Morse index and first Betti number for self-shrinkers in higher codimension

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1 Preliminaries

1.1 f-minimal submanifold

Let $\mathbf{x} : (\Sigma^n, g) \to (\mathbb{R}^m, \bar{g})$ be an oriented closed submanifold which is isometrically embedded into the Euclidean space with standard inner product $\bar{g} = \langle \cdot, \cdot \rangle$. We regard $\mathbf{x}$ as the position vector of the submanifold $\Sigma^n \subset \mathbb{R}^m$. The *second fundamental form* B of $\Sigma^n \subset \mathbb{R}^m$ is defined by

$$B(X, Y) := (\overline{\nabla}_X Y)^\perp,$$

for $X, Y \in \Gamma(T\Sigma)$, where $\overline{\nabla}$ denotes the Levi-Civita connection on $\mathbb{R}^m$. The second fundamental form B is a $\Gamma(N\Sigma)$-valued symmetric 2-tensor on Σ^n, where $N\Sigma$ denotes the normal bundle of Σ^n. The g-trace of B is called the *mean curvature vector field* of the submanifold $\Sigma^n \subset \mathbb{R}^m$, i.e.,

$$H := \operatorname{tr}_\Sigma B = \sum_{i=1}^n B(e_i, e_i),$$

where $\{e_1, \dots, e_n\}$ is a local orthonormal frame on Σ^n. For a smooth normal vector field $N \in \Gamma(N\Sigma)$, the *shape operator* in the direction of N is defined by

$$A^N(X) = -(\overline{\nabla}_X N)^\top, \quad X \in \Gamma(T\Sigma).$$

Then we have the relation

$$\langle A^N(X), Y \rangle = \langle B(X, Y), N \rangle = \langle B(Y, X), N \rangle = \langle A^N(Y), X \rangle.$$

This means that the shape operator $A^N : \Gamma(T\Sigma) \to \Gamma(T\Sigma)$ is self-adjoint w.r.t. g.

In the following, we put a weight function f on the ambient space $\mathbb{R}^m$, that is we consider a smooth metric measure space $(\mathbb{R}^m, \bar{g}, e^{-f} \operatorname{vol}^{\bar{g}})$, where $f \in C^\infty(\mathbb{R}^m)$ and $\operatorname{vol}^{\bar{g}}$ is the Riemannian volume measure on $(\mathbb{R}^m, \bar{g})$. An f-minimal submanifold $\mathbf{x} : \Sigma^n \to \mathbb{R}^m$ is defined to be a critical point of the weighted volume functional

$$\operatorname{vol}_f^g(\Sigma, \mathbf{x}) := \int_\Sigma e^{-\mathbf{x}^* f} \operatorname{dvol}^g = \int_\Sigma e^{-f} \operatorname{dvol}^g,$$

where vol^g is the Riemannian volume measure on (Σ^n, g). In the following, we will omit $\bar{g}$ and g from the notations $\operatorname{vol}^{\bar{g}}$ and vol^g since we can easily identify them by the context.

Now we consider a normal variation $\mathbf{x}_t : \Sigma^n \to (\mathbb{R}^m, \bar{g}, e^{-f} \operatorname{vol})$ of an embedding $\mathbf{x}_0 = \mathbf{x}$ with an associated variation vector field $N := \frac{d}{dt}\big|_{t=0} \mathbf{x}_t \in \Gamma(N\Sigma)$. Then we have the first variation formula

$$\delta\Sigma_f(N) := \frac{d}{dt}\Big|_{t=0} \operatorname{vol}_f(\Sigma, \mathbf{x}_t) = -\int_\Sigma \langle H_f, N \rangle e^{-f} \operatorname{dvol},$$

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where

$$H_f := H + \nabla^\perp f$$

is called the *weighted mean curvature vector field.* By the first variation formula, the definition of the f-minimality is equivalent to say that $\mathbf{x}$ satisfies the equation $H_f = 0$.

Remark 1. If we take f as

$$f = \frac{|\bullet|^2}{4},$$

then the f-minimal equation becomes

$$H_f = H + \frac{\mathbf{x}^\perp}{2} = 0$$

since $(\overline{\nabla} f)(\mathbf{x}) = \frac{\mathbf{x}}{2}$. This is nothing but the definition of *self-shrinkers.* The standard sphere $\mathbb{S}^n(\sqrt{2n}) \subset \mathbb{R}^m$ and cylinders $\mathbb{S}^k(\sqrt{2k}) \times \mathbb{R}^{n-k} \subset \mathbb{R}^m$ $(k = 1, \dots, n-1)$ are basic examples of self-shrinkers.

