-Euclidean at the suspension points, then Σ is homeomorphic to $\mathbb{S}^m$.

[Proc. AMS]

- Mazur indicated that Thm 1.8 would follow from the form of the generalized Schoenflies theorem in [Bull. AMS (1959)].

1.4 Proof of Theorem 1.7

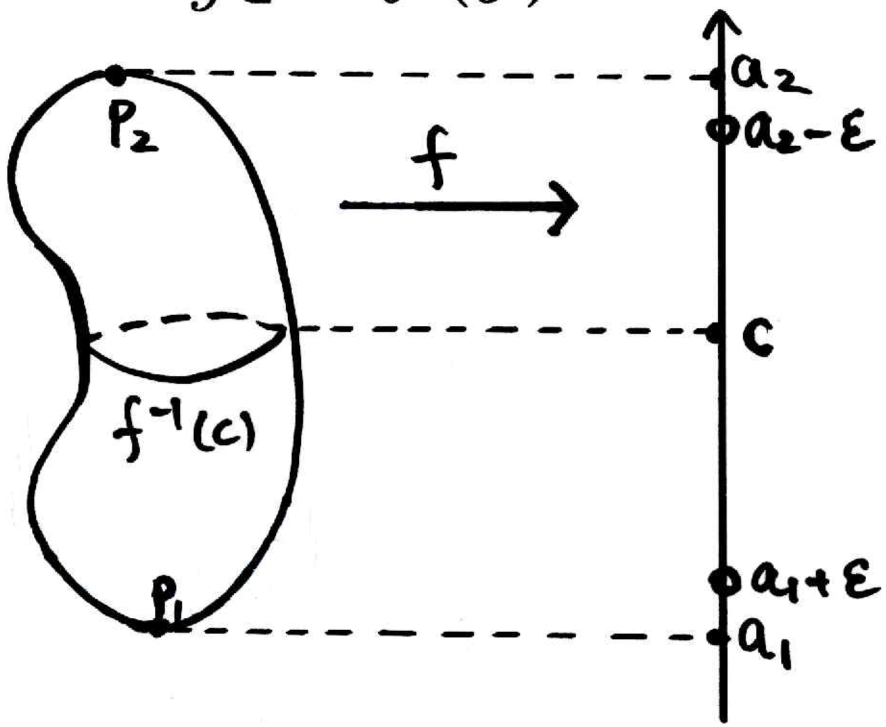
$p_1, p_2 \in M$: exactly two critical points of f.

By the max-min value thoerem, we can assume

$$a_1 := f(p_1) = \min_{x \in M} f(x),$$
$$a_2 := f(p_2) = \max_{y \in M} f(y).$$

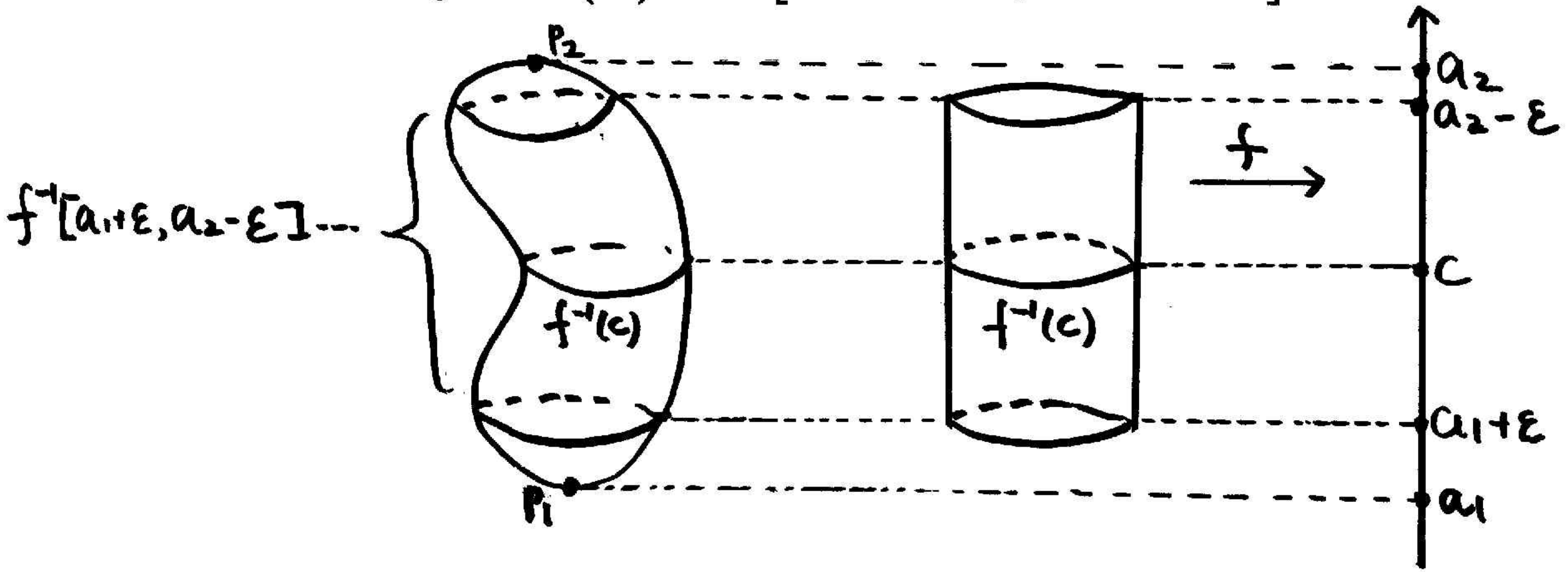
Let $c := \dfrac{a_1 + a_2}{2}$.

Choose $\varepsilon > 0$ so that $c \in (a_1 + \varepsilon, a_2 - \varepsilon)$.



Since f has no critical point on $f^{-1}[a_1+\varepsilon, a_2-\varepsilon]$, by the Fundamental theorem,

$$f^{-1}[a_1+\varepsilon, a_2-\varepsilon] \overset{\text{diffeo}}{\simeq} f^{-1}(a_1+\varepsilon)\times[a_1+\varepsilon, a_2-\varepsilon] \overset{\text{diffeo}}{\simeq} f^{-1}(c)\times[a_1+\varepsilon, a_2-\varepsilon].$$

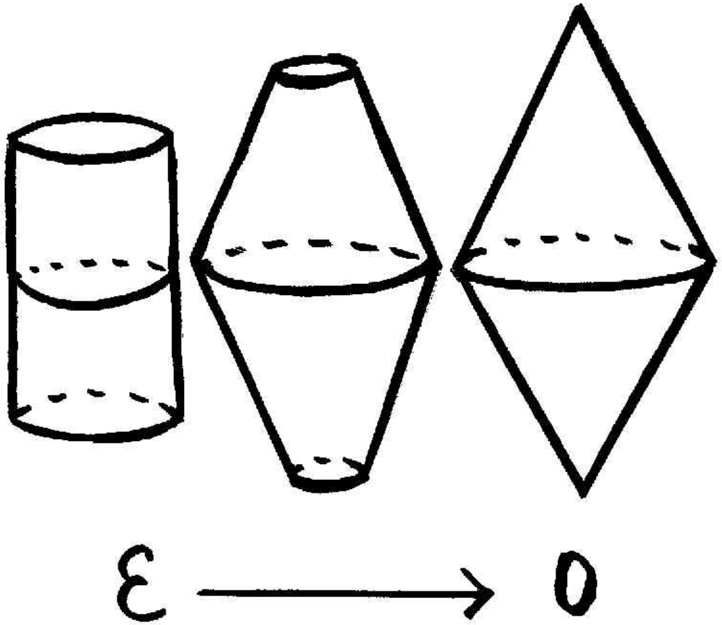


Since $f^{-1}(a_1+\varepsilon) \to p_1$ and $f^{-1}(a_2-\varepsilon) \to p_2$ as $\varepsilon \to 0$, we have

$$M = \lim\nolimits_{\varepsilon\downarrow 0} f^{-1}[a_1+\varepsilon, a_2-\varepsilon]$$

