On instability of F-Yang-Mills connections

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This is a joint work with Kazuto Shintani (arXiv:2301.04291)

小池直之先生と私

- 学部
  - 数学研究 (2003 年度)
  - 卒業研究 (2004 年度)
- 大学院
  - 修士課程 (2004 年度〜)
  - 博士課程 (2006 年度〜)

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F-Yang-Mills connection

- M: a connected closed Riemannian manifold, G: a compact Lie group
- P: a prin. fiber bdl. over M with str. grp. G, $\mathfrak{g}_P = P \times_{\mathrm{Ad}} \mathfrak{g}$: the adjoint bundle
- $R^\nabla \in \Omega^2(\mathfrak{g}_P)$: the curvature 2-form of a connection ∇ on P

Definition (F-Yang-Mills connection, Dong-Wei (2011))

Let $F : [0, \infty) \to \mathbb{R}$ be a nonnegative strictly increasing C^2-function.

The F-Yang-Mills functional $\mathcal{YM}_F$ is defined by

$$\mathcal{YM}_F(\nabla) = \int_M F(\frac{1}{2}\|R^\nabla\|^2)dv, \quad \nabla \in \mathcal{C} := \{\text{the space of connections}\}$$

A critical point (connection) ∇ of $\mathcal{YM}_F$ is called an F-Yang-Mills connection on P.

Example: (1) $F(t) = t$: F-Yang-Mills connection $=$ Yang-Mills connection

(2) $F_p(t) = \dfrac{1}{p}(2t)^{p/2}$, $p \geq 2$: F_p-Yang-Mills connection $=$ p-Yang-Mills connection

(3) $F_e(t) = \exp(t)$: F_e-Yang-Mills connection $=$ exponential Yang-Mills connection

First variational formula

Any connection ∇ satisfies

$$d^{\nabla} R^{\nabla} = 0,$$

which is called the Bianchi identity.

Let δ^{∇} denote the formal adjoint operator of d^{∇}.

Theorem (First variational formula)

Let $\nabla \in \mathcal{C}$ and ∇^t be a C^{∞}-curve in $\mathcal{C}$ with $\nabla^0 = \nabla$ and $\alpha = \frac{d}{dt}\Big|_{t=0} \nabla^t \in \Omega^1(\mathfrak{g}_P)$

Then we have:

$$\frac{d}{dt}\Big|_{t=0} \mathcal{YM}_F(\nabla^t) = \int_M \langle \delta^{\nabla}(F'(\frac{1}{2}\|R^{\nabla}\|^2)R^{\nabla}), \alpha \rangle dv$$

It follows from this formula that ∇ is an F-Yang-Mills connection if and only if

$$\delta^{\nabla}(F'(\frac{1}{2}\|R^{\nabla}\|^2)R^{\nabla}) = 0,$$

which is called the F-Yang-Mills equation.

Instability of F-Yang-Mills connections

Definition (Weakly stable, unstable)

An F-Yang-Mills connection is said to be weakly stable if

$$\left.\frac{d^2}{dt^2}\right|_{t=0} \mathcal{YM}_F(\nabla^t) \geq 0$$

for any C^∞-curve ∇^t with $\nabla^0 = \nabla$.

An F-Yang-Mills connection ∇ is said to be unstable if ∇ is not weakly stable.

Remark

As shown in the next page, any flat connection is a weakly stable F-Yang-Mills connection (by the assumption of the function F and 2nd v.f.).

Second variational formula

$\mathcal{R}^\nabla : \Omega^1(\mathfrak{g}_P) \to \Omega^1(\mathfrak{g}_P)$: the Weitzenböck curvature of ∇, which satisfies

$$\langle \mathcal{R}^\nabla(\alpha), \alpha \rangle = \langle [\alpha \wedge \alpha], R^\nabla \rangle, \quad \alpha \in \Omega^1(\mathfrak{g}_P)$$

Theorem (Second variational formula)

Let $\nabla \in \mathcal{C}$ be an F-Yang-Mills connection.

Then we have:

$$\left.\frac{d^2}{dt^2}\right|_{t=0} \mathcal{YM}_F(\nabla^t) = \int_M F''(\frac{1}{2}\|R^\nabla\|^2)\langle d^\nabla \alpha, R^\nabla \rangle^2 dv + \int_M F'(\frac{1}{2}\|R^\nabla\|^2)\left\{\langle \mathcal{R}^\nabla(\alpha), \alpha \rangle + \|d^\nabla \alpha\|^2\right\} dv$$

Any flat connection ∇ is weakly stable because the 2nd v.f. yields

$$\left.\frac{d^2}{dt^2}\right|_{t=0} \mathcal{YM}_F(\nabla^t) = \int_M F'(0)\|d^\nabla \alpha\|^2 dv \geq 0$$

Simons' Theorem for the instability of YM connections

The aim of this talk is to generalize the following Simons' Theorem for Yang-Mills connections to F-Yang-Mills connections:

Theorem (Simons' Theorem (1977))

If $n > 4$, then any non-flat Yang-Mills connection over the n-sphere S^n is unstable.

