非負曲率を持つ概エルミート多様体上の負の小平次元
Negative Kodaira dimension on almost Hermitian manifolds with nonnegative curvature

川村昌也
Masaya Kawamura

椙山女学園大学
Sugiyama Jogakuen University

11/21/2023

Table of Contents

1. Kodaira dimension on almost complex manifolds

Complexified tangent vector bundle

Let (M, J) be a real $2n$-dimensional almost complex manifold.
Let TM be the real tangent vector bundle and let $\Lambda^1 M$ be the dual of TM. We consider the complexified tangent vector bundle:

$$T^{\mathbb{C}}M = TM \otimes_{\mathbb{R}} \mathbb{C} = T^{1,0}M \oplus T^{0,1}M,$$

$$T^{1,0}M = \{X - \sqrt{-1}JX \big| X \in TM\}, \quad T^{0,1}M = \{X + \sqrt{-1}JX \big| X \in TM\}.$$

We have that

$$\Lambda^1 M \otimes_{\mathbb{R}} \mathbb{C} = \Lambda^{1,0}M \oplus \Lambda^{0,1}M,$$

$$\Lambda^{1,0}M = \{\eta + \sqrt{-1}J\eta \big| \eta \in \Lambda^1 M\}, \ \Lambda^{0,1}M = \{\eta - \sqrt{-1}J\eta \big| \eta \in \Lambda^1 M\}.$$

Define

$$\Lambda^{p,q}M := \Lambda^p(\Lambda^{1,0}M) \otimes \Lambda^q(\Lambda^{0,1}M).$$

Then we have

$$\Lambda^r M \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{p+q=r} \Lambda^{p,q}M.$$

Exterior differential operator

On an almost complex manifold (M, J), we split the exterior differential operator

$$d : \Lambda^p M \otimes_{\mathbb{R}} \mathbb{C} \to \Lambda^{p+1} M \otimes_{\mathbb{R}} \mathbb{C},$$

into four components

$$d = A + \partial + \bar{\partial} + \bar{A}$$

with

$$\partial : \Lambda^{p,q} M \to \Lambda^{p+1,q} M, \quad \bar{\partial} : \Lambda^{p,q} M \to \Lambda^{p,q+1} M,$$
$$A : \Lambda^{p,q} M \to \Lambda^{p+2,q-1} M, \quad \bar{A} : \Lambda^{p,q} M \to \Lambda^{p-1,q+2} M.$$

Define the canonical bundle

$$\mathcal{K}_M := \Lambda^n(\Lambda^{1,0}M).$$

We extend the operator

$$\bar{\partial} : \Gamma(M, \mathcal{K}_M) \to \Gamma(M, \Lambda^{n,1}M) \cong \Gamma(M, \Lambda^{0,1}M \otimes \mathcal{K}_M)$$

to

$$\bar{\partial}_m : \Gamma(M, \mathcal{K}_M^{\otimes m}) \to \Gamma(M, \Lambda^{0,1}M \otimes \mathcal{K}_M^{\otimes m})$$

by $\bar{\partial}_1 := \bar{\partial}$ and for $m \in \mathbb{Z}_{\geq 2}$ inductively by the product rule

$$\bar{\partial}_m(s_1 \otimes s_2) = \bar{\partial}s_1 \otimes s_2 + s_1 \otimes \bar{\partial}_{m-1}s_2$$

for $s_1 \in \Gamma(M, \mathcal{K}_M)$ and $s_2 \in \Gamma(M, \mathcal{K}_M^{\otimes(m-1)})$. the operator $\bar{\partial}_m$ satisfies the Leibniz rule $\bar{\partial}_m(fs) = \bar{\partial}f \otimes s + f\bar{\partial}_m s$ for any smooth function f and any section $s \in \Gamma(M, \mathcal{K}_M^{\otimes m})$. Hence, $\bar{\partial}_m$ is a pseudoholomorphic structure on the pluricanonical bundle $\mathcal{K}_M^{\otimes m}$.

Kodaira dimension on almost complex manifolds

Define the space of pseudoholomorphic sections of $\mathcal{K}_M^{\otimes m}$ for $m \in \mathbb{Z}_{\geq 1}$ by

$$H^0(M, \mathcal{K}_M^{\otimes m}) := \{s \in \Gamma(M, \mathcal{K}_M^{\otimes m}) | \bar{\partial}_m s = 0\}.$$

We define the m-th plurigenus

$$P_m(M, J) := \dim_{\mathbb{C}} H^0(M, \mathcal{K}_M^{\otimes m})$$

and the kodaira dimension of an almost complex manifold (M, J):

$$\kappa(M, J) := \begin{cases} -\infty, & \text{if } P_m(M, J) = 0 \text{ for any } m \in \mathbb{Z}_{\geq 1} \\ \limsup_{m \to \infty} \dfrac{\log P_m(M, J)}{\log m}, & \text{otherwise.} \end{cases}$$

Example 1. (The Kodaira-Thurston surface)

$$M := \mathbb{S}^1 \times (\Gamma\backslash \mathrm{Nil}^3),\ \ \mathrm{Nil}^3 := \{A \in GL(3,\mathbb{R}) | A = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}\},$$

Γ is the subgroup in Nil^3 consisting of element with integer entries, acting by left multiplication. Let t be the coordinate of $\mathbb{S}^1$.
For $\forall a \neq 0 \in \mathbb{R}$, the almost complex structure J_a is defined by

$$J_a(\frac{\partial}{\partial t}) = \frac{\partial}{\partial x},\ \ J_a(\frac{\partial}{\partial x}) = -\frac{\partial}{\partial t},$$

$$J_a(\frac{\partial}{\partial y} + x\frac{\partial}{\partial z}) = \frac{