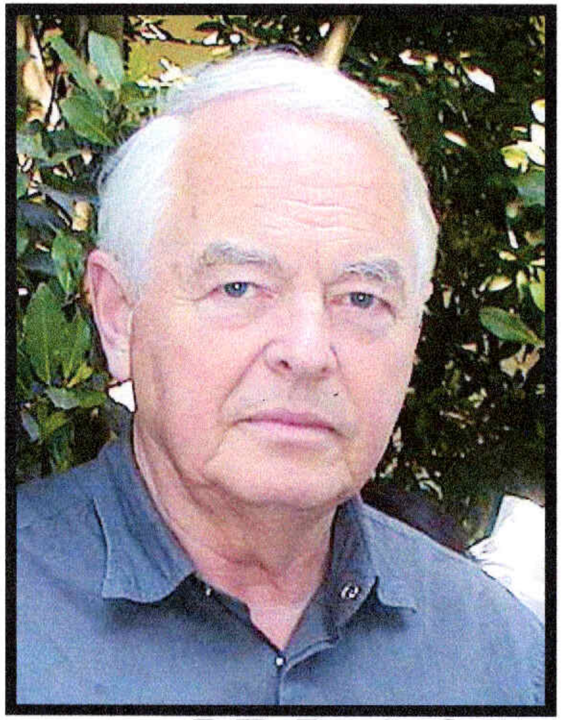
$$\overset{\text{homeo}}{\simeq} \frac{f^{-1}(c)\times[a_1,a_2]}{(f^{-1}(c)\times\{a_1\}, f^{-1}(c)\times\{a_2\})}$$

$$= \text{the suspension } \Sigma \text{ over } f^{-1}(c).$$

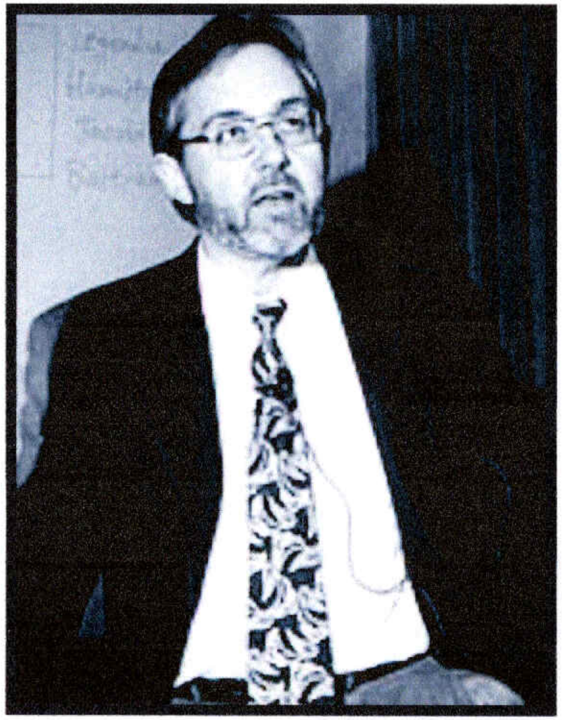


Since Σ is locally m-Euclidean at suspension pts of Σ, Mazur-Brown's theorem shows that M is homeomorphic to $\mathbb{S}^m$. $\square$

2 Nonsmooth Analysis



R.T. Rockafellar



F.H. Clarke

2.1 Differential of Lipschitz maps

M: m-dim Riemannian manifold,

N: n-dim Riemannian manifold,

$F : M \to N$: Lipschitz map.

Note that, by Rademacher's thm,

$\exists E_F \subset M$: Lebesgue measure zero

s.t. $\exists dF$ on $M \setminus E_F$.

$B_r(x)$: open metric ball with center $x \in M$,

or $x \in N$, and radius $r > 0$.

Definition 2.1

For each $p \in M$, let

$$\mathcal{L}(p, F(p))$$

be the set of all linear maps from T_pM to $T_{F(p)}N$.

Note that $\mathcal{L}(p, F(p))$ is a nm-dim vector space over $\mathbb{R}$.

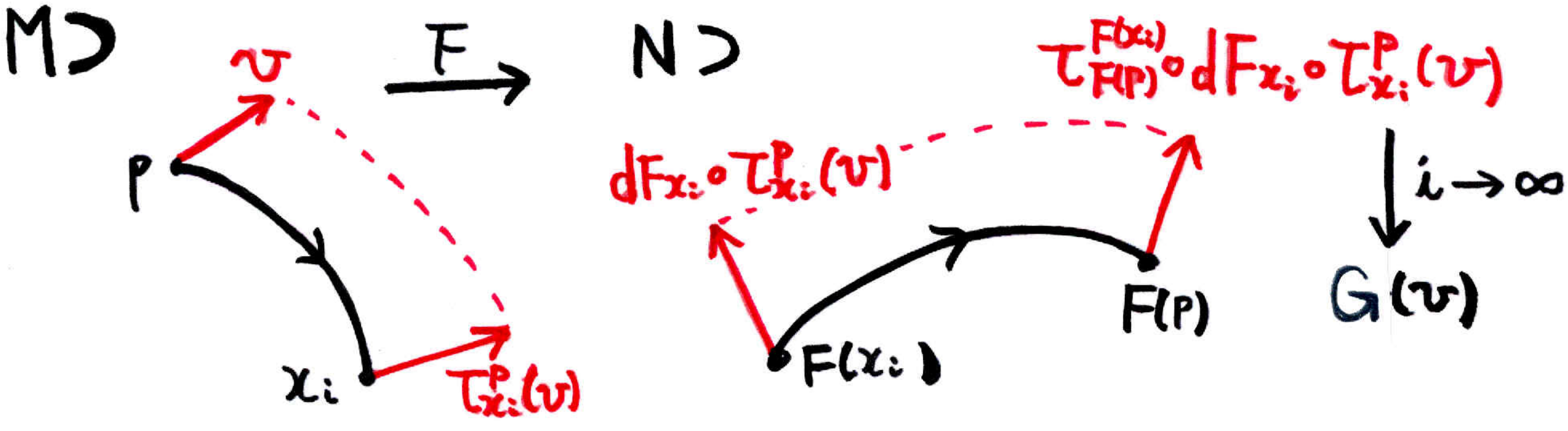
We topologize $\mathcal{L}(p, F(p))$ with the operator norm $\|\cdot\|$, i.e., $\forall G \in \mathcal{L}(p, F(p))$

$$\|G\| := \sup_{\|v\| \leq 1} \|G(v)\|$$

Definition 2.2 (Kondo (2022))

For each $p \in M$, let

$$K_F(p) := \left\{ G \in \mathcal{L}(p, F(p)) \;\middle|\; \begin{array}{l} \exists \{x_i\}_{i \in \mathbb{N}} \subset B_{r(p)}(p) \setminus E_F \text{ s.t. } x_i \to p, \\ \tau^{F(x_i)}_{F(p)} \circ dF_{x_i} \circ \tau^p_{x_i} \to G \text{ as } i \to \infty \end{array} \right\}.$$



Definition 2.3 (Kondo (2022))

For each $p \in M$, let

$$\begin{aligned} \partial F(p) &:= \text{the convex hull of } K_F(p) \\ &:= \text{the set of all convex combinations} \\ &\quad\ \text{of points in } K_F(p) \\ &= \text{the smallest convex set containing } K_F(p). \end{aligned}$$

We call $\partial F(p)$ the generalized differential of F at p.

