Negative Kodaira dimension
on compact almost Hermitian manifolds with nonnegative curvature

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Abstract

In recent years, there has been a growing interest in non-Kähler geometry, which has led to the study of almost Hermitian manifolds vigorously. Recently, the concept of the Kodaira dimension has been extended to general almost complex manifolds and its relation to the curvature has been studied. In this talk, we will consider the conditions for the Kodaira dimension to be negative on almost Hermitian manifolds with positive sectional curvature. First of all, as one of main results, we show some conditions for having the negative Kodaira dimension on almost Kähler manifolds with positive sectional curvature. We also consider a parabolic flow, which is called positive Hermitian curvature flow, introduced by Y.Ustinovskiy, on almost Hermitian manifolds. We show that the Griffith non-negativity of curvature is preserved along this flow and show that the Griffith non-negativity of curvature is conserved along this flow. As an application of this result, we will introduce a condition for the Kodaira dimension being negative.

1 Conditions for the negative Kodaira dimension

Let (M^{2n}, J) be an almost complex manifold of real dimension $2n$ with $n \geq 3$ and let g be an almost Hermitian metric on M. Let $\{e_r\}$ be an arbitrary local $(1,0)$-frame around a fixed point $p \in M$ and let $\{\theta^r\}$ be the associated coframe. Then the associated real $(1,1)$-form ω with respect to g takes the local expression $\omega = \sqrt{-1} g_{r\bar{k}} \theta^r \wedge \theta^{\bar{k}}$. We will also refer to ω as to an almost Hermitian metric. Notice that we use the Einstein convention omitting the symbol of sum over repeated indexes. It is well-known that there exists a unique affine connection ∇ preserving g and J on M whose torsion has vanishing $(1,1)$-part (cf. [6]), which is called the Chern connection. Now let ∇ be the Chern connection on M. The curvature Ω splits in $\Omega = H + R + \bar{H}$, where $R \in \Gamma(\Lambda^{1,1}(M) \otimes \mathrm{End}(T^{1,0}M))$, $H \in \Gamma(\Lambda^{2,0}(M) \otimes \mathrm{End}(T^{1,0}M))$, $\bar{H} \in \Gamma(\Lambda^{0,2}(M) \otimes \mathrm{End}(T^{1,0}M))$, and the curvature form can be expressed by $\Omega^i_j = d\gamma^i_j + \gamma^i_s \wedge \gamma^s_j$. In terms of e_r's, we have

$$R_{i\bar{j}k}{}^r = \Omega^r_k(e_i, e_{\bar{j}}) = e_i(\Gamma^r_{\bar{j}k}) - e_{\bar{j}}(\Gamma^r_{ik}) + \Gamma^r_{is}\Gamma^s_{\bar{j}k} - \Gamma^r_{\bar{j}s}\Gamma^s_{ik} - B^s_{i\bar{j}}\Gamma^r_{sk} + B^{\bar{s}}_{\bar{j}i}\Gamma^r_{\bar{s}k}, \tag{1.1}$$

$$H_{ijk}{}^r = \Omega^r_k(e_i, e_j) = e_i(\Gamma^r_{jk}) - e_j(\Gamma^r_{ik}) + \Gamma^r_{is}\Gamma^s_{jk} - \Gamma^r_{js}\Gamma^s_{ik} - B^s_{ij}\Gamma^r_{sk} - B^{\bar{s}}_{ij}\Gamma^r_{\bar{s}k}, \tag{1.2}$$

where Γ's are the Christoffel symbols with respect to the Chern connection ∇ and B's are the structure coefficients of Lie bracket (cf. [8§2]). We define that $R_{i\bar{j}k\bar{l}} := R_{i\bar{j}k}{}^r g_{r\bar{l}}$, $H_{ijk\bar{l}} := H_{ijk}{}^r g_{r\bar{l}}$, $H_{\bar{i}\bar{j}k\bar{l}} := H_{\bar{i}\bar{j}k}{}^r g_{r\bar{l}}$. We define the Chern scalar curvature s_ω and the Riemannian type scalar curvature $\hat{s}_\omega$ of the metric ω with respect to the Chern connection:

$$s_\omega := g^{i\bar{j}} g^{k\bar{l}} R_{i\bar{j}k\bar{l}}, \quad \hat{s}_\omega := g^{i\bar{l}} g^{k\bar{j}} R_{i\bar{j}k\bar{l}}. \tag{1.3}$$

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We have shown the following crucial lemma for the relation between two scalar curvatures above.

Lemma 1.1. (cf. [13, Lemma 4.4]) Let (M^{2n}, J, ω) be a real $2n$-dimensional compact almost Hermitian manifold with $n \geq 2$. Then we have

$$s_\omega - \hat{s}_\omega = \langle \bar{\partial}\bar{\partial}^* \omega, \omega \rangle + T^{\bar{r}}_{si} T^{s}_{\bar{r}\bar{i}}, \tag{1.4}$$

where $\langle \bar{\partial}\bar{\partial}^* \omega, \omega \rangle = g^{i\bar{j}} g^{p\bar{q}} (\bar{\partial}\bar{\partial}^* \omega)_{i\bar{q}} \overline{\omega_{j\bar{p}}} = g^{i\bar{j}} g^{p\bar{q}} (-\sqrt{-1} \nabla_{\bar{q}} w_i) \overline{\sqrt{-1} g_{j\bar{p}}} = -g^{i\bar{j}} \nabla_{\bar{j}} w_i$.

Definition 1.1. (cf. [23]) An almost Hermitian manifold (M^{2n}, J, ω) is called almost Kähler if $d\omega = 0$. When an almost Hermitian metric ω is almost Kähler, the triple (M^{2n}, J, ω) is called an almost Kähler manifold.