1.2 Second variation formula for f-minimal submanifolds

Let $\mathbf{x} : \Sigma^n \to \mathbb{R}^m$ be a smoothly embedded closed f-minimal submanifold and $\mathbf{x}_t : \Sigma^n \to \mathbb{R}^m$ be its normal variation with $\mathbf{x}_0 = \mathbf{x}$ and associated variation vector field $N \in \Gamma(N\Sigma)$. Then we know the second variation formula (see [7] or [6]):

$$\delta^2\Sigma_f(N,N) := \frac{d^2}{dt^2}\Big|_{t=0} \mathrm{vol}_f(\Sigma, \mathbf{x}_t) = \int_\Sigma \left\langle \Delta^\perp N + \nabla^\perp_{\nabla f} N - \mathcal{A}(N) - (\overline{\nabla}_N \overline{\nabla} f)^\perp, N \right\rangle e^{-f}\,\mathrm{dvol},$$

where $\nabla^\perp$ is the normal connection, $\Delta^\perp N$ is the *normal Laplacian* acting on $\Gamma(N\Sigma)$ and $\mathcal{A}$ is *Simons' operator* defined by

$$\mathcal{A}(N) := \sum_{i=1}^n B(A^N(e_i), e_i) = \sum_{i,j=1}^n \langle B(e_i, e_j), N\rangle B(e_i, e_j), \quad N \in \Gamma(N\Sigma),$$

for some local orthonormal frame $\{e_i\}$ on (Σ^n, g). Note also that our definition of $\Delta^\perp$ is

$$\Delta^\perp N := -\sum_{i=1}^n (\nabla^\perp_{e_i}\nabla^\perp_{e_i} N - \nabla^\perp_{\nabla_{e_i} e_i} N).$$

Define the weighted normal Laplace operator by

$$\Delta^\perp_f N := \Delta^\perp N + \nabla^\perp_{\nabla f} N, \quad N \in \Gamma(N\Sigma),$$

then from the second variation formula, we know that the *Jacobi operator* (or *stability operator*) for our variational problem can be written as

$$LN = \Delta^\perp_f N - \mathcal{A}(N) - (\overline{\nabla}_N \overline{\nabla} f)^\perp, \quad N \in \Gamma(N\Sigma). \tag{1}$$

Remark 2. Since

$$\mathrm{Ric}_f = \mathrm{Ric} + \mathrm{Hess}_f = \mathrm{Hess}_f,$$

the Jacobi operator for f-minimal submanifolds can be written as

$$LN = \Delta^\perp_f N - \mathcal{A}(N) - \mathrm{Ric}_f(N)^\perp,$$

where $\mathrm{Ric}_f(X)$ for $X \in \Gamma(T\mathbb{R}^m)$ is the uniquely determined vector field on $\mathbb{R}^m$ by the relation

$$\langle \mathrm{Ric}_f(X), Y\rangle = \mathrm{Ric}_f(X, Y) \quad \text{for all} \quad Y \in \Gamma(T\mathbb{R}^m).$$

Remark 3. For a self-shrinker $\mathbf{x} : \Sigma^n \to \mathbb{R}^m$ with $f = \frac{|\bullet|^2}{4}$, the Jacobi operator is

$$LN = \Delta_f^\perp N - \mathcal{A}(N) - \frac{1}{2}N, \quad N \in \Gamma(N\Sigma)$$

and the second variation formula becomes

$$\delta^2\Sigma_f(N,N) = \int_\Sigma \langle \Delta_f^\perp N - \mathcal{A}(N) - \frac{1}{2}N, N\rangle e^{-f}\,\mathrm{dvol}. \tag{2}$$

1.3 Some formulas for f-minimal immersions

We will denote by $\mathcal{P}$ the set of all parallel vector fields on $\mathbb{R}^m$, that is, $V \in \mathcal{P}$ is nothing but a constant vector field on $\mathbb{R}^m$. Of course, we can identify a vector $V \in \mathbb{R}^m$ with a parallel vector field $V \in \mathcal{P}$. By a direct computation, we have the following.

Lemma 4. *Let $\mathbf{x} : \Sigma^n \to \mathbb{R}^m$ be a submanifold (not necessarily be f-minimal), $V \in \mathcal{P}$ and $X \in \Gamma(T\Sigma)$. Then the following relations hold:*

$$\nabla_X V^\top = A^{V^\perp}(X) \tag{3}$$

$$\nabla_X^\perp V^\perp = -B(X, V^\top) \tag{4}$$

Using (3), (4) and the Codazzi equation, we obtain the following Lemma.

Lemma 5. *Let $\mathbf{x} : \Sigma^n \to \mathbb{R}^m$ be an f-minimal submanifold and $V \in \mathcal{P}$. Then we have*

$$\Delta_f^\perp V^\perp = \mathcal{A}(V^\perp) - (\overline{\nabla}_{V^\top}\overline{\nabla} f)^\perp. \tag{5}$$

Moreover, if $\mathbf{x} : \Sigma^n \to \mathbb{R}^m$ is a self-shrinker, then

$$\Delta_f^\perp V^\perp = \mathcal{A}(V^\perp). \tag{6}$$

2 Hodge Laplacian for f-minimal immersions

Let $\Omega^1(\Sigma)$ be the space of 1-forms. The *Hodge Laplacian* acting on $\Omega^1(\Sigma)$ is given by

$$\Delta^{[1]} := d\delta + \delta d.$$

Moreover, we consider

$$\delta_f := \delta + i_{\nabla f}.$$

Then, the *weighted Hodge Laplacian* acting on $\Omega^1(\Sigma)$ is defined by

$$\Delta_f^{[1]} := d\delta_f + \delta_f d.$$

By the classical Weitzenböck formula, we have the following.