Remark 2.4

Fix $p \in M$.

- If F is of class C^1, then $\partial F(p) = \{dF_p\}$.
- $\partial F(p)$ is a compact convex set in $\mathcal{L}(p, F(p))$.
- ∂F is upper semicontinuous, which means

  $\forall \varepsilon > 0$, $\exists \mu(p,\varepsilon) \in (0, r(p))$

  s.t. $\forall x \in B_{\mu(p,\varepsilon)}(p)$,

$$\tau_{F(p)}^{F(x)} \circ \partial F(x) \circ \tau_x^p \subset U_\varepsilon(\partial F(p)).$$

Definition 2.5 (Kondo (2022))
$p \in M$: non-singular point of F in the sense of Clarke

$$\overset{\text{def}}{\iff} \forall f \in \partial F(p),\ \mathrm{rank}(f) = \min\{m, n\}.$$

- Remark that, if $p \in M$ is non-singular for F in the sense of Clarke, then $\exists \lambda(p) > 0$ s.t. every $x \in B_{2\lambda(p)}(p)$ is non-singular for F in that sense.

Definition 2.5 (Kondo (2022))

$p \in M$: non-singular point of F in the sense of Clarke

$$\overset{\text{def}}{\iff} \forall f \in \partial F(p),\ \mathrm{rank}(f) = \min\{m, n\}.$$

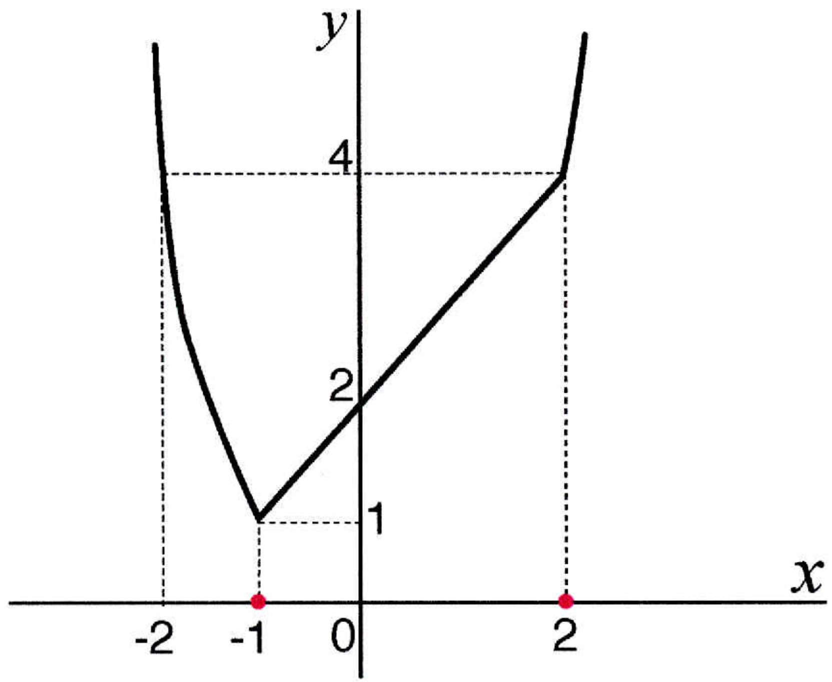
- Note, in the case of $N = \mathbb{R}$, i.e., F is a Lipschitz function,

  $p \in M$: singular point of F in the sense of Clarke

  $$\overset{\text{iff}}{\iff} \exists g \in \partial F(p) \quad \text{s.t.} \quad \mathrm{rank}(g) = 0.$$

Example 2.6

$$f(x) := \begin{cases} x^2, & x \geq 2, \\ x+2, & x \in [-1,2], \\ x^2, & x \leq -1, \end{cases}$$



which is a convex function on $\mathbb{R}$,

and is not differentiable at $x = -1, 2$.

Note that for $\forall p \in \mathbb{R} \setminus \{-1, 2\}$

$$df_p\left(\frac{d}{dx}\Big|_p\right) = \frac{df}{dx}(p) = f'(p).$$

Since $$f'(x) := \begin{cases} 2x, & x > 2, \\ 1, & x \in (-1, 2), \\ 2x, & x < -1, \end{cases}$$

we see, for each $\lambda \in [0, 1]$,

$$(1 - \lambda) \lim_{x \downarrow -1} f'(x) + \lambda \lim_{x \uparrow -1} f'(x) = 1 - 3\lambda,$$

$$(1 - \lambda) \lim_{x \downarrow 2} f'(x) + \lambda \lim_{x \uparrow 2} f'(x) = 4 - 3\lambda.$$

Thus $\partial f(-1) = \{1 - 3\lambda \mid \lambda \in [0, 1]\} = [-2, 1]$,

$$\partial f(2) = \{4 - 3\lambda \mid \lambda \in [0, 1]\} = [1, 4].$$

Since $0 \in \partial f(-1) = [-2, 1]$,

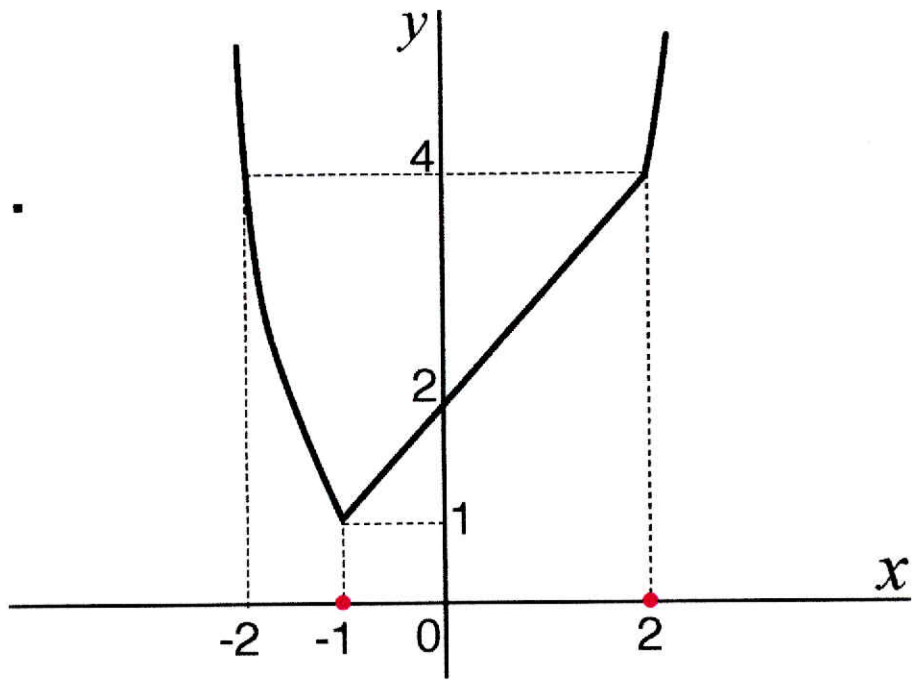
$x = -1$ is singular for f,

and $1 = f(-1) = \min_{x \in \mathbb{R}} f(x)$.