Lemma 1.2. (cf. [23]) The almost Kählerity is equivalent to

$$T^k_{ij} = 0, \quad T^{\bar{k}}_{ij} + T^{\bar{j}}_{ki} + T^{\bar{i}}_{jk} = 0 \quad \text{for all } i, j, k = 1, \ldots, n. \tag{1.5}$$

Lemma 1.3. (cf. [13, Lemma 1.3]) Let (M^{2n}, J, ω) be an almost Kähler manifold. Then we have that $T^{\bar{r}}_{ij} T^{i}_{\bar{r}\bar{j}} \geq 0$. The equality $T^{\bar{r}}_{ij} T^{i}_{\bar{r}\bar{j}} = 0$ holds if and only if the almost Kähler manifold is Kähler.

Recall the definition of a semi-Kähler metric on almost complex manifolds.

Definition 1.2. (cf. [9]) Let (M^{2n}, J) be an almost complex manifold. An almost Hermitian metric ω is called semi-Kähler if the metric ω satisfies $d\omega^{n-1} = 0$.

Since the nonnegativity of $T^{\bar{r}}_{ij} T^{i}_{\bar{r}\bar{j}}$ plays an important role, it is meaningful to investigate what conditions give us the nonnegativity. The following characterization of the semi-Kählerity gives that we have $T^{\bar{q}}_{ij} T^{i}_{\bar{q}\bar{j}} = 0$ for a Kähler-like semi-Kähler metric.

Proposition 1.1. (cf. [9, Theorem 1.1]) Let (M^{2n}, J, ω) be a compact Kähler-like almost Hermitian manifold with $n \geq 2$. Then (M^{2n}, J, ω) is semi-Kähler if and only if $T^{\bar{q}}_{ik} T^{i}_{\bar{q}\bar{l}} = 0$ for all $k, l = 1, \ldots, n$.

We recall the definition of Kodaira dimension on an almost complex manifold by following [2]. Let (M, J) be a compact $2n$-dimensional smooth manifold equipped with an almost complex structure J. Let $\pi^{p,q}$ be the projection to the set of smooth section of $\Lambda^{p,q} M$: $\Gamma(M, \Lambda^{p,q} M)$, where $\Lambda^{p,q} M$ is the bundle of (p, q)-forms on M. The $\bar{\partial}$ and ∂ operator can be defined by:

$$\bar{\partial} = \pi^{p,q+1} \circ d : \Gamma(M, \Lambda^{p,q} M) \to \Gamma(M, \Lambda^{p,q+1} M),$$

$$\partial = \pi^{p+1,q} \circ d : \Gamma(M, \Lambda^{p,q} M) \to \Gamma(M, \Lambda^{p+1,q} M),$$

where d is the exterior differential. Both $\bar{\partial}$ and ∂ satisfy the Leibniz rule, but in general $\bar{\partial}^2$ and ∂^2 may not be zero. Applying $\bar{\partial}$ to a smooth section of the canonical line bundle $\mathcal{K}_M := \Lambda^n(\Lambda^{1,0} M) = \Lambda^{n,0} M$ we have

$$\bar{\partial} : \Gamma(M, \mathcal{K}_M) \to \Gamma(M, \Lambda^{n,1} M) \cong \Gamma(M, (T^* M)^{0,1} \otimes \mathcal{K}_M).$$

We can extend the $\bar{\partial}$ to an operator $\bar{\partial}_m : \Gamma(M, \mathcal{K}_M^{\otimes m}) \to \Gamma(M, (T^*M)^{0,1} \otimes \mathcal{K}_M^{\otimes m})$, $\bar{\partial}_1 := \bar{\partial}$, inductively by the product rule for $m \in \mathbb{Z}_{\geq 2}$, $s_1 \in \Gamma(M, \mathcal{K}_M)$ and $s_2 \in \Gamma(M, \mathcal{K}_M^{\otimes(m-1)})$,

$$\bar{\partial}_m(s_1 \otimes s_2) = \bar{\partial} s_1 \otimes s_2 + s_1 \otimes \bar{\partial}_{m-1} s_2.$$

Then, the operator $\bar{\partial}_m$ satisfies the Leibniz rule $\bar{\partial}_m(fs) = \bar{\partial} f \otimes s + f \bar{\partial}_m s$ for any smooth function $f \in C^\infty(M, \mathbb{R})$ and any smooth section $s \in \Gamma(M, \mathcal{K}_M)$ of $\mathcal{K}_M^{\otimes m}$. Hence, $\bar{\partial}_m$ is a pseudoholomorphic structure on $\mathcal{K}_M^{\otimes m}$. For $m \in \mathbb{Z}_{\geq 1}$, the space of pseudoholomorphic sections of $\mathcal{K}_M^{\otimes m}$ is defined to be (cf. [2, Definition 2.1])

$$H^0(M, \mathcal{K}_M^{\otimes m}) = \{s \in \Gamma(M, \mathcal{K}_M^{\otimes m}) : \bar{\partial}_m s = 0\}.$$

The Kodaira dimension on an almost complex manifold (M, J) is defined as follows.

