TOTALLY COMPLEX SUBMANIFOLDS AND R-SPACES

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A submanifold N immersed in a quaternionic Kähler manifold (M^{4n}, g, Q) ([4]) is called a *totally complex submanifold* if N has an open covering $N = \bigcup_\lambda U_\lambda$ such that there exists a smooth section J^λ of the quaternionic Kähler structure Q on each neighborhood U_λ such that $J^\lambda \circ J^\lambda = -\mathrm{Id}_{T_pM}$, $J^\lambda(T_pN) = T_pN$ and $\tilde{\nabla}_X J^\lambda = 0$ $(\forall X \in T_pN, \forall p \in U_\lambda)$ ([3], [6]). They form a special class of minimal submanifolds. Note that a totally complex submanifold is of even dimension 2ℓ and $\ell \leq n$. If $\ell = n$, then N is called a *maximal dimensional* totally complex submanifold.

The n-dimensional quaternionic projective space $\mathbb{H}P^n$ over the quaternionic number field $\mathbb{H} = \mathbb{R}1 + \mathbb{R}\mathrm{i} + \mathbb{R}\mathrm{j} + \mathbb{R}\mathrm{k}$ has the quaternionic Kähler structure naturally induced from $\mathbb{H}^{n+1}$ and the Hopf fibration $\pi : S^{4n+3}(1) \longrightarrow \mathbb{H}P^n$.

The $(2n+1)$-dimensional complex propjective space $\mathbb{C}_\mathrm{i}P^{2n+1}$ over $\mathbb{C}_\mathrm{i} = \mathbb{R}1+\mathbb{R}\mathrm{i}$ with the unit imaginary unit i and the $(2n+1)$-dimensional complex propjective space $\mathbb{C}_\mathrm{j}P^{2n+1}$ over $\mathbb{C}_\mathrm{j} = \mathbb{R}1 + \mathbb{R}\mathrm{j}$ with the unit imaginary unit i play a role of *twistor spaces* for $\mathbb{H}P^n$.

Now let $\varphi : N^{2\ell} \longrightarrow \mathbb{H}P^n$ $(1 \leq \ell \leq n)$ be a totally complex submanifold immersed in a quaternionic projective space. Totally complex submanifolds induce several special types of minimal submanifolds in each manifold equipped with different geometric structures. Especially maximal dimensional totally complex submanifolds produce minimal submanifolds in other spaces such as complex Legendre submanifolds in $\mathbb{C}_\mathrm{j}P^{2n+1}$, minimal Legendrian submanifolds in $S^{4n+3}(1)$ and minimal Lagrangian submanifoldfs in $\mathbb{C}_\mathrm{i}P^{2n+1}$ as in the following diagram:

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$$
\begin{tikzcd}
& \mathbb{H}^{n+1} \arrow[d, phantom, "\bigcup"] & & & \\
& S^{4n+3}(1) \arrow[rr, "\pi_{\mathrm{i}}", "S^3"'] \arrow[dr, "\pi_{\mathrm{j}}"] \arrow[drrr, "\pi"] & & \mathbb{C}_{\mathrm{i}}P^{2n+1} \arrow[dr, "p_{\mathrm{i}}"] & \\
\widehat{N} = \pi^{-1}(N) \arrow[ur] \arrow[dr, phantom, "\bigcup"] \arrow[ddrrrr, bend right, "\pi"', "S^3"] & & \mathbb{C}_{\mathrm{j}}P^{2n+1} \arrow[rr, "p_{\mathrm{j}}"'] & & \mathbb{H}P^n \\
& \widetilde{L}^{2n+1} = \pi_{\mathrm{j}}^{-1}(\widetilde{N}) \arrow[uu, "\text{min.}", "\text{Leg.}"'] \arrow[rr, dash] \arrow[dr] & & L^{2n+1} = \pi_{\mathrm{i}}(\widetilde{L}) \arrow[uu, "\text{min.}", "\text{Lag.}"'] \arrow[dr] & \\
& & \widetilde{N}^n \arrow[uu, "\text{cplx.}", "\text{Leg.}"'] \arrow[rr] & & N^{2n} \arrow[uu, "\text{tot.}", "\text{cplx.}"']
\end{tikzcd}
$$

The inverse image of a totally complex submanifold N $\widehat{N}^{2\ell+3} = \pi^{-1}(N^{2\ell}) = \{(p, \mathbf{x}) \in N \times S^{4n+3}(1) \mid \varphi(p) = \pi(\mathbf{x})\}$ under the Hopf fibration $\pi : S^{4n+3}(1) \subset \mathbb{H}P^n$, is a $(2\ell+3)$-dimensional minimal submanifold of $S^{4n+3}(1)$.