Since $0 \notin \partial f(2) = [1, 4]$,

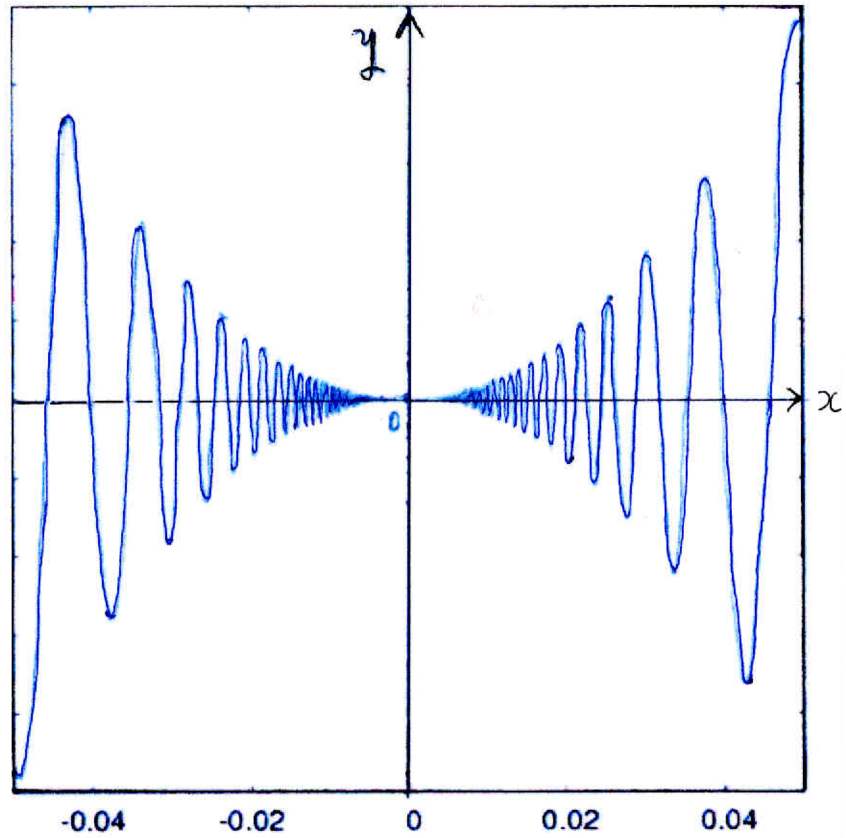
$x = 2$ is nonsingular for f,

and f is increasing near $x = 2$.



Example 2.7

$$g(x) := \begin{cases} x^2 \sin \frac{1}{x} & (x \neq 0), \\ 0 & (x = 0), \end{cases}$$



which is locally Lipschitz.

g is differentiable on $\mathbb{R}$,

but is not C^1 at $x = 0$,

for $g'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$.

Since $|\cos(1/x)| \leq 1$, we can choose a sequence of lines with the same slope tangent to $y = g(x)$.

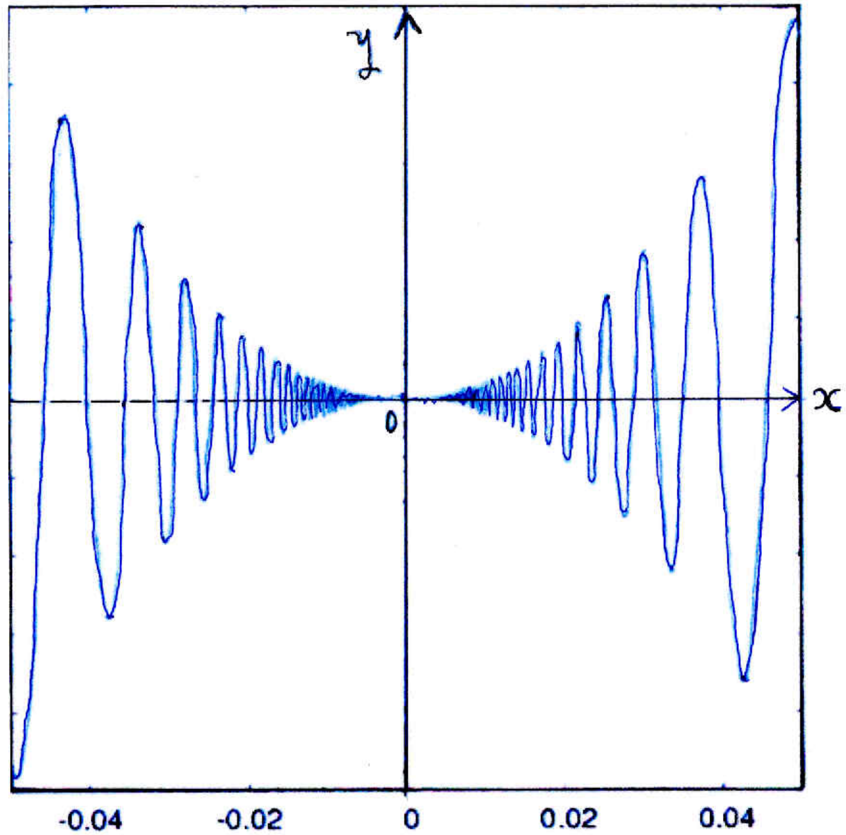
For instance, $\forall i \in \mathbb{N}$, let

$$\frac{1}{x_i} := \frac{\pi}{3} + 2(i-1)\pi.$$

We see

$\lim_{i\to\infty} x_i = 0$ and

$$\lim_{i\to\infty} g'(x_i) = -\frac{1}{2}.$$



Take the convex hull $\partial g(0)$ of the set

$$K_g(0) := \left\{ \alpha \in \mathbb{R} = T_0\mathbb{R} \;\middle|\; \begin{array}{l} \exists \{x_i\}_{i\in\mathbb{N}} \subset \mathbb{R} \setminus \{0\} \text{ s.t. } x_i \to 0, \\ g'(x_i) \to \alpha \text{ as } i \to \infty \end{array} \right\}.$$

Since $|\cos(1/x)| \leq 1$, we see

$$\partial g(0) = [-1, 1].$$

As $0 \in \partial g(0)$, $x = 0$ is a singular point of g.

Definition 2.8 (Kondo (2022))

$F : M \to \mathbb{R}$: Lipschitz function, $p \in M$.

$$\divideontimes_F(p) := \Big\{ w \in T_pM \;\Big|\; \begin{array}{l} \exists \{x_i\}_{i\in\mathbb{N}} \subset B_{r(p)}(p) \setminus E_F \ \text{ s.t. } x_i \to p, \\ \nabla F(x_i) \to w \text{ as } i \to \infty \end{array} \Big\},$$

$\circledast_F(p) :=$ the convex hull of $\divideontimes_F(p)$.

We call $\circledast_F(p)$ the generalized gradient of F at p.

Definition 2.9 (Kondo (2022))

$F : M \to \mathbb{R}$: Lipschitz function.

$p \in M$: critical point of F in the sense of Clarke

$\overset{\text{def}}{\iff}$ $o_p \in \circledast_F(p)$

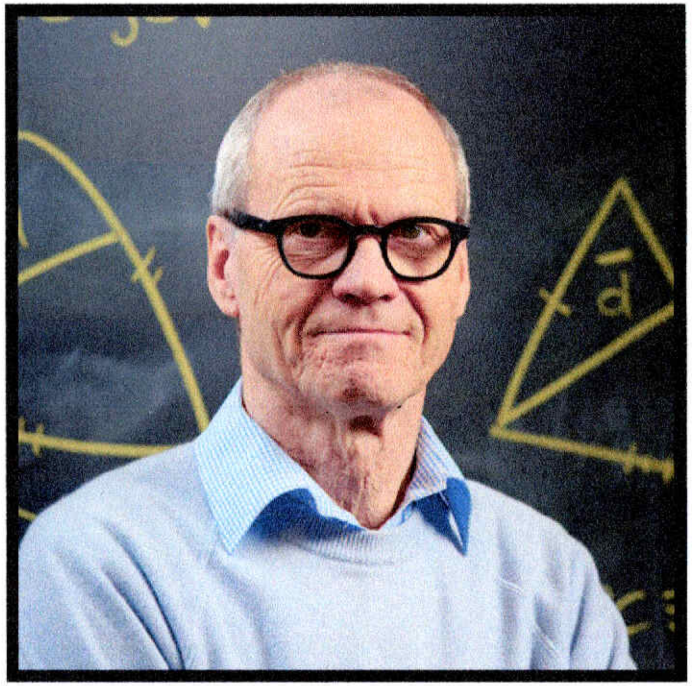
where o_p denotes the origin of T_pM.