- Histroical remark: This theorem was originally announced in Symposium at Tokyo by Simons (1977). After him, Bourguignon-Lawson (1981) gave the proof of this theorem.
- In order to extend Simons' Theorem, we will observe their proof. The method of their proof is based on the average method for Yang-Mills connections.

Bourguignon-Lawson's proof

The average method is based on the following lemma.

Lemma

Assume that M is isometrically immersed in $\mathbb{R}^N$. For the standard basis $\{E_A\}$ of $\mathbb{R}^N$, we denote by V_A the tangent comp. of E_A w.r.t. $M \subset \mathbb{R}^N$. For any F-Yang-Mills connection ∇, if

$$\sum_A I_\nabla(\iota_{V_A} R^\nabla) < 0,$$

then ∇ is unstable. Here, I_∇ denotes the index form of ∇.

In the case when $M = S^n \subset \mathbb{R}^{n+1}$ and $F(t) = t$,

- $I_\nabla(\alpha) = \int_{S^n} \left\{ \langle \mathcal{R}^\nabla(\alpha), \alpha \rangle + \|d^\nabla \alpha\|^2 \right\} dv$ for $\alpha \in \Omega^1(\mathfrak{g}_P)$

Then, Bourguignon-Lawson proved:

$$\sum_A I_\nabla(\iota_{V_A} R^\nabla) = 2(4-n) \int_{S^n} \|R^\nabla\|^2 dv$$

Index form of an F-Yang-Mills connection and degree of F'

- In general, for an F-Yang-Mills connection ∇, its index form I_∇ is described by

$$I_\nabla(\alpha) = \int_M F''(\frac{1}{2}\|R^\nabla\|^2)\langle d^\nabla\alpha, R^\nabla\rangle^2 dv + \int_M F'(\frac{1}{2}\|R^\nabla\|^2)\left\{\langle \mathcal{R}^\nabla(\alpha),\alpha\rangle + \|d^\nabla\alpha\|^2\right\} dv$$

  for all $\alpha \in \Omega^1(\mathfrak{g}_P)$

- In order to evaluate $\sum_A I_\nabla(\iota_{V_A}R^\nabla)$, we introduce:

$$d_{F'} = \sup_{0<t<\infty} \frac{tF''(t)}{F'(t)}$$

Example:

(1) $F(t) = t$: $d_{F'} = 0$

(2) $F_p(t) = \frac{1}{p}(2t)^{p/2}$, $p \geq 2$: $d_{F'_p} = \frac{p-2}{2}$

(3) $F_e(t) = \exp(t)$: $d_{F'_e} = \infty$

Instability of F-YM connections

The following theorem is the main result of this talk.

Theorem (B.-Shintani, arXiv:2301.0429)

Assume that $d_{F'}$ is finite. If

$$n > 4d_{F'} + 4,$$

then any non-flat F-Yang-Mills connection over the n-sphere S^n is unstable.

Pf (Sketch). We have the following inequality:

$$\sum_A I_\nabla(\iota_{V_A} R^\nabla) \leq 2(4d_{F'} + 4 - n) \int_{S^n} \|R^\nabla\|^2 dv$$

Our theorem is an extension of the following as well as Simons' one.

Theorem (Chen-Zhou (2007))

If $n > 2p$, then any non-flat p-Yang-Mills connection over the n-sphere S^n is unstable.

Instability of exponential YM connections

For an exponential Yang-Mills connection ∇, a similar calculation of $\sum_A I_\nabla(\iota_{V_A} R^\nabla)$ shows

$$\sum_A I_\nabla(\iota_{V_A} R^\nabla) = 2\int_{S^n} \exp(\frac{1}{2}\|R^\nabla\|^2)\|R^\nabla\|^2 \left\{2\|R^\nabla\|^2 - (n-4)\right\} dv$$

Then we have:

Proposition

Let ∇ be a non-flat exponential Yang-Mills connection over S^n. Assume that $n > 4$. If

$$\|R^\nabla\| < \sqrt{\frac{n-4}{2}},$$

then ∇ is unstable.

YM theory and Harmonic map theory (1 of 2)

- There are strong similarities between YM theory and harmonic map theory.
- Our concern is a counterpart of our results in harmonic map theory.
- Ara (1999) introduced the notion of F-harmonic maps as a generalization of harmonic maps, p-harmonic maps and so on. In fact, he derived the following theorem.