Definition 1.3. (cf. [2, Definition 1.2]) We define the m^{th}-plurigenus of (M, J) by

$$P_m(M, J) := \dim_{\mathbb{C}} H^0(M, \mathcal{K}_M^{\otimes m}).$$

The Kodaira dimension of (M, J) is defined by

$$\kappa(M) := \begin{cases} -\infty, & \text{if } P_m(M, J) = 0 \text{ for any } m \geq 1 \\ \\ \limsup_{m \to \infty} \dfrac{\log P_m(M, J)}{\log m}, & \text{otherwise.} \end{cases}$$

From the definition of the Kodaira dimension, we have $\kappa(X_1 \times X_2) = \kappa(X_1) + \kappa(X_2)$ for any two compact almost complex manifolds (X_1, J_1), (X_2, J_2) (cf [2, Corollary 6.9]). By taking direct products of the Kodaira-Thurston surface $X = \mathbb{S}^1 \times (\Gamma \backslash \mathrm{Nil}^3)$, where Nil^3 is the Heisenberg group and Γ is the subgroup in Nil^3 consisting of element with integer entries, acting by left multiplication, with copies of 2-torus $\mathbb{T}^2$, we have compact $2n$-manifolds with non-integrable almost complex structure and $\kappa = -\infty$ or 0. On the other hand, by taking direct products of the 4-manifold $X = \mathbb{T}^2 \times \Sigma_g$ with copies of 2-torus $\mathbb{T}^2$ or a compact Riemann surface Σ_g with genus $g \geq 2$, we get compact $2n$-manifolds with non-integrable almost complex structures and $\kappa = 1, 2, \ldots, n-1$ (cf. [2, §6]).

Proposition 1.2. (cf. [2, Theorem 6.10]) There are examples of compact $2n$-dimensional non-integral almost complex manifolds (M^{2n}, J) with Kodaira dimension $\kappa(M)$ lying among $\{-\infty, 0, 1, \ldots, n-1\}$ for $n \geq 2$.

We introduce the definition of a Gauduchon metric on an almost Hermitian manifold in the following.

Definition 1.4. (cf. [12, Definition 1.1]) Let (M^{2n}, J, ω) be a real $2n$-dimensional almost Hermitian manifold. An almost Hermitian metric ω is called Gauduchon if ω satisfies that $\partial\bar{\partial}\omega^{n-1} = 0$.

We have the following classical Gauduchon's theorem.

Proposition 1.3. (cf. [5]) Let (M^{2n}, J, ω) be a compact almost Hermitian manifold with $n \geq 2$. Then there exists a smooth function u, unique up to addition of a constant, such that the conformal almost Hermitian metric $e^u\omega$ is Gauduchon.

We now introduce the following result.

Proposition 1.4. (cf. [3, Theorem 4.3]) Let (M^{2n}, J) be a real $2n$-dimensional compact almost complex manifold with $n \geq 2$. If one of the following is satisfied:

(i) M admits an almost Hermitian metric ω with $s_\omega > 0$ everywhere,

(ii) M admits a Gauduchon metric with positive total scalar curvature,

then $\kappa(M) = -\infty$.

In 1990s, S.-T. Yau proposed the following question (cf. [22, Problem 67]). Let HCF(ω) denote the holomorphic sectional curvature of the metric ω.

Question 1.1. If (M, ω) is a compact Kähler manifold with HSC$(\omega) > 0$, does M have negative Kodaira dimension, i.e., $\kappa(M) = -\infty$?

X. Yang has given an answer for Yau's question in a general setting.

Proposition 1.5. (cf. [21, Theorem 1.2]) Let (M, ω) be a compact Hermitian manifold with semipositive holomorphic sectional curvature. If the holomorphic sectional curvature is not identically zero, then M has Kodaira dimension $-\infty$. In particular, if (M, ω) has HSC$(\omega) > 0$, then $\kappa(M) = -\infty$.

At this point, we would like to ask the following more general question.

Question 1.2. What about the almost Hermitian case?

We define the holomorphic sectional curvature on almost Hermitian manifold (M, J, ω) with the associated almost Hermitian metric g with respect to ω: for a point $p \in M$ and a non-zero $(1,0)$-vector $\xi \in T_p^{1,0}M$, the holomorphic sectional curvature $\mathcal{H}^g$ of ω at the point p and the direction ξ is define by

$$\mathcal{H}_p^g(\xi) := R^g(\xi, \bar{\xi}, \xi, \bar{\xi})|_p = R^g_{i\bar{j}k\bar{l}}|_p \xi^i \xi^{\bar{j}} \xi^k \xi^{\bar{l}}.$$

We write HSC$(\omega) > 0$ when we have that $\mathcal{H}_p^g(\xi) > 0$ for any point $p \in M$ and any non-zero $(1,0)$-vector $\xi \in T_p^{1,0}M$.

Applying the formula (1.1), we have the following proposition.

Proposition 1.6. (cf. [13, Proposition 1.12])Let (M^{2n}, J, ω) be a compact almost Hermitian manifold with $n \geq 2$, $T^{\bar{r}}_{ij} T^{i}_{\bar{r}\bar{j}} \geq 0$ and HCF$(\omega) > 0$. Then, we have that $\kappa(M) = -\infty$.

Combining Proposition 1.5 with Lemma 1.3 for the case of $n \geq 3$, we have the following.

Theorem 1.1. (cf. [13, Theorem 1.1]) Let (M^{2n}, J, ω) be a compact almost Kähler manifold with $n \geq 3$, and HCF$(\omega) > 0$. Then, $\kappa(M) = -\infty$.

In the case of $n = 2$, we have $T_{ij}^{\bar{r}} T_{\bar{r}\bar{j}}^{i} \geq 0$ automatically as follows.

Lemma 1.4. (cf. [13, Lemma 4.5]) On a real 4-dimensional almost Hermitian manifold, we have that $T_{ij}^{\bar{r}} T_{\bar{r}\bar{j}}^{i} \geq 0$. The equality $T_{ij}^{\bar{r}} T_{\bar{r}\bar{j}}^{i} = 0$ holds if and only if $T_{12}^{\bar{1}} = T_{12}^{\bar{2}}$.