$$
\begin{tikzcd}
\widehat{N}^{2\ell+3} = \pi^{-1}(N) = \bigcup_{\lambda\in\Lambda} \pi^{-1}(U_\lambda) \arrow[r, "\hat{\varphi}"] \arrow[d, "\pi"', "Sp(1)"] & S^{4n+3}(1) \subset \mathbb{H}^{n+1} \arrow[d, "\pi"', "Sp(1)"] \\
N^{2\ell} = \bigcup_{\lambda\in\Lambda} U_\lambda \arrow[r, "\varphi"] & \mathbb{H}P^n
\end{tikzcd}
$$

Then we proved the following:

Theorem 0.1 ([1], [2]). *Assume that N is a maximal dimensional totally complex submanifold of $\mathbb{H}P^n$. Then there exists a canonical connection $\nabla^c = \nabla^{\widehat{N}} - D$ (a metric connection satisfying $\nabla^c D = 0$ different from the Levi-Civita connection $\nabla^{\widehat{N}}$) on the tangent vector bundle $T\widehat{N}$ such that the parallelism of the second fundamental form $\alpha^{\widehat{N}}$ of $\widehat{N}$ with respect to ∇^c*

$$
\begin{aligned}
(\nabla^c_X \alpha^{\widehat{N}})(Y, Z) :=& \nabla^{\perp}_X(\alpha^{\widehat{N}}(Y, Z)) - \alpha^{\widehat{N}}(\nabla^c_X Y, Z) - \alpha^{\widehat{N}}(Y, \nabla^c_X Z) \\
=& 0 \quad (\forall X, Y, Z \in T\widehat{N})
\end{aligned}
$$

is equivalent to the parallelism of the second fundamental form α^N of a totally complex submanifold N with respect to the Levi-Civita connection in the usual sense

$$\begin{aligned}(\nabla^*_X\alpha^N)(Y,Z) :=&\nabla^\perp_X(\alpha^N(Y,Z)) - \alpha^N(\nabla^N_X Y, Z) - \alpha^N(Y, \nabla^N_X Z)\\ =&0 \quad (\forall X, Y, Z \in TN).\end{aligned}$$

Thus applying Olmos-Sánchez's theorem on differential geometric characterization of standardly embedded R-spaces ([5]) to the canonical connection ∇^c which we constructed and the observation of $Sp(n+1)\times Sp(1)$-symmetry of $\widehat{N}$, we obtain

Corollary 0.1 ([1], [2])**.** *If N is a maximal dimensional totally complex submanifold $\mathbb{H}P^n$ with $\nabla^*\alpha^N = 0$, then $\widehat{N}$ is obtained as an orbit (standardly embedded R-space) of the isotropy representation (s-representation) of a quaternionic Kähler symmetric pair (U, K).*

Moreover, we can determine concretely such R-spaces by complutation of Lie algebras and root systems for each quaternionic Kähler symmetric pair (U, K). Therefore we obtain a classification of maximal dimensional totally complex submanifolds with $\nabla^*\alpha^N = 0$ of $\mathbb{H}P^n$. This provides a new geometric proof by a different method for totally complex submanifolds of $\mathbb{H}P^n$ with parallel second fundamental form due to Kazumi Tsukada in 1985.

In the case when $(U, K) = (G_2, SO(4))$, a totally complex submanifold N^2 of $\mathbb{H}P^1 = S^4$ with parallel second fundamental form is the Veronese minimal surface $\mathbb{R}P^2 \subset S^4$ and the corresponding minimal Lagrangian submanifold L^3 of $\mathbb{C}P^3$ is an $SU(2)$-orbit, which is well-known as the *River Chiang Lagrangian*.

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References

[1] J.-T.Cho, K.Hashimoto and Y.Ohnita, *On quaternionic Kähler structures and totally complex submanifolds of quaternionic projective spaces*, Proceedings of The 24th International Workshop on Differential Geometry & Related

Fields **24** (2023) 217–238. edited by Y.-J.Suh, Y. Ohnita and Hyunjin Lee. OCAMI, RIRCM, Chosun University, JSPS, NRF.

[2] J.-T. Cho, K. Hashimoto and Y. Ohnita, *Totally complex submanifolds and R-spaces*, in preparation.

[3] S. Funabashi, *Totally complex submanifolds of a quaternionic Kaehlerian manifold*, Kodai Math. J. **2** (1979), 314–336.

[4] S. Ishihara, *Quaternion Kählerian manifolds*, J. Differential Geom. **9** (1974), 483–500.

[5] C. Olmos and C. Sánchez, *A geometric characterization of the orbits of s-representations*. J. reine angew. Math. **420** (1991), 195–202.

[6] K. Tsukada, *Parallel submanifolds in a quaternion projective space*. Osaka J. Math. **22** (1985), 187–241.

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