Theorem (Ara (1999))

Let M be a closed Riemannian manifold. For any F-harmonic map $\phi : M \to S^n$, if

$$\int_M \|d\phi\|^2 \left\{ \|d\phi\|^2 F''(\frac{1}{2}\|d\phi\|^2) + (2-n)F'(\frac{1}{2}\|d\phi\|^2) \right\} dv < 0, \tag{1}$$

then ϕ is unstable.

YM theory and Harmonic map theory (2 of 2)

- In the case when $d_{F'}$ is finite, the l.h.s of the inequality (1) satisfies

$$\int_M \|d\phi\|^2 \left\{ \|d\phi\|^2 F''(\frac{1}{2}\|d\phi\|^2) + (2-n)F'(\frac{1}{2}\|d\phi\|^2) \right\} dv$$
$$\leq \int_M \|d\phi\|^2 F'(\frac{1}{2}\|d\phi\|^2) \left\{ 2d_{F'} + (2-n) \right\} dv$$

  Theorefore, it follows from Ara's theorem that, if

  (2) $$n > 2d_{F'} + 2,$$

  then any non-constant F-harmonic map $\phi : M \to S^n$ is unstable.

- The inequality (2) is an extension of Leung's theorem (1982) for the instability of harmonic maps.

Further direction

- Our concern is to find an instability theorem for other base spaces.
- Kobayashi-Ohnita-Takeuchi (1986) developed the average method to study the instability of Yang-Mills connections in the case when the base space is an irreducible compact Riemannian symmetric space. In fact, they proved:

Theorem (Kobayashi-Ohnita-Takeuchi (1986))

Let M be an irreducible compact Riemannian symmetric space with canonical metric. If

$$\lambda_1 - 1 + 2\mu < 0, \tag{3}$$

then any non-flat Yang-Mills connection over M is unstable. Here,

λ_1: the first eigenvalue of the Laplacian of M

μ: the maximum eigenvalue of the curvature operator on $\Lambda^2 T_x M$ $(x \in M)$

Remark. By the irreducibility, M can be realized as a minimal submanifold of the standard sphere by the first standard immersion (Takahashi's theorem).

Instability of F-Yang-Mills connections

After them, Shintani extened the inequality (3) to F-Yang-Mills connections as follows. Let M be an irreducible compact Riemannian symmetric space with canonical metric, and denote by γ the maximum eigenvalue of the shape operator of the minimal isometric immersion from M to $S^{N-1}(r)$ ($r = \sqrt{\dim(M)/\lambda_1}$).

Theorem (Shintani (2023))

Assume that $d_{F'} \geq 0$. If

(4) $$\lambda_1 - 1 + 2\mu + 4d_{F'} \left\{ (\mathrm{codim}(M) - 1)\gamma + \frac{\lambda_1}{\dim(M)} \right\} < 0,$$

then any non-flat F-Yang-Mills connection over M is unstable. Here, $\gamma := \max_{x \in M} \{ \|A_\xi\| \mid \xi \in T_x^\perp M \}$ (A_ξ: the shape operator of $M \subset S^{N-1}(r)$)

His theorem is an extension of the result by Kawagoe (2015) for p-Yang-Mills connections as well as Kobayashi-Ohnita-Takeuchi's one.

Problems

Now, we give two problems:

(1) Classify irreducible compact Riemannian symmetric spaces M satisfying Shintani's inequality (4).

(2) Can we find a weakly stable F-Yang-Mills connection over M if M does not satisfies the inequality (4)?

Remark.

- In the case when $F(t) = t$, the above problems were solved by Kobayashi-Ohnita -Takeuchi (1986). In fact, they proved that the followings are the only irreducible compact Riemannian symmetric spaces satisfying the inequality (1):
  - S^n $(n > 4)$
  - the Cayely projective plane $F_4/Spin(9)$
  - E_6/F_4

Instability of FYM connections over symmetric R-spaces

A partial solution of Problem (1) for F-Yang-Mills connections is given as follows.

- Let M be an irreducible symmetric R-space, i.e., an irreducible compact Riemannian symmetric space which is realized as an orbit of the isotropy representation of some Riemannian symmetric space U/K.
- In this setting, we can obtain the γ in the inequlity (4) by means of the restricted root system of U/K. Then, we have:

Theorem (B. (2023))

If $0 \leq d_{F'} < 1/6$, then any non-flat F-Yang-Mills connection over the Cayely projective plane $F_4/Spin(9)$ is unstable.

In fact, S^n $(0 \leq d_{F'} < (n-4)/4)$ and the $F_4/Spin(9)$ $(0 \leq d_{F'} < 1/6)$ are the only irreducible symmetric R-spaces satisfying the inequality (4).

Remark. E_6/F_4 is not a symmetric R-space.

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Thank you for your attention!