Theorem 1.2. (cf. [13, Theorem 1.2]) Let (M^4, J, ω) be a real 4-dimensional compact almost Hermitian manifold with $\mathrm{HCF}(\omega) > 0$. Then, $\kappa(M) = -\infty$.

Note that in [15, Theorem 1.1], it has shown that if a compact Hermitian manifold has $\mathrm{HSC}(\omega) > 0$, then the Kodaira dimension is negative. Since one has $T_{ij}^{\bar{r}} = 0$ for all $i, j, r = 1, \ldots, n$ in the complex case, the result of Theorem 1.1 can be considered as a generalization of [15, Theorem 1.1].

A quasi-Kähler structure is an almost Hermitian structure whose real $(1,1)$-form ω satisfies $(d\omega)^{(1,2)} = \bar{\partial}\omega = 0$, which is equivalent to the original definition of quasi-Kählerianity: $D_X J(Y) + D_{JX} J(JY) = 0$ for all vector fields X, Y, where D is the Levi-Civita connection with respect to the metric ω. By letting $\mathcal{K}$, $\mathcal{QK}$, and $\mathcal{H}$ denote the class of Kähler manifolds, the class of quasi-Kähler manifolds, and the class of Hermitian manifolds respectively, we have that $\mathcal{K} = \mathcal{H} \cap \mathcal{QK}$ (cf. [7]).

From Lemma 1.4 and the formula (1.1), we have the following corollary.

Corollary 1.1. If $\hat{s}_\omega > 0$ on a real 4-dimensional compact quasi-Kähler (equivalently almost Kähler, or semi-Kähler) manifold (M^4, J, ω), then $\kappa(M) = -\infty$.

Note that the quasi-Kählerity implies $\alpha_\omega = J\delta\omega = 0$, where $\delta := - * d *$, since we have $d * \omega = \frac{1}{(n-1)!} d\omega^{n-1} = \frac{1}{(n-1)!}(\partial + \bar{\partial})\omega^{n-1} = 0$, where we used $A\omega^{n-1} = \bar{A}\omega^{n-1} = 0$. Then, we have the following Lemma.

Lemma 1.5. (cf. [4, Corollary 4.5]) Let (M^4, J, ω) be a real 4-dimensional quasi-Kähler manifold. Then, $\hat{s}_\omega = \frac{1}{2}\tilde{s} + \frac{1}{32}|N|^2 \geq \frac{1}{2}\tilde{s}$, where $\tilde{s}$ is the Riemannian scalar curvature with respect to the Levi-Civita connection, and N is the Nijenhuis tensor of the almost complex structure J.

Combining Corollary 1.1 and Lemma 1.5, we obtain the following.

Corollary 1.2. If $\tilde{s} > 0$ on a real 4-dimensional compact quasi-Kähler manifold (M^4, J, ω), then $\kappa(M) = -\infty$.

Since we have $T_{ri}^{\bar{k}} T_{\bar{k}\bar{i}}^{r} \geq 0$ on an almost Kähler manifold, we have the following result.

Corollary 1.3. If $\hat{s}_\omega > 0$ on a compact almost Kähler manifold (M^{2n}, J, ω) with $n \geq 2$, then $\kappa(M) = -\infty$.

Since we have $(d\omega)^- = 0$ and $\alpha_\omega = 0$ on an almost Kähler manifold, where $(d\omega)^-$ is the sum of $(3,0)$ and $(0,3)$ components of $d\omega$, we have the following lemma.

Lemma 1.6. (cf. [4, Theorem 4.3]) Let (M^{2n}, J, ω) be an almost Kähler manifold of real dimension $2n$. Then, for $t = 0$, $\hat{s}_\omega = \frac{1}{2}\tilde{s} + \frac{1}{32}|N^0|^2 \geq \frac{1}{2}\tilde{s}$, where $N^0 := N - \mathfrak{b}N$, $\mathfrak{b}N$ is the skew-symmetric part of N.

Combining Corollary 1.3 and Lemma 1.6, we have the following result.

Corollary 1.4. If $\tilde{s} > 0$ on a compact almost Kähler manifold (M^{2n}, J, ω) with $n \geq 2$, then $\kappa(M) = -\infty$.

2 The t-Gauduchon metric

Let (M, J, g) be an almost Hermitian manifold. Let TM be the real tangent vector bundle and we consider the complexified tangent vector bundle $T^{\mathbb{C}}M = TM \otimes_{\mathbb{R}} \mathbb{C}$, which can be decomposed as the eigenspaces of J with eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$ respectively, by $T^{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$. It is known that there exists a unique affine connection preserving g and J whose torsion has vanishing $(1,1)$-part, which is called the Chern connection and denoted by ${}^c\nabla$. Let D be the Levi-Civita connection and ${}^LD$ be the restriction to $T^{1,0}M$, which is called the Lichnerowicz connection.