Remark 2.10 (Kondo (2022))

$p \in M$: singular for F in the sense of Clarke

$\overset{\text{iff}}{\iff}$ $p \in M$: critical for F in that sense.

2.2 Example: Grove-Shiohama's theory



K. Grove

K. Shiohama

X: m-dim complete Riemannian manifold,

d: distance function of X.

Fix $p \in X$.

Define the function $d_p : X \to \mathbb{R}$ by

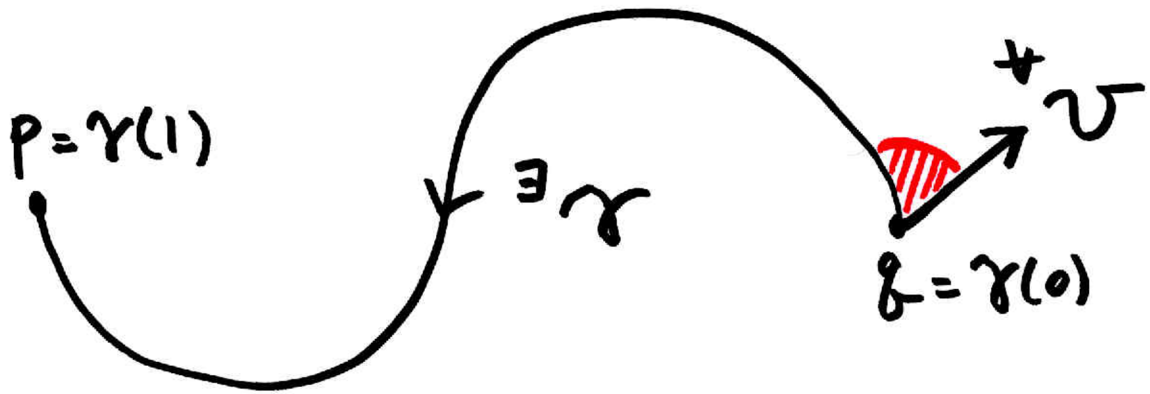
$$d_p(x) := d(p, x), \quad \forall x \in X.$$

Note that d_p is a 1-Lipschitz function.

- $q \in X$: critical point of d_p in the sense of Grove-Shiohama

$\overset{\text{def}}{\Longleftrightarrow}$ $\forall v \in T_q X$,

$\exists \gamma : [0,1] \to X$: minimal geodesic joining q to p s.t. $\angle(v, \gamma'(0)) \leq \pi/2$.

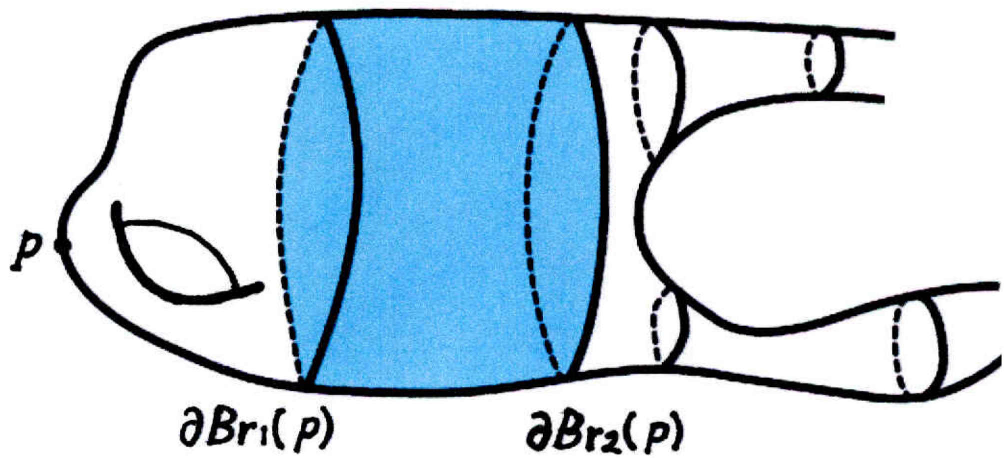


For convienience p is also called critical for d_p.

- **Gromov's isotopy lemma** (1981)

  If $0 < r_1 < r_2 \leq \infty$, and if d_p has no critical points in the sense of GS on $\overline{B_{r_2}(p)} \setminus B_{r_1}(p)$, then $\overline{B_{r_2}(p)} \setminus B_{r_1}(p) \overset{\text{homeom}}{\simeq} \partial B_{r_1}(p) \times [r_1, r_2]$.

  [Comment. Math. Helv.]



- **Diameter sphere theorem** (Grove-Shiohama)

  If the sectional curvature of X is bounded from below by 1, and if the diameter of X is greater than $\pi/2$, then X is homeom. to $\mathbb{S}^m$.

  [Ann. of Math. (1977)]

- **Reeb's sphere theorem for distance functions**

  Fix $p \in X$. If X is closed, and if d_p has exactly two critical points p, q in the sense of GS, then X is homeomorphic to $\mathbb{S}^m$.

- Remark that, for any fixed $p \in X$,

  $q \in X$: critical for d_p in the sense of GS

  $\overset{\text{iff}}{\Longleftrightarrow}$ $q \in X$: critical for d_p in the sense of Clarke

  $\overset{\text{iff}}{\Longleftrightarrow}$ $q \in X$: singular for d_p in that sense.

  [Kondo, J. Math. Soc. Japan (2022)]

3 Main Theorem

Main Theorem (Kondo (2022))

M: m-dim closed Riemannian manifold,

$F : M \to \mathbb{R}$: Lipschitz function.

If F has exactly two singular points in the sense of Clarke, then M is homeomorphic to $\mathbb{S}^m$.