Lemma 6 ([4])**.** *Let $(\Sigma^n, g, e^{-f}\,\mathrm{vol})$ be a weighted Riemannian manifold and $\omega \in \Omega^1(\Sigma)$. Then*

$$\Delta_f^{[1]}\omega = \Delta_f\omega + \mathrm{Ric}_f^\Sigma(\omega), \tag{7}$$

where $\Delta_f\omega = \Delta\omega + (\nabla\omega)(\nabla f, \cdot) \in \Omega^1(\Sigma)$ *and* $\mathrm{Ric}_f^\Sigma(\omega) = \mathrm{Ric}_f^\Sigma(\omega^\sharp, \cdot) = \mathrm{Ric}^\Sigma(\omega^\sharp, \cdot) + \mathrm{Hess}_f^\Sigma(\omega^\sharp, \cdot) \in \Omega^1(\Sigma)$.

If $X \in \Gamma(T\Sigma)$, we can consider its dual 1-form $X^\flat \in \Omega^1(\Sigma)$. The weighted Hodge Laplacian of X is defined as the unique vector field satisfying

$$g(\Delta_f^{[1]} X, Y) = (\Delta_f^{[1]} X^\flat)(Y), \quad Y \in \Gamma(T\Sigma).$$

Or equivalently, we can say that

$$\Delta_f^{[1]} X := (\Delta_f^{[1]} X^\flat)^\sharp.$$

Then the above Weitzenböck formula becomes

$$\Delta_f^{[1]} X = \Delta_f X + \mathrm{Ric}_f^\Sigma(X), \quad X \in \Gamma(T\Sigma), \tag{8}$$

where $\mathrm{Ric}_f^\Sigma(X) = (\mathrm{Ric}_f^\Sigma(X, \cdot))^\sharp \in \Gamma(T\Sigma)$. Now, using the Gauss equation, the Weitzenböck formula (8) for f-minimal submanifolds yields the following.

Lemma 7. *Let $\mathbf{x} : \Sigma^n \to \mathbb{R}^m$ be an f-minimal submanifold. For any $X \in \Gamma(T\Sigma)$, we have*

$$\Delta_f^{[1]} X = \Delta_f X + (\overline{\nabla}_X \overline{\nabla} f)^\top - \sum_{i=1}^n A^{B(e_i, X)}(e_i). \tag{9}$$

In particular, if $\mathbf{x} : \Sigma^n \to \mathbb{R}^m$ is a self-shrinker, then the following holds:

$$\Delta_f^{[1]} X = \Delta_f X + \frac{1}{2} X - \sum_{i=1}^n A^{B(e_i, X)}(e_i). \tag{10}$$

On the other hand, using (3), (4) and the Codazzi equation, we can compute the Laplacian of the tangent part of a parallel vector field $V \in \mathcal{P}$.

Lemma 8. *Let $\mathbf{x} : \Sigma^n \to \mathbb{R}^m$ be a smooth submanifold (not necessarily be f-minimal) and $V \in \mathcal{P}$, then it holds that*

$$\Delta V^\top = \sum_{i=1}^n \left\{ A^{B(e_i, V^\top)}(e_i) - \langle \nabla_{e_i}^\perp H, V^\perp \rangle e_i \right\}. \tag{11}$$

In addition, for an f-minimal submanifold $\mathbf{x} : \Sigma^n \to \mathbb{R}^m$, we have

$$\Delta_f V^\top := \Delta V^\top + \nabla_{\nabla f} V^\top = \sum_{i=1}^n A^{B(e_i, V^\top)}(e_i) + (\overline{\nabla}_{V^\perp} \overline{\nabla} f)^\top. \tag{12}$$

and

$$\Delta_f^{[1]} V^\top = (\overline{\nabla}_V \overline{\nabla} f)^\top = (\overline{\nabla}_{V^\top} \overline{\nabla} f)^\top + (\overline{\nabla}_{V^\perp} \overline{\nabla} f)^\top. \tag{13}$$

Moreover, if $\mathbf{x} : \Sigma^n \to \mathbb{R}^m$ is a self-shrinker, then we have

$$\Delta_f^{[1]} V^\top = \frac{1}{2}(V^\top)^\top + \cancel{\frac{1}{2}(V^\perp)^\top} = \frac{1}{2} V^\top. \tag{14}$$

Finally, using f-minimality, (3), (9) and (12), we can compute the following.

Lemma 9. *Let $\mathbf{x} : \Sigma^n \to \mathbb{R}^m$ be a smooth submanifold (not necessarily be f-minimal) and $V \in \mathcal{P}$. Then for any $X \in \Gamma(T\Sigma)$, we have*

$$\nabla \langle V, X \rangle = \sum_{i=i}^n \langle V, \nabla_{e_i} X \rangle e_i + A^{V^\perp}(X). \tag{15}$$

Moreover, if $\mathbf{x} : \Sigma^n \to \mathbb{R}^m$ *is* f*-minimal then we have*

$$\begin{aligned} \Delta_f \langle V, X\rangle &:= \Delta\langle V, X\rangle + \langle \nabla f, \nabla\langle V, X\rangle\rangle \\ &= -\operatorname{Hess}_f(V^\top, X) + \operatorname{Hess}_f(V^\perp, X) + \langle V, \Delta_f^{[1]} X\rangle + 2A^2(V^\top, X) - 2\langle A^{V^\perp}, \nabla X\rangle, \end{aligned} \tag{16}$$

where

$$\langle A^{V^\perp}, \nabla X\rangle := \sum_{i=1}^{n} \langle A^{V^\perp}(e_i), (\nabla X)(e_i)\rangle.$$