$$\begin{array}{ccc} \Gamma(M, T^{\mathbb{C}}M) & \xrightarrow{D} & \Gamma(M, (T^{\mathbb{C}}M)^* \otimes T^{\mathbb{C}}M) \\ \cup & & \\ \Gamma(M, T^{1,0}M) & \xrightarrow{{}^LD = D|_{T^{1,0}M}} & \Gamma(M, (T^{\mathbb{C}}M)^* \otimes T^{1,0}M), \end{array}$$

which is given by

$${}^LD_XY = D_XY - \frac{1}{2}J(D_XJ)Y \text{ for } X, Y \in \Gamma(TM).$$

Notice that if the metric g is quasi-Kähler, the Chern connection ${}^c\nabla$ is given by (cf. [6], [20, (10)])

$${}^c\nabla = D - \frac{1}{2}JDJ = {}^LD.$$

As in the Hermitian case (cf. [1]), we define the t-Gauduchon connection for $t \in \mathbb{R}$ on (M, J, g) by

$${}^t\nabla := t{}^c\nabla + (1-t){}^LD.$$

The t-Gauduchon connection ${}^t\nabla$ is reduced to ${}^LD$ when the manifold is quasi-Kähler (cf. [6]). We define

$${}^ts_\omega := g^{i\bar{j}}g^{k\bar{l}}\,{}^tR^g_{i\bar{j}k\bar{l}}, \quad {}^t\hat{s}_\omega := g^{i\bar{l}}g^{k\bar{j}}\,{}^tR^g_{i\bar{j}k\bar{l}}.$$

Note that ${}^1s_\omega = s_\omega$, ${}^1\hat{s}_\omega = \hat{s}_\omega$. We compute that

$${}^ts_\omega = ts_\omega + (1-t)(\hat{s}_\omega + T^{\bar{r}}_{ki}T^{k}_{\bar{r}\bar{i}}),$$

$${}^t\hat{s}_\omega = t\hat{s}_\omega + (1-t)(s_\omega - T^{\bar{r}}_{ki}T^{k}_{\bar{r}\bar{i}}) - \Big(\frac{1-t}{2}\Big)^2(|T'|^2 + |w|^2),$$

where $w_i = g^{r\bar{s}}T_{ir\bar{s}}$.

Theorem 2.1. (cf. [14, Theorem 1.1]) Let (M^{2n}, J, ω) be a compact semi-Kähler manifold with $n \geq 2$. Then, we have that for any $t \in \mathbb{R}$, ${}^ts_\omega = s_\omega$.

Combining Proposition 1.1 with Theorem 1.1, we have the following Corollary.

Corollary 2.1. (cf. [14, Corollary 1.1]) Let (M^{2n}, J, ω) be a compact semi-Kähler manifold with $n \geq 2$. If ${}^ts_\omega > 0$ for some $t \in \mathbb{R}$, then $\kappa(M) = -\infty$.

3 The positive Hermitian curvature flow

The almost Hermitian flow (AHF) with an almost Hermitian initial metric ω_0 on (M, J) is as follows:

$$\text{(AHF)} \quad \begin{cases} \dfrac{\partial}{\partial t}\omega(t) = \partial\partial^*_{g(t)}\omega(t) + \bar{\partial}\bar{\partial}^*_{g(t)}\omega(t) - P(\omega(t)), \\ \\ \omega(0) = \omega_0, \end{cases}$$

where $\partial^*_{g(t)}$ and $\bar{\partial}^*_{g(t)}$ are the L^2-adjoint operators of ∂ and $\bar{\partial}$ with respect to metrics $g(t)$, and P is one of the Ricci-type curvatures of the Chern curvature, which is called the first Chern-Ricci curvature and locally given by $P_{i\bar{j}} = g^{k\bar{l}}R_{i\bar{j}k\bar{l}}$.

Let $\{e_i\}$ be a local $(1,0)$-frame around a point. The author proved the short-time existence and the uniqueness of the solution to the parabolic flow (AHF) which follows from the standard parabolic theory since the manifold is supposed to be compact. This flow (AHF) coincides with the pluriclosed flow if J is integrable and also the initial metric is pluriclosed. Note that $\partial\partial^*_g\omega + \bar{\partial}\bar{\partial}^*_g\omega$ involves the second derivatives of g; $g^{k\bar{l}}e_k e_{\bar{l}}(g_{i\bar{j}}) - g^{k\bar{l}}e_{\bar{j}}e_i(g_{k\bar{l}})$ (cf. [8, Proposition 3.1]).

Lemma 3.1. (cf. [12, Lemma 2.3]) One has

$$R_{i\bar{j}k\bar{l}} = g(([\nabla_{e_i}, \nabla_{e_{\bar{j}}}] - \nabla_{[e_i, e_{\bar{j}}]})e_k, e_{\bar{l}}).$$

Lemma 3.2. (cf. [12, Lemma 1.1]) One has

$$\begin{aligned} P_{i\bar{j}} \;=\; & -g^{k\bar{l}}e_{\bar{j}}e_i(g_{k\bar{l}}) + g^{k\bar{l}}e_i(\Gamma^s_{\bar{j}k})g_{s\bar{l}} + g^{k\bar{l}}e_{\bar{j}}(\Gamma^{\bar{s}}_{i\bar{l}})g_{k\bar{s}} - g^{k\bar{l}}\Gamma^r_{\bar{j}k}\Gamma^{\bar{s}}_{i\bar{l}}g_{r\bar{s}} + g^{k\bar{l}}\Gamma^r_{ik}\Gamma^{\bar{s}}_{\bar{j}\bar{l}}g_{r\bar{s}} \\ & + g^{k\bar{l}}\Gamma^r_{\bar{j}i}\Gamma^s_{rk}g_{s\bar{l}} - g^{k\bar{l}}\Gamma^{\bar{r}}_{i\bar{j}}\Gamma^s_{\bar{r}k}g_{s\bar{l}} + g^{k\bar{l}}\Gamma^s_{\bar{j}k}e_i(g_{s\bar{l}}) + g^{k\bar{l}}\Gamma^{\bar{s}}_{i\bar{l}}e_{\bar{j}}(g_{k\bar{s}}). \end{aligned} \tag{3.1}$$