[J. Math. Soc. Japan]

♣ The key result in the proof is

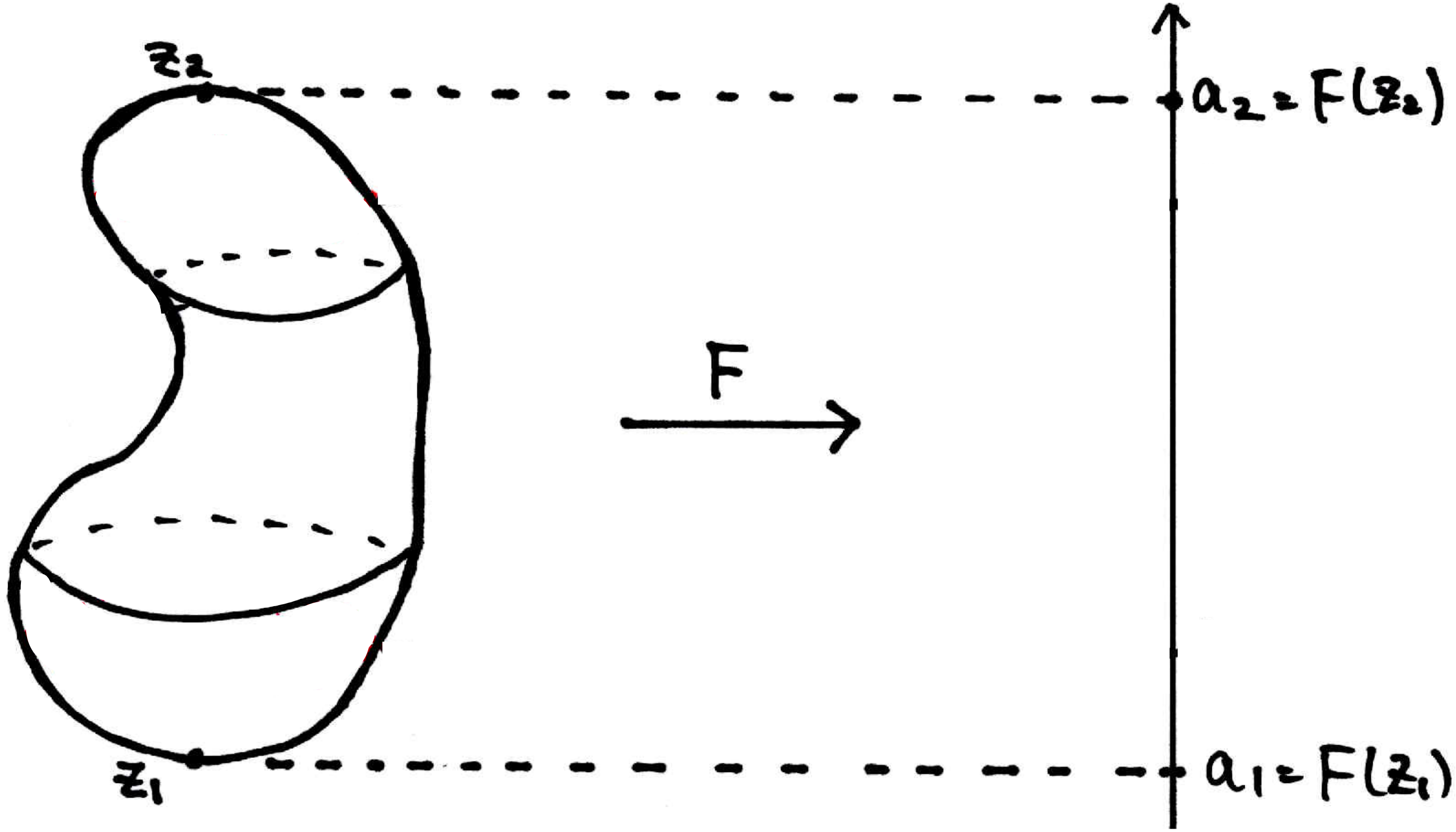
Mazur-Brown's theorem.

4 Proof of Main Theorem

$z_1, z_2 \in M$: singular for F in the sense of Clarke

Since z_1, z_2 are critical for F in that sense,
by the max-min value thm, we can assume

$$a_1 := F(z_1) = \min_{x \in M} F(x),$$
$$a_2 := F(z_2) = \max_{y \in M} F(y).$$


z_2
$a_2 = F(z_2)$
F
z_1
$a_1 = F(z_1)$

4 Proof of Main Theorem

$z_1, z_2 \in M$: singular for F in the sense of Clarke

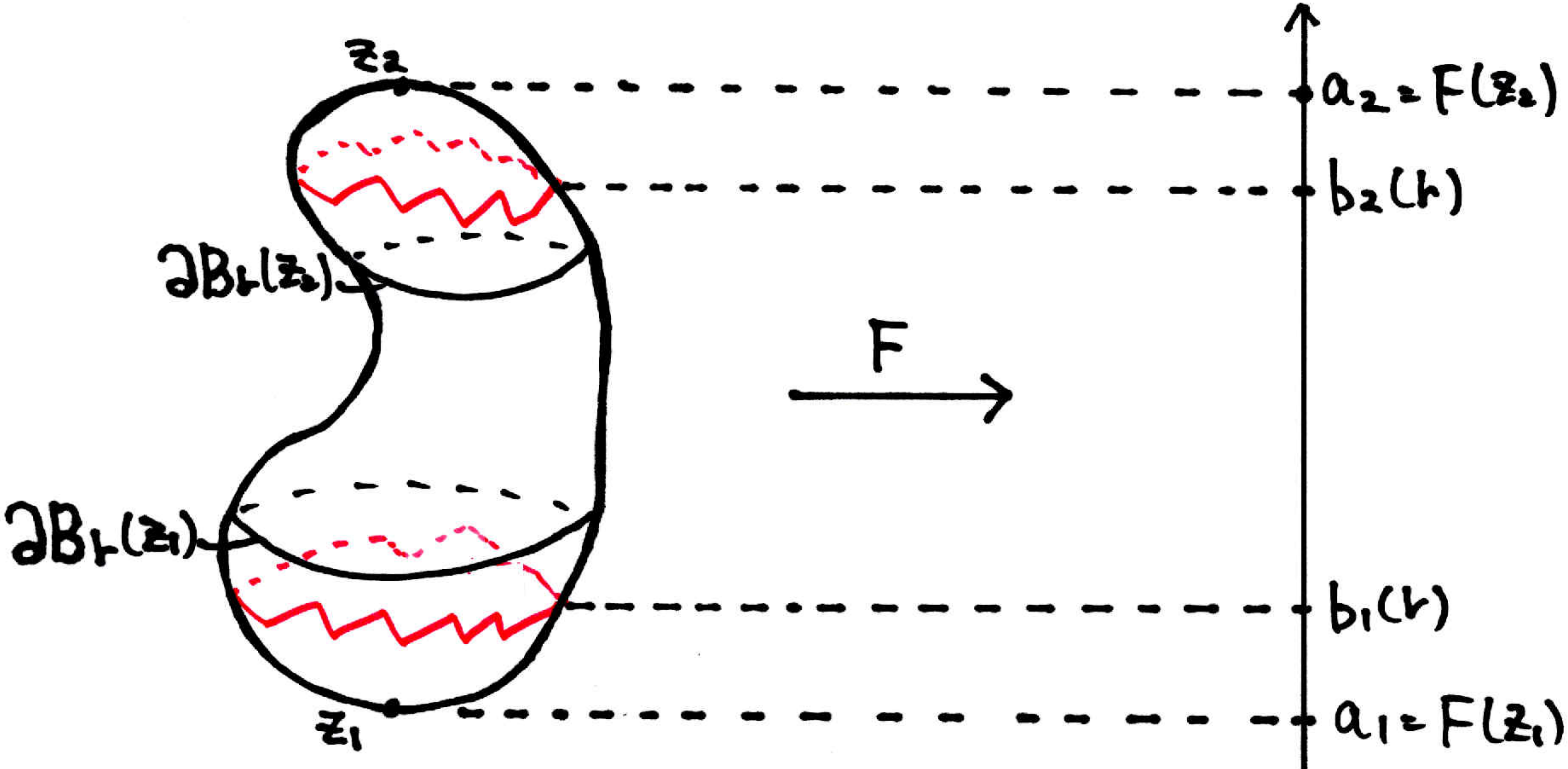
Since z_1, z_2 are critical for F in that sense,
by the max-min value thm, we can assume

$$a_1 := F(z_1) = \min_{x \in M} F(x),$$
$$a_2 := F(z_2) = \max_{y \in M} F(y).$$

For any $r > 0$ with $B_r(z_1) \cap B_r(z_2) = \emptyset$,

$$\exists b_i(r) \in (a_1, a_2) \text{ s.t. } F^{-1}(b_i(r)) \subset B_r(z_i)$$

where $i = 1, 2$.