For a self-shrinker, it additionally holds that

$$\Delta_f\langle V, X\rangle = -\frac{1}{2}\langle V, X\rangle + \langle V, \Delta_f^{[1]} X\rangle + 2A^2(V^\top, X) - 2\langle A^{V^\perp}, \nabla X\rangle.$$

3 Index estimate for f-minimal submanifolds

3.1 Test functions

For $X \in \Gamma(T\mathbb{R}^m)$ and $V, W \in \mathcal{P}$, we define the *test function*

$$\Phi_{V,W}(X) := \langle W, X\rangle V^\perp - \langle V, X\rangle W^\perp.$$

Sometimes we just write $\Phi(X)$ omitting V, W for simplicity. Using formulas (4), (5) and (16), we can compute

Proposition 10. *Let* $\mathbf{x} : \Sigma^n \to \mathbb{R}^m$ *be an* f*-minimal submanifold,* $X \in \Gamma(T\Sigma)$ *and* $V, W \in \mathcal{P}$*. Then we have*

$$\begin{aligned} L\Phi(X) = &-\Phi((\overline{\nabla}_X \overline{\nabla} f)^\top) - (\overline{\nabla}_{\Phi(X)} \overline{\nabla} f)^\perp + \Phi(\Delta_f^{[1]} X) \\ &+ 2\Phi(A^{B(X,e_i)}(e_i)) + 2B\left(\nabla\langle W, X\rangle, V^\top\right) - 2B(\nabla\langle V, X\rangle, W^\top) + \mathcal{Z}, \end{aligned}$$

where

$$\mathcal{Z} := -2\Phi(B(e_i, \nabla_{e_i} X)) + \Phi((\overline{\nabla}_X \overline{\nabla} f)^\perp) - \langle W, X\rangle(\overline{\nabla}_{V^\top} \overline{\nabla} f)^\perp + \langle V, X\rangle(\overline{\nabla}_{W^\top} \overline{\nabla} f)^\perp.$$

3.2 Integration formulas

Let $\mathcal{U} \subset \mathcal{P}$ be a set of all parallel vector fields with unit length. Then $\mathcal{U}$ can be identified with the unit hypersphere $\mathbb{S}^{m-1} \subset \mathbb{R}^m$. We put the normalized volume measure

$$\mathrm{d}\hat{\sigma} := \frac{1}{|\mathbb{B}^m|}\mathrm{d}\sigma \quad \text{on} \quad \mathcal{U},$$

where $|\mathbb{B}^m|$ is the volume of the unit ball $\mathbb{B}^m \subset \mathbb{R}^m$ and $\mathrm{d}\sigma$ is the standard volume element on the sphere. The following proposition is an elementary application of the divergence theorem.

Proposition 11. *For any vector fields* $X, Y \in \Gamma(T\mathbb{R}^m)$*, we have*

$$\int_{\mathcal{U}} \langle X, V\rangle\langle Y, V\rangle \mathrm{d}\hat{\sigma}(V) = \langle X, Y\rangle.$$

Lemma 12. *For any* $X, Y \in \Gamma(T\mathbb{R}^m)$*, the following holds:*

$$\int_{\mathcal{U}} |V^\perp|^2 \mathrm{d}\hat{\sigma}(V) = \operatorname{codim}(\Sigma) = m - n,$$

$$\int_{\mathcal{U}\times\mathcal{U}} \langle \Phi_{V,W}(X), \Phi_{V,W}(Y)\rangle \mathrm{d}\hat{\sigma}(V)\mathrm{d}\hat{\sigma}(W) = 2(m-n)\langle X, Y\rangle - 2\langle X, Y^\perp\rangle.$$

As a direct consequence of this lemma, for any $X \in \Gamma(T\Sigma)$ and $N \in \Gamma(N\Sigma)$, we see

$$\int_{\mathcal{U}\times\mathcal{U}} \langle \Phi(X), \Phi(X)\rangle \mathrm{d}\hat{\sigma}(V)\mathrm{d}\hat{\sigma}(W) = 2(m-n)|X|^2, \quad \int_{\mathcal{U}\times\mathcal{U}} \langle \Phi(X), \Phi(N)\rangle \mathrm{d}\hat{\sigma}(V)\mathrm{d}\hat{\sigma}(W) = 0.$$

Now we list other integral formulas which will play important roles in our index estimate. These are derived by Lemma 12, identity (15) and Proposition 11.