On the other hand, since $P_{i\bar{j}}$ involves only $-g^{k\bar{l}}e_{\bar{j}}e_i(g_{k\bar{l}})$, the right-hand side of the parabolic flow (AHF) involves the second derivative of g; $g^{k\bar{l}}e_k e_{\bar{l}}(g_{i\bar{j}})$. Therefore, since we have

$$\begin{aligned} g^{k\bar{l}}e_k e_{\bar{l}}(g_{i\bar{j}}) &= g^{k\bar{l}}(e_k e_{\bar{l}} - [e_k, e_{\bar{l}}]^{(0,1)})g_{i\bar{j}} + g^{k\bar{l}}[e_k, e_{\bar{l}}]^{(0,1)}(g_{i\bar{j}}) \\ &= g^{k\bar{l}}\partial_k\partial_{\bar{l}}g_{i\bar{j}} + g^{k\bar{l}}B^{\bar{s}}_{k\bar{l}}e_{\bar{s}}(g_{i\bar{j}}), \end{aligned}$$

and since g is positive definite, the right-hand side of (AHF) is strictly elliptic, which implies that the flow (AHF) is strictly parabolic. From the standard parabolic theory, we obtain the short-time existence result since the manifold is supposed to be compact.

Proposition 3.1. (cf. [8, Theorem 1.1]) Given a compact almost Hermitian manifold (M, J, ω_0), there exists a unique solution to (AHF) with initial condition ω_0 on $[0, \varepsilon)$ for some $\varepsilon > 0$.

Denote by $S = S(g)$ one of the Ricci-type curvatures of the Chern curvature with respect to g, called the second Chern-Ricci curvature and locally given by $S_{i\bar{j}} = g^{k\bar{l}}R_{k\bar{l}i\bar{j}}$.

We proved that a solution of the almost Hermitian flow with initial condition g_0 is equivalent to a solution of the following parabolic flow on a compact almost complex manifold with an almost Hermitian metric, we call it the almost Hermitian curvature flow (AHCF):

$$\text{(AHCF)} \quad \begin{cases} \dfrac{\partial}{\partial t} g(t) = -S(g(t)) - Q^7(g(t)) - Q^8(g(t)) + BT'(g(t)) + \bar{e}(T')(g(t)), \\ \\ g(0) = g_0, \end{cases}$$

where Q^7, Q^8 are quadratics in the torsion of the Chern connection (cf. [22, pg. 712])

$$Q^7_{i\bar{j}} := g^{r\bar{s}} g^{k\bar{l}} T_{irk} T_{\bar{s}\bar{l}\bar{j}}, \quad Q^8_{i\bar{j}} := g^{r\bar{s}} g^{k\bar{l}} T_{irk} T_{\bar{j}\bar{l}\bar{s}},$$

also $w_i := g^{r\bar{s}} T_{ir\bar{s}}$,

$$BT'_{i\bar{j}} := g^{r\bar{s}} g^{p\bar{q}} B^{j}_{\bar{s}p} T_{ir\bar{q}} + g^{p\bar{q}} B^{r}_{\bar{q}i} T_{pr\bar{j}} + g^{s\bar{r}} B^{p}_{\bar{r}s} T_{pi\bar{j}} + B^{r}_{\bar{j}i} w_r,$$

and

$$\bar{e}(T')_{i\bar{j}} := -g^{r\bar{l}} e_{\bar{l}}(T^s_{ri}) g_{s\bar{j}} + g^{r\bar{l}} e_{\bar{j}}(T^s_{ri}) g_{s\bar{l}}.$$

These components are defined using an arbitrary unitary frame.

We introduce the following parabolic flow on a compact complex manifold the Hermitian curvature flow $(\mathrm{HCF})_{Q^1}$, which is a special case of the Hermitian curvature flow, whose torsion quadratic is taken arbitrarily, introduced in [17]:

$$(\mathrm{HCF})_{Q^1} \qquad \begin{cases} \dfrac{\partial}{\partial t} g(t) = -S(g(t)) + Q^1(g(t)), \\ \\ g(0) = g_0, \end{cases}$$

where g_0 is a pluriclosed metric (i.e., the associated real $(1,1)$-form ω_0 satisfies that $\partial\bar{\partial}\omega_0 = 0$) and Q^1 is a quadratic in the torsion of the Chern connection which is defined by $Q^1_{i\bar{j}} := g^{r\bar{s}} g^{k\bar{l}} T_{ik\bar{s}} T_{\bar{j}\bar{l}r}$. The solution to $(\mathrm{HCF})_{Q^1}$ coincides with the solution to the pluriclosed flow starting at the same initial metric g_0 (Streets-Tian identifiability theorem) and preserves the pluriclosedness (cf. [16, Proposition 3.3, Theorem 3.4]). We have the following relation between $(\mathrm{HCF})_{Q^1}$ and (AHCF).

Proposition 3.2. (cf. [8, Proposition 1.1]) (AHCF) coincides with the flow $(\mathrm{HCF})_{Q^1}$ starting at a pluriclosed metric if J is integrable.

The following result is the generalized version of Streets-Tian identifiability theorem.

Proposition 3.3. (cf. [8, Theorem 1.2]) Let (M, J, g_0) be a compact almost Hermitian manifold with the associated real $(1,1)$-form ω_0. Then a solution to (AHCF) with initial condition g_0 is equivalent to a solution to (AHF) starting at the initial condition ω_0.

Hence, we have the short-time existence of (AHCF).