Since M is compact, there are finite number of strongly convex open balls

$$\exists B_{r(p_1)}(p_1), \cdots, \exists B_{r(p_k)}(p_k) \subset M$$

which cover M, where $r(p_i) \in (0, \mathrm{inj}(M)/2)$.

Fix $\varepsilon \in (0, \mathrm{inj}(M)/2)$.

$F_\varepsilon^{(i)}$: local convolution smoothing of F
on $B_{r(p_i)}(p_i)$, $i = 1, \cdots, k$, defined by

$$F_\varepsilon^{(i)}(q) :=$$

$$\int_{y \in T_{p_i}M} \rho_\varepsilon^{(i)}(y)(F \circ \exp_{p_i})(\exp_{p_i}^{-1} q - y)dy$$

for all $q \in B_{r(p_i)}(p_i)$.

Fix $\varepsilon \in (0, \mathrm{inj}(M)/2)$.

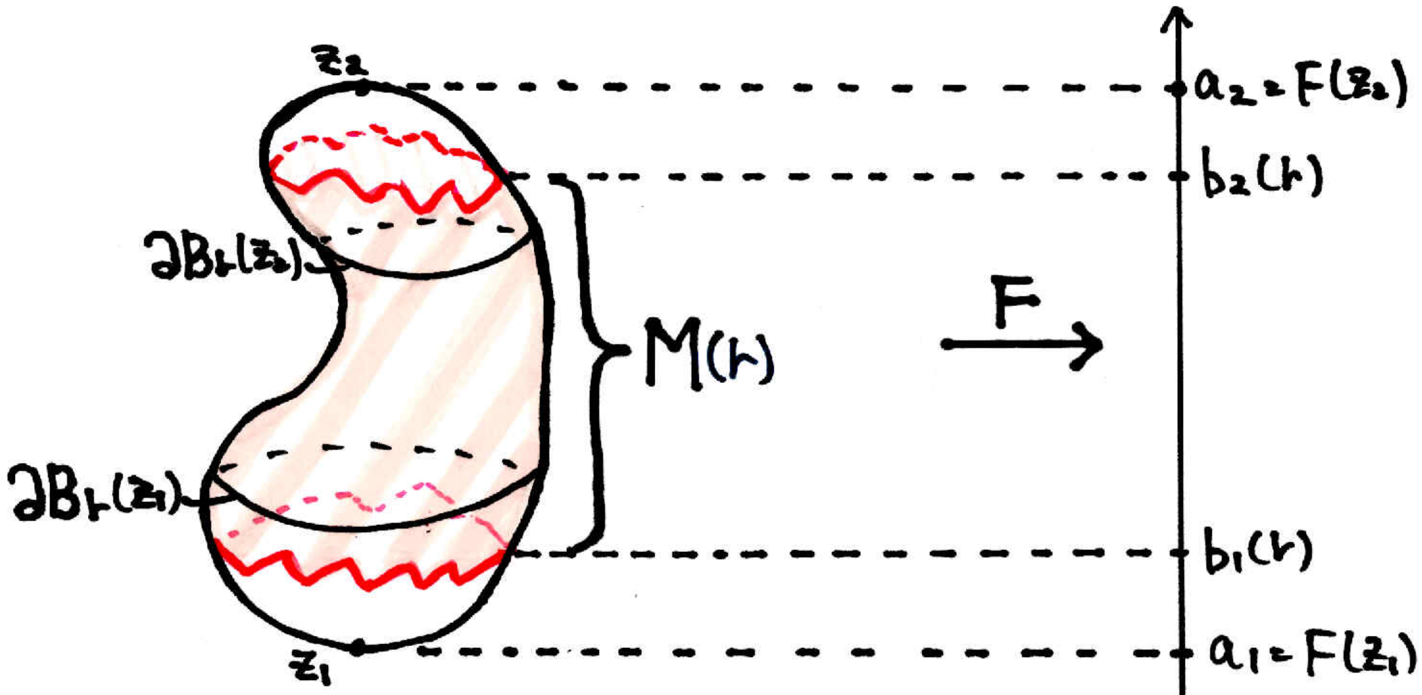
$F_\varepsilon^{(i)}$: local convolution smoothing of F
on $B_{r(p_i)}(p_i)$, $i = 1, \cdots, k$.

For a C^∞ partition of unity $\{\psi_i\}_{i=1}^k$ subordinate to $\{B_{r(p_i)}(p_i)\}_{i=1}^k$, let

$$F_\varepsilon(x) := \textstyle\sum_{i=1}^k \psi_i(x) \cdot F_\varepsilon^{(i)}(x), \ \forall x \in M.$$

Note that $F_\varepsilon \rightrightarrows F$ uniformly as $\varepsilon \downarrow 0$, i.e., F_ε is a global smoothing of F.

Let $M(r) := F^{-1}([b_1(r), b_2(r)])$.