Lemma 13. *For any $X \in \Gamma(T\Sigma)$ and $N \in \Gamma(T\Sigma)$, it holds that*

$$\begin{aligned}
-\int_{\mathcal{U}\times\mathcal{U}} \langle \Phi(X), \Phi((\overline{\nabla}_X \overline{\nabla} f)^\top)\rangle &= -2(m-n)\,\mathrm{Hess}_f(X,X), \\
\int_{\mathcal{U}\times\mathcal{U}} \langle \Phi(X), \Phi(\Delta_f^{[1]} X)\rangle &= 2(m-n)\langle X, \Delta_f^{[1]} X\rangle, \\
2\int_{\mathcal{U}\times\mathcal{U}} \langle \Phi(X), \Phi(A^{B(e_i,X)}(e_i))\rangle &= 4(m-n)A^2(X,X), \\
2\int_{\mathcal{U}\times\mathcal{U}} \langle B(\nabla\langle W,X\rangle, V^\top) - B(\nabla\langle V,X\rangle, W^\top)\rangle &= -4A^2(X,X), \\
-\int_{\mathcal{U}\times\mathcal{U}} \langle \Phi(X), (\overline{\nabla}_{\Phi(X)} \overline{\nabla} f)^\perp\rangle &= -2|X|^2 \sum_{\alpha=1}^{m-n} \mathrm{Hess}_f(\nu_\alpha, \nu_\alpha), \\
\int_{\mathcal{U}\times\mathcal{U}} \langle \Phi(X), \mathcal{Z}\rangle &= 0.
\end{aligned}$$

Remark 14. The Gauss equation and f-minimality of $\mathbf{x} : \Sigma^n \to \mathbb{R}^m$ implies

$$A^2(X,X) = \mathrm{Hess}_f(X,X) - \mathrm{Ric}_f^\Sigma(X,X).$$

3.3 Eigenvalue comparison theorem for f-minimal immersions

Let $\mathbf{x} : \Sigma^n \to (\mathbb{R}^m, \bar{g}, e^{-f}\,\mathrm{vol})$ be a compact smoothly immersed f-minimal submanifold. In this section, we assume

$$\mathrm{codim}(\Sigma) = m - n = 2, \quad \mathrm{Hess}_f = \mathrm{Ric}_f \geq K > 0, \quad \mathrm{Ric}_f^\Sigma \geq -\kappa, \quad K > \kappa \geq 0.$$

Let $\{N_i\}_{i=1}^\infty$ be an orthonormal system of $\Gamma(N\Sigma)$ which consists of eigensections of the stability operator L with corresponding eigenvalues $\{\mu_i\}_{i=1}^\infty$. For a positive interger $k \geq 1$, we want to find a non-zero vector field $X \in \Gamma(T\Sigma)$ satisfying

$$\int_\Sigma \langle \Phi_{V,W}(X), N_1\rangle e^{-f}\,\mathrm{dvol} = \cdots = \int_\Sigma \langle \Phi_{V,W}(X), N_{k-1}\rangle e^{-f}\,\mathrm{dvol} = 0 \tag{17}$$

for any pair of $(V,W) \in \mathcal{P} \times \mathcal{P}$. Since $\Phi_{V,W}(\cdot)$ is skew-symmetric for $(V,W) \in \mathcal{P}\times\mathcal{P}$, finding a non-zero solution X to (17) is equivalent to find a non-zero solution to a homogeneous system with

$$\tilde{d}(k) = \frac{1}{2}m(m-1)(k-1)$$

equations and d unknown variables. Of course, such a solution exists whenever the number of valuables $d \geq \tilde{d}(k) + 1$. Let

$$d(k) := \tilde{d}(k) + 1 = \frac{1}{2}m(m-1)(k-1) + 1.$$

Now we consider a subspace

$$E^d := \bigoplus_{i=1}^d \mathcal{V}_i \subset \Gamma(T\Sigma)$$

which is the direct sum of first d eigenspaces $\mathcal{V}_1, \dots, \mathcal{V}_d$ of $\Delta_f^{[1]}$ with corresponding eigenvalues $\lambda_1, \dots, \lambda_d$. By the argument above, we can find a non-zero vector field $X \in E^{d(k)}$ satisfying (17). Then by the min-max principle, we have

$$\begin{aligned}\mu_k \int_\Sigma |\Phi(X)|^2 e^{-f}\,\mathrm{dvol} &\le \int_\Sigma \langle \Phi(X), L\Phi(X)\rangle e^{-f}\,\mathrm{dvol} \\ &\le \int_\Sigma \big\langle \Phi(X), -\Phi((\overline{\nabla}_X \overline{\nabla} f)^\top) - (\overline{\nabla}_{\Phi(X)} \overline{\nabla} f)^\perp + \Phi(\Delta_f^{[1]} X) \\ &\qquad + 2\Phi(A^{B(e_i,X)}(e_i)) + 2B(\nabla\langle W, X\rangle, V^\top) - 2B(\nabla\langle V, X\rangle, W^\top) + \mathcal{Z}\big\rangle e^{-f}\,\mathrm{dvol}\end{aligned}$$