Proposition 3.4. (cf. [8, Theorem 1.3]) Given a compact almost Hermitian manifold (M, J, g_0), there exists a unique solution to (AHCF) with initial almost Hermitian metric g_0 on $[0, \varepsilon)$ for some $\varepsilon > 0$.

We consider the following type of Hermitian curvature flow, which is called Ustinovskiy's flow or positive Hermitian curvature flow, on a compact almost complex manifold. We call this flow *positive HCF*. The flow will be denoted by $(\mathrm{HCF})_+$ in what follows.

$$(\mathrm{HCF})_+ \qquad \begin{cases} \dfrac{\partial}{\partial t} g(t) = -S(g(t)) - Q(g(t)), \\ \\ g(0) = g_0, \end{cases}$$

where g_0 is an almost Hermitian metric and Q is a quadratic in the torsion of the Chern connection which is defined by $Q_{i\bar{j}} := \frac{1}{2} g^{i\bar{j}} g^{k\bar{l}} T_{ik\bar{s}} T_{\bar{j}\bar{l}r}$.

We compute by using the formula in Lemma 3.1,

Lemma 3.3. (cf. [12, Lemma 1.2]) One has

$$\begin{aligned} S_{i\bar{j}} \;=\; & -g^{k\bar{l}} e_k e_{\bar{l}}(g_{i\bar{j}}) + g^{k\bar{l}} B^s_{k\bar{l}} e_s(g_{i\bar{j}}) + g^{k\bar{l}} B^{\bar{s}}_{k\bar{l}} e_{\bar{s}}(g_{i\bar{j}}) + g^{k\bar{l}} e_k(B^s_{\bar{l}i}) g_{s\bar{j}} + g^{k\bar{l}} e_{\bar{l}}(B^{\bar{s}}_{k\bar{j}}) g_{i\bar{s}} \\ & - B^s_{\bar{l}i} B^{\bar{r}}_{k\bar{j}} g_{s\bar{r}} + g^{k\bar{l}} \Gamma^r_{ki} \Gamma^{\bar{s}}_{\bar{l}\bar{j}} g_{r\bar{s}} - g^{k\bar{l}} B^s_{k\bar{l}} \Gamma^r_{si} g_{r\bar{j}} - g^{k\bar{l}} B^{\bar{s}}_{k\bar{l}} B^r_{\bar{s}i} g_{r\bar{j}} + g^{k\bar{l}} B^s_{\bar{l}i} e_k(g_{s\bar{j}}) + g^{k\bar{l}} B^{\bar{s}}_{k\bar{j}} e_{\bar{l}}(g_{i\bar{s}}). \end{aligned}$$

Note that the term which has the second order derivative of g in the equation $(\mathrm{HCF})_+$ is only the second Chern-Ricci curvature S, and $-S_{i\bar{j}}$ involves the second derivatives of g; $g^{k\bar{l}} e_k e_{\bar{l}}(g_{i\bar{j}})$. Since the metric g is positive definite, the right-hand side of $(\mathrm{HCF})_+$ is strictly elliptic, which implies that the equation $(\mathrm{HCF})_+$ is strictly parabolic and hence we can have the short-time unique existence of the solution to $(\mathrm{HCF})_+$ from the standard parabolic theory since the manifold is supposed to be compact.

Proposition 3.5. On a compact almost Hermitian manifold (M, J, g_0), there exists a unique solution of $(\mathrm{HCF})_+$ with initial metric g_0 on $[0, \varepsilon)$ for some $\varepsilon > 0$.

The differences between (AHCF) and $(\mathrm{HCF})_+$ in the estimate argument are the terms include $Q^7 + Q^8 - BT' - \bar{e}(T')$ in the (AHCF) case and Q in the case of $(\mathrm{HCF})_+$. Since we can estimate especially $\bar{\nabla}\nabla Q$ by applying Lemma 3.5, we have the following regularity result and the blow-up at a finite singular time for $(\mathrm{HCF})_+$ as in [10] for (AHCF). The long-time existence obstruction will be used for estimating the evolution equation of the Chern curvature R along $(\mathrm{HCF})_+$ in order to prove our main theorem.

Proposition 3.6. (cf. [10, Theorem 1.1], [12, Proposition 1.6]) Let $(M^{2n}, J, g(t))$ be a solution to $(\mathrm{HCF})_+$ for a maximal time interval $[0, \tau_{\max})$ on a compact almost Hermitian manifold which starts at the initial almost Hermitian metric g_0. The following statements (i), (ii) hold.

(i) We choose arbitrary $0 < \tau < \tau_{\max}$. Assume that, for a positive constants α with $\alpha/\tau > 1$, the following inequalities hold:

$$\sup_{M\times[0,\tau)} |R|_{g(t)} \le \frac{\alpha}{\tau}, \quad \sup_{M\times[0,\tau)} |T'|^2_{g(t)} \le \frac{\alpha}{\tau}, \quad \sup_{M\times[0,\tau)} |\nabla T'|_{g(t)} \le \frac{\alpha}{\tau}.$$

Then, for any $m \in \mathbb{N}$, the following inequalities hold:

$$|\nabla^m R|_{g(t)} \le \frac{C_{m,n,\alpha}}{\tau \cdot t^{\frac{m}{2}}}, \quad |\nabla^{m+1} T'|_{g(t)} \le \frac{C_{m,n,\alpha}}{\tau \cdot t^{\frac{m}{2}}}$$

for any $t \in (0, \tau]$ on M, where $C_{m,n,\alpha}$ is some positive constant depending only on m, n and α.