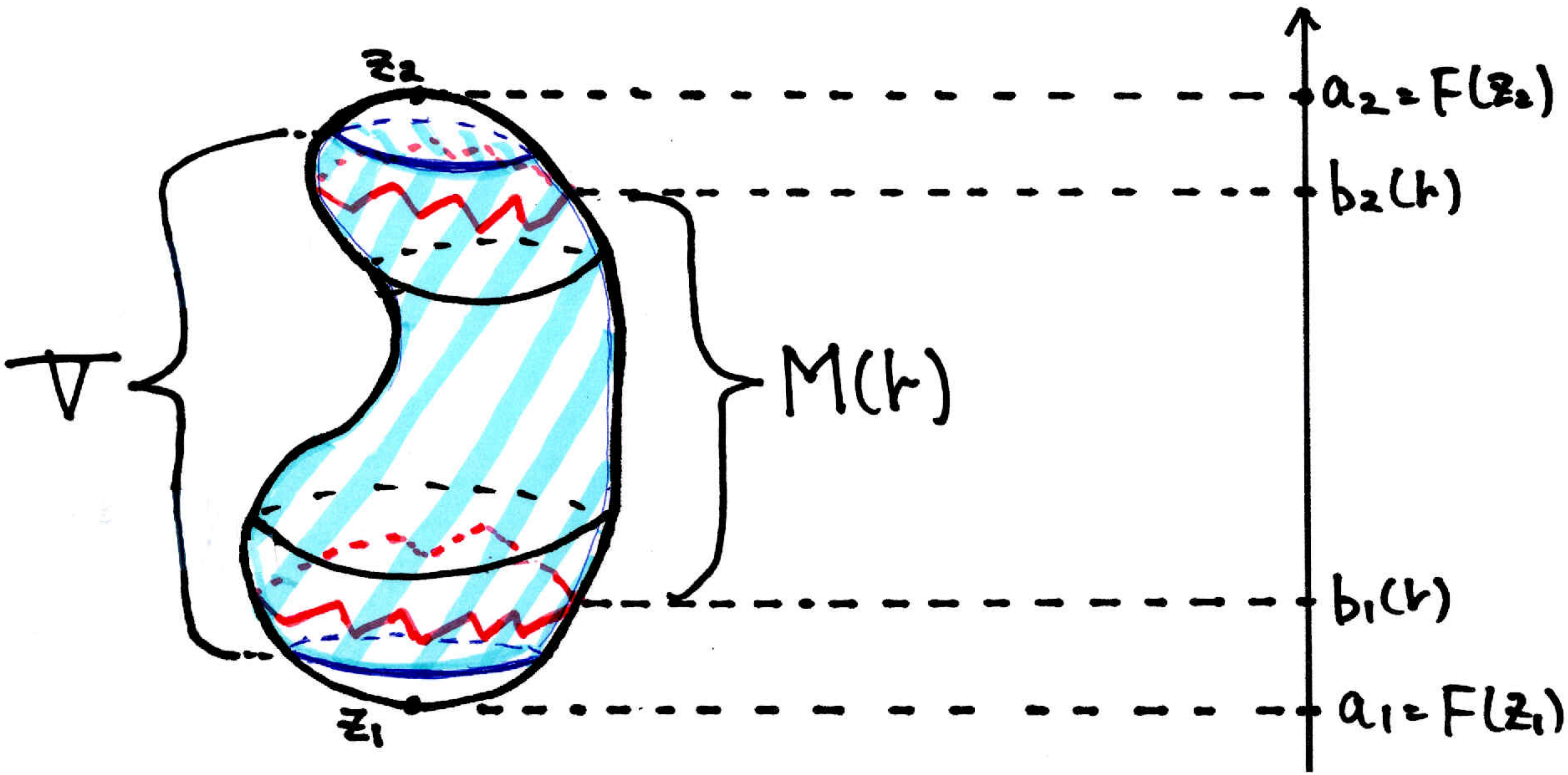


Since F has no singular point on $M(r)$,

$\forall q \in M(r)$, $\exists \lambda(q) > 0$, $\exists \varepsilon(q) \in (0, \mathrm{inj}(M)/2)$

s.t. $\forall \varepsilon \in (0, \varepsilon(q))$, $\nabla F_\varepsilon \neq 0$ on $B_{\lambda(q)}(q)$.

Thus $\exists V \subset M$: open, $\exists \varepsilon_0 \in (0, \mathrm{inj}(M)/2)$
s.t. $M(r) \subset V$, and that
$\forall \varepsilon \in (0, \varepsilon_0)$, F_ε has no critical point on V.

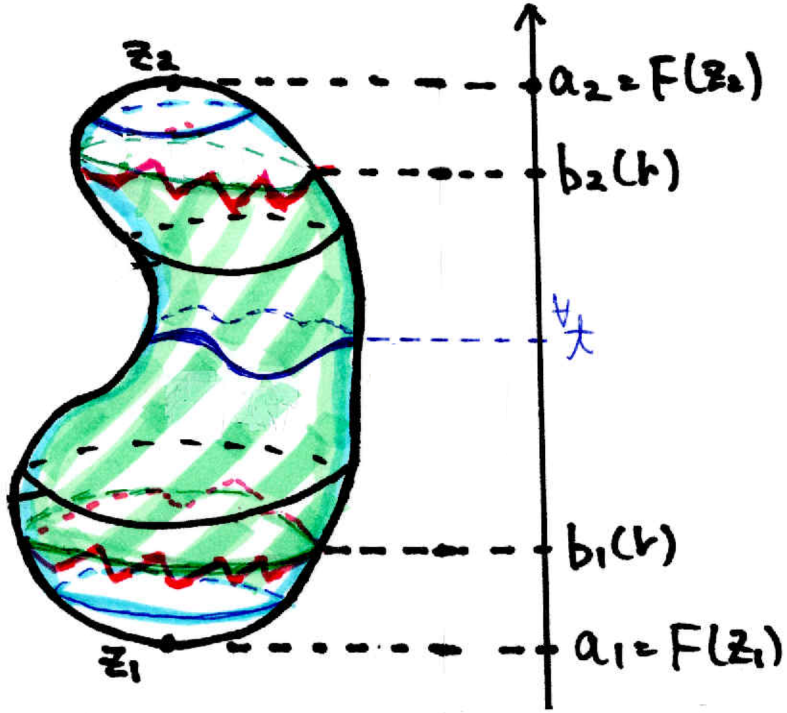


Since $F_\varepsilon \rightrightarrows F$ as $\varepsilon \downarrow 0$, $\exists \varepsilon_1 \in (0, \varepsilon_0)$ s.t. $\forall \varepsilon \in (0, \varepsilon_1)$,

$$F_\varepsilon^{-1}(b_i(r)) \subset B_r(z_i) \cap V \ (i = 1, 2).$$

In particular $M_\varepsilon(r) := F_\varepsilon^{-1}([b_1(r), b_2(r)]) \subset V$.

Since F_ε has no critical point on $M_\varepsilon(r)$, $\forall t \in [b_1(r), b_2(r)]$, $F_\varepsilon^{-1}(t)$ is a compact regular submanifold of M of codim 1.



Fix $c := (a_1 + a_2)/2$ and $\varepsilon \in (0, \varepsilon_1)$.

W.o.l.g., we can assume $c \in (b_1(r), b_2(r))$.

By the Fundamental Thm,

$$M_\varepsilon(r) \overset{\text{diffeo}}{\simeq} F_\varepsilon^{-1}(c) \times [b_1(r), b_2(r)].$$

Since $\lim_{r\downarrow 0} b_i(r) = a_i$ and $F_\varepsilon^{-1}(b_i(r)) \subset B_r(z_i)$,

we have $\quad \lim_{r\downarrow 0} F_\varepsilon^{-1}(b_i(r)) = z_i$ $(i = 1, 2)$.

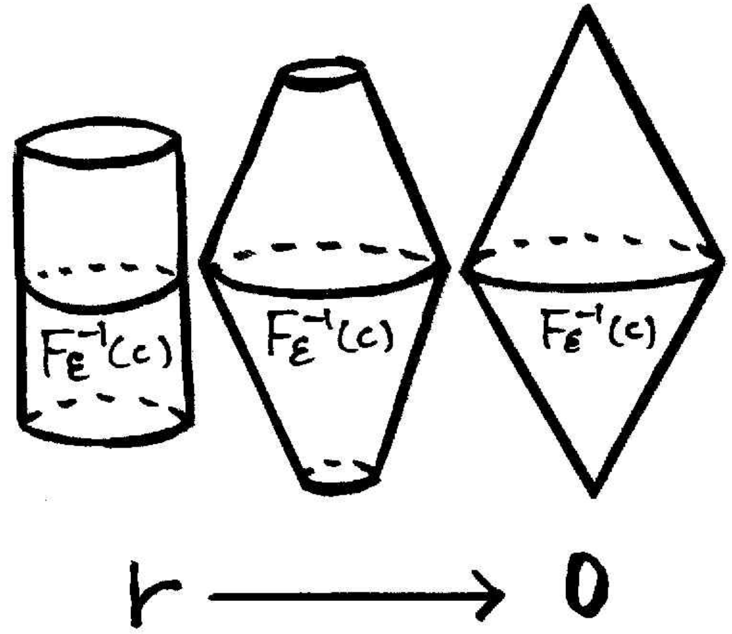
In particular $\quad \lim_{r\downarrow 0} M_\varepsilon(r) = M$.

We have

$$M = \lim_{r \downarrow 0} M_\varepsilon(r)$$

$$\overset{\text{homeo}}{\simeq} \frac{F_\varepsilon^{-1}(c) \times [a_1, a_2]}{(F_\varepsilon^{-1}(c) \times \{a_1\}, F_\varepsilon^{-1}(c) \times \{a_2\})}$$

$$= \text{the suspension } \Sigma \text{ over } F_\varepsilon^{-1}(c).$$



Since Σ is locally m-Euclidean at suspension pts of Σ, Mazur-Brown's theorem shows that M is homeomorphic to $\mathbb{S}^m$. $\square$