Integrating both sides of this inequality w.r.t. $(V,W) \in \mathcal{U} \times \mathcal{U}$, Fubini's theorem and Lemma 13 imply

$$\begin{aligned}&2(m-n)\mu_k \int_\Sigma |X|^2 e^{-f}\,\mathrm{dvol} \\ &\quad\le \{2(m-n)-4\} \int_\Sigma \mathrm{Hess}_f(X,X) e^{-f}\,\mathrm{dvol} - 2\sum_{\alpha=1}^{m-n} \int_\Sigma |X|^2 \,\mathrm{Hess}_f(\nu_\alpha,\nu_\alpha) e^{-f}\,\mathrm{dvol} \\ &\qquad + 2(m-n) \int_\Sigma \langle X, \Delta_f^{[1]} X\rangle e^{-f}\,\mathrm{dvol} - 4(m-n-1) \int_\Sigma \mathrm{Ric}_f^\Sigma(X,X) e^{-f}\,\mathrm{dvol} \\ &\quad\le \{2(m-n)-4\} \int_\Sigma \mathrm{Hess}_f(X,X) e^{-f}\,\mathrm{dvol} - 2\sum_{\alpha=1}^{m-n} \int_\Sigma |X|^2 \,\mathrm{Hess}_f(\nu_\alpha,\nu_\alpha) e^{-f}\,\mathrm{dvol} \\ &\qquad + 2(m-n)\lambda_{d(k)} \int_\Sigma |X|^2 e^{-f}\,\mathrm{dvol} - 4(m-n-1) \int_\Sigma \mathrm{Ric}_f^\Sigma(X,X) e^{-f}\,\mathrm{dvol}\,. \quad (18)\end{aligned}$$

where we have used

$$\int_\Sigma \langle X, \Delta_f^{[1]} X\rangle e^{-f}\,\mathrm{dvol} \le \lambda_{d(k)} \int_\Sigma |X|^2 e^{-f}\,\mathrm{dvol},$$

since $X \in E^{d(k)}$. Now we will use the assumption

$$\mathrm{codim}(\Sigma) = m-n = 2, \quad \mathrm{Hess}_f \ge K > 0, \quad \mathrm{Ric}_f^\Sigma \ge -\kappa, \quad K > \kappa \ge 0$$

to obtain

$$4\mu_k \int_\Sigma |X|^2 e^{-f}\,\mathrm{dvol} \le -4(K-\kappa) \int_\Sigma |X|^2 e^{-f}\,\mathrm{dvol} + 4\lambda_{d(k)} \int_\Sigma |X|^2 e^{-f}\,\mathrm{dvol},$$

hence

$$\mu_k \le -(K-\kappa) + \lambda_{d(k)}.$$

This completes the proof of the eigenvalue comparison theorem between L and $\Delta_f^{[1]}$.

Theorem 15. *Let* $\mathbf{x} : \Sigma^n \to (\mathbb{R}^m, \bar{g}, e^{-f}\,\mathrm{vol})$ *be a closed smoothly embedded f-minimal submanifold. Assume that*

$$\mathrm{codim}(\Sigma) = m-n = 2, \quad \mathrm{Hess}_f = \mathrm{Ric}_f \ge K > 0, \quad \mathrm{Ric}_f^\Sigma \ge -\kappa, \quad K > \kappa \ge 0.$$

Then for all $k \ge 1$, we have

$$\mu_k \le -(K-\kappa) + \lambda_{d(k)} \quad \textit{with} \quad d(k) := \frac{m(m-1)(k-1)}{2} + 1,$$

where $\{\mu_i\}$ and $\{\lambda_j\}$ are eigenvalues of the operators L and $\Delta_f^{[1]}$, respectively.

If $\mathbf{x} : \Sigma \to (\mathbb{R}^m, \bar{g}, e^{-f}\,\mathrm{vol})$ with $f = \frac{|\bullet|^2}{4}$ is a self-shrinker, then

$$\mathrm{Hess}_f = \frac{1}{2}\bar{g}.$$

Hence, without codimension restriction, (18) reduces to

$$\begin{aligned} &2(m-n)\mu_k \int_\Sigma |X|^2 e^{-f}\,\mathrm{dvol} \\ &\quad \le \{-2 + 2(m-n)\lambda_{d(k)}\} \int_\Sigma |X|^2 e^{-f}\,\mathrm{dvol} - 4(m-n-1)\int_\Sigma \mathrm{Ric}_f^\Sigma(X,X) e^{-f}\,\mathrm{dvol}\,. \end{aligned}$$

Moreover, if $\mathrm{Ric}_f^\Sigma \ge \kappa$, then we have

$$\mu_k \le \lambda_{d(k)} - \frac{1}{m-n} + \frac{2(m-n-1)\kappa}{m-n}.$$

Note that the sum of the last two terms is negative if

$$0 \le \kappa < \frac{1}{2(m-n-1)}.$$

Corollary 16. *Let $\mathbf{x} : \Sigma^n \to (\mathbb{R}^m, \bar{g}, e^{-f}\,\mathrm{vol})$ be a closed embedded self-shrinker with $f = \frac{|\bullet|^2}{4}$. Assume that*

$$\mathrm{Ric}_f^\Sigma \ge -\kappa, \quad \kappa \ge 0.$$

Then for all $k \ge 1$, we have

$$\mu_k \le \lambda_{d(k)} - \frac{1}{m-n} + \frac{2(m-n-1)\kappa}{m-n}, \quad d(k) := \frac{m(m-1)(k-1)}{2} + 1,$$

where $\{\mu_i\}$ and $\{\lambda_j\}$ are eigenvalues of the operators L and $\Delta_f^{[1]}$, respectively.

3.4 Index estimate for f-minimal submanifolds

Finally, we use the same method as Impera–Rimoldi–Savo [4] to estimate the Morse index of L below by the first Betti number $b_1 = b_1(\Sigma)$. For any positive number $a > 0$, let

$$\mathcal{N}_{\Delta_f}(a) := \#\{\text{positive eigenvalues of } \Delta_f \text{ which are less than } a\}.$$

Theorem 17. *Let $\mathbf{x} : \Sigma^n \to (\mathbb{R}^{n+2}, \bar{g}, e^{-f}\,\mathrm{vol})$ be a closed smoothly embedded f-minimal submanifold of codimention two. Assume that*

$$\mathrm{Ric}_f \ge K > 0, \quad \mathrm{Ric}_f^\Sigma \ge -\kappa \quad (K > \kappa \ge 0).$$

Then we have

$$\mathrm{ind}_f(\Sigma) \ge \frac{2}{(n+1)(n+2)} \left\{\mathcal{N}_{\Delta_f}(K-\kappa) + b_1(\Sigma)\right\}.$$

Proof. Let $l \in \mathbb{Z}$ be any fixed positive integer and choose $\beta \in \mathbb{Z}$ such that

$$\frac{(l-1)(n+1)(n+2)}{2} \le \beta \le \frac{l(n+1)(n+2)}{2}.$$

Then it is easy to see that the largest integer $k \in \mathbb{Z}$ satisfying

$$d(k) := \frac{(n+1)(n+2)}{2}(k-1) + 1 \leq \beta$$

is $k = l$. Therefore, for such k, we have $k \geq \frac{2\beta}{(n+1)(n+2)}$.

Now let

$$\beta := \#\{\text{eigenvalues of } \Delta_f^{[1]} \text{ which are less than } K - \kappa\}.$$

Choose $k \in \mathbb{Z}$ as the largest integer which satisfies $d(k) \leq \beta$. Above observation shows that

$$k \geq \frac{2\beta}{(n+1)(n+2)}.$$

Then Theorem 15 implies that

$$\mu_1 \leq \cdots \leq \mu_k \leq -(K-\kappa) + \lambda_{d(k)} \leq -(K-\kappa) + \lambda_\beta < 0.$$

Therefore, the Morse index, i.e., the number of negative eigenvalues of L is estimated as

$$\mathrm{ind}_f(\Sigma) \geq k \geq \frac{2\beta}{(n+1)(n+2)}. \tag{19}$$

Now we will associate β with $b_1(\Sigma)$ and $\mathcal{N}_{\Delta_f}(K-\kappa)$. Let $\gamma := \mathcal{N}_{\Delta_f}(K-\kappa)$ and $u_1, \ldots, u_\gamma \in C^\infty(\Sigma)$ be L^2_f-orthogonal eigenfunctions of Δ_f with associated positive eigenvalues which are less than $K - \kappa > 0$. By the Stokes formula, we compute

$$\int_\Sigma \langle \mathrm{d}u_i, \mathrm{d}u_j \rangle e^{-f}\, \mathrm{dvol} = \int_\Sigma (\Delta_f u_i) u_j\, e^{-f}\, \mathrm{dvol} = \lambda_i \delta_{ij},$$

so that $\mathrm{d}u_1, \ldots, \mathrm{d}u_\gamma \in \Omega^1(\Sigma)$ are also orthogonal. Moreover, as the weighted Hodge Laplacian $\Delta_f^{[1]}$ commutes with exterior derivative d, we see

$$\Delta_f^{[1]} \mathrm{d}u_i = \mathrm{d}\Delta_f u_i = \lambda_i \mathrm{d}u_i,$$

i.e., $\mathrm{d}u_1, \ldots, \mathrm{d}u_\gamma$ are eigenforms of $\Delta_f^{[1]}$ associated to the eigenvalues $0 < \lambda_1 \leq \cdots \leq \lambda_\gamma < K - \kappa$. As $\mathrm{d}u_1, \ldots, \mathrm{d}u_\gamma$ are all perpendicular to the space of f-harmonic 1-forms $\mathcal{H}^1_f(\Sigma)$ whose dimension coincides with $b_1(\Sigma)$, it follows that $\beta \geq b_1(\Sigma) + \gamma$. Combining this with (19), we have

$$\mathrm{ind}_f(\Sigma) \geq \frac{2}{(n+1)(n+2)}(b_1(\Sigma) + \gamma)$$

which is the desired one. □

3.5 Index estimate for self-shrinkers in higher codimension

In this subsection, we will estimate the Morse index of self-shrinkers in higher codimension below by the first Betti number $b_1 = b_1(\Sigma)$.

Corollary 18. *Let $\mathbf{x} : \Sigma^n \to (\mathbb{R}^m, \bar{g}, e^{-f}\,\mathrm{vol})$ be a closed embedded self-shrinker with $f = \frac{|\bullet|^2}{4}$. Assume that*

$$\mathrm{Ric}_f^\Sigma > \frac{-1}{2(m-n-1)}.$$

Then we have

$$\mathrm{ind}_f(\Sigma) \geq \frac{2\, b_1(\Sigma)}{m(m-1)}.$$

Proof. Let $k \in \mathbb{Z}$ be the lagest integer satisfying $d(k) \leq b_1(\Sigma)$. Then by Corollary 16 and the lower bound on Ric_f^Σ imply $\mu_k < 0$. So,

$$\mathrm{ind}_f(\Sigma) \geq k.$$

On the other hand, as in the proof of Theorem 15, we have $k \geq \frac{2}{m(m-1)}$. Therefore, we can estimate

$$\mathrm{ind}_f(\Sigma) \geq k \geq \frac{2\, b_1(\Sigma)}{m(m-1)}.$$

This completes the proof. □

Remark 19. The lower bound assumption on Ric_f^Σ is equivalent to say that $B^2(X,X) < |X|^2$, which is very restrictive. However, in codimension one, such an assumption on Ric_f^Σ can be removed (see [4]). Moreover, for a codimension one self-shrinker $\mathbf{x} : \Sigma^n \to \mathbb{R}^{n+1}$, Impera–Rimoldi–Savo showed much better estimate that

$$\mathrm{ind}_f(\Sigma) \geq \frac{2\, b_1(\Sigma)}{n(n+1)} + n + 1.$$

This follows from the eigenvalue comparison theorem for codimension one self-shrinkers and the fact that all self-shrinkers (even if they are in higher codimension) satisfy

$$Ly^\perp = -\frac{1}{2} y^\perp$$

for any vector $y \in \mathbb{R}^{n+1}$.

4 Examples

Here we compute Ric_f^Σ explicitly. Recall that on a self-shrinker $\mathbf{x} : \Sigma^n \to (\mathbb{R}^m, \bar{g}, e^{-f}\,\mathrm{vol})$ with $f = \frac{|\bullet|^2}{4}$, we have

$$\mathrm{Hess}_f^\Sigma(X,X) = \mathrm{Hess}_f(X,X) + \langle B(X,X), \nabla^\perp f\rangle = \frac{1}{2}|X|^2 + \frac{1}{2}\langle B(X,X), \mathbf{x}^\perp\rangle$$

for any $X \in \Gamma(T\Sigma)$.

Example 20. Let $\Sigma^n = \mathbb{S}^n(\sqrt{2n}) \subset \mathbb{R}^m$. Since the normal part $\mathbf{x}^\perp$ of the position vector $\mathbf{x} \in \mathbb{S}^n$ is $\mathbf{x}$ itself,

$$\begin{aligned} \mathrm{Hess}_f^\Sigma(X,X) &= \frac{1}{2}|X|^2 + \frac{1}{2}\langle \overline{\nabla}_X X, \mathbf{x}^\perp\rangle \\ &= \frac{1}{2}|X|^2 - \frac{1}{2}\langle X, \overline{\nabla}_X \mathbf{x}^\perp\rangle \\ &= \frac{1}{2}|X|^2 - \frac{1}{2}\langle X, \overline{\nabla}_X X\rangle \\ &= \frac{1}{2}|X|^2 - \frac{1}{2}\langle X, X\rangle = 0. \end{aligned}$$

So that, we have

$$\mathrm{Ric}_f^\Sigma(X,X) = \mathrm{Ric}^\Sigma(X,X) = \frac{n-1}{2n}|X|^2.$$

This can be generalized to *spherical self-shrinkers*. A spherical self-shrinker is a self-shrinker $\mathbf{x} : \Sigma^n \to \mathbb{R}^m$ which is contained in some sphere $\mathbb{S}^l \subset \mathbb{R}^m$ with appropriate radius. It is well-known that being a spherical self-shrinker is equivalent to being minimal submanifold of the sphere.

Example 21. Let $\mathbf{x} : \Sigma^n \to \mathbb{R}^m$ be a spherical self-shrinker. Then again the position vector of Σ^n satisfies $\mathbf{x} = \mathbf{x}^\perp$ since $\mathbf{x}$ is outward pointing to the sphere. By the same computation we have $\mathrm{Hess}_f^\Sigma(X, X) = 0$, so that

$$\mathrm{Ric}_f^\Sigma(X, X) = \mathrm{Ric}^\Sigma(X, X).$$

For example, Clifford torus $\mathbf{x} : \mathbb{T}^2 \to \mathbb{R}^4$ defined by

$$x(\theta, \phi) = \sqrt{2}(\cos\theta, \sin\theta, \cos\phi, \sin\phi) \in \mathbb{S}^1(\sqrt{2}) \times \mathbb{S}^1(\sqrt{2}) \subset \mathbb{S}^3(2) \subset \mathbb{R}^4$$

is a spherical self-shrinker. Since $\mathrm{Ric}^\Sigma = 0$ on the Clifford torus, we see

$$\mathrm{Ric}_f^\Sigma(X, X) = \mathrm{Ric}^\Sigma(X, X) = 0.$$

Now, we can apply Corollary 18 to obtain

$$\mathrm{ind}_f(\Sigma) \geq \frac{2\, b_1(\mathbb{T}^2)}{4 \cdot 3} = \frac{1}{3}.$$

The Morse index is an integer, so we conclude that $\mathrm{ind}_f(\Sigma) \geq 1$. Unfortunately, this estimate is useless. In fact, for every self-shrinkers in $\mathbb{R}^m$, we already know $\mathrm{ind}_f(\Sigma) \geq m + 1$ because the mean curvature vector field H and normal part $y^\perp$ of any vector $y \in \mathbb{R}^m$ have negative eigenvalues -1 and $-1/2$, respectively.

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