(ii) If $\tau_{\max} < \infty$, then
$$\limsup_{t \to \tau_{\max}} \max \left\{ |R|_{C^0(g(t))}, |T'|^2_{C^0(g(t))}, |\nabla T'|_{C^0(g(t))} \right\} = \infty.$$

We adopt the argument of Ustinovskiy in [18], [19] and prove that the positive HCF preserves Griffiths positivity (non-negativity) of the Chern curvature R, where we denote Ω the curvature of the Chern connection ∇. By Proposition 1.5, there exists unique solution to $(\mathrm{HCF})_+$ on some time interval $[0, \tau)$ for some $0 < \tau < \infty$. From (2) in Proposition 1.4, if τ is equal to the finite explosion time $\tau_{\max}$ of $(\mathrm{HCF})_+$, then one has that $\limsup_{t \to \tau} \max\{\max_M |R|_{g(t)}, \max_M |T'|^2_{g(t)}, \max_M |\nabla T'|_{g(t)}\} = \infty$. For this reason, we choose τ whichi is the time smaller than the explosion time $\tau_{\max}$. Our main result is the following.

Theorem 3.1. (cf. [12, Theorem 1.1]) Let $g(t)$, $t \in [0, \tau)$ be the solution to $(\mathrm{HCF})_+$ on a compact almost Hermitian manifold (M, J, g_0) with $g(0) = g_0$ for any $\tau < \tau_{\max} < \infty$, where $\tau_{\max}$ is the finite explosion time of the HCF in Proposition 1.4. Assume that the Chern curvature $R(g_0)$ is Griffiths non-negative (resp. positive), i.e., for any $\xi, \eta \in \Gamma(T^{1,0}M)$:
$$R(g_0)(\xi, \bar{\xi}, \eta, \bar{\eta}) \geq 0 \quad (\text{resp. } > 0).$$
Then for $t \in [0, \tau)$, the Chern curvature $R(g(t))$ remains Griffiths non-positive (resp. positive). If, moreover, the Chern curvature $R(g_0)$ is Griffiths positive at least at one point, then for any $t \in (0, \tau)$, the Chern curvature $R(g(t))$ is Griffiths positive everywhere on M.

Note that the following condition "non-quasi-Kähler" means that the almost complex structure J admits no quasi-Kähler metric.

Corollary 3.1. (cf. [12, Corollary 1.1]) Suppose (M, J, g_0) is a compact non-quasi-Kähler almost Hermitian manifold with Griffiths non-negative Chern curvature $R(g_0)$. Moreover, if the Chern curvature $R(g_0)$ is Griffiths positive at least at one point, then J cannot be integrable.

Proof. By the short-time existence given by Proposition 3.5, there exists a short-time solution $g(t)$ to $(\mathrm{HCF})_+$ starting from the metric g_0. By Theorem 3.1, for any $t \in (0, \tau)$, the metric $g(t)$ has the Griffiths positive Chern curvature $R(g(t))$. Now, let us assume that the almost complex structure J is integrable. Under this assumption, the first Chern-Ricci form $\omega_{i\bar{j}} := \sqrt{-1} g^{k\bar{l}} R_{i\bar{j}k\bar{l}}$ representing the first Chern class of the anticanonical bundle, is strictly positive. Hence, from the result [18, Proposition 6.1], the manifold M must be biholomorphic to the projective space $\mathbb{CP}^n$, which is contradictory to that J does not admit any Kähler metrics since we have the relation $\mathcal{K} = \mathcal{H} \cap \mathcal{QK}$. Therefore, the almost complex structure J cannot be integrable under these assumptions. $\square$

Here, we note the similer result in [11, Corollary 1.1].

Corollary 3.2. (cf. [11, Corollary 1.1]) Suppose (M, J, g_0) is a compact non-quasi-Kähler almost Hermitian manifold with Griffiths non-positive Chern curvature. Moreover, if the metric g_0 has the first Chern-Ricci curvature which is negative at some point, then J cannot be integrable.

As an application of Theorem 3.1, we have the following condition for the negative Kodaira dimension.

Corollary 3.3. (cf. [12, Corollary 1.3]) Let (M, J, g_0) be a compact almost Hermitian manifold with $g_0 \in \mathcal{G}(M)$. Then, one has $\kappa(M) = -\infty$.

Proof. Let $g(t)$, $t \in [0, \tau)$ be a solution to the AHCF on (M, J, g_0) with the initial condition $g(0) = g_0 \in \mathcal{G}(M)$. By applying Theorem 3.1, $R(g(t))_{i\bar{i}j\bar{j}} > 0$ for all $t \in (0, \tau)$ and all $i, j = 1, \ldots, n$ on M. Then we have $s(g(t)) > 0$ for all $t \in (0, \tau)$ everywhere on M. By applying Proposition 1.4 (i), we have the desired result. $\square$

Since for the standard almost complex structure J on $\mathbb{S}^6$, which is constructed by using the cross product of $\mathbb{R}^7$ (consider $\mathbb{S}^6$ as a homogeneous space of the exceptional Lie group G_2 and $\mathbb{S}^6 \cong G_2/SU(3)$) applying to the tangent space of $\mathbb{S}^6$, we have $P_m(\mathbb{S}^6, J) = 1$ for any $m \geq 1$ and then $\kappa(\mathbb{S}^6) = 0$ (cf. [2, Theorem 1.6]), from Corollary 3.3, we conclude that $(\mathbb{S}^6, J)$ does not admit metrics $g \in \mathcal{G}(\mathbb{S}^6)$. It is known that the standard almost complex structure J constructed by the cross product of $\mathbb{R}^7$ is not integrable since the Nijenhuis tensor of J is nowhere vanishing (cf. [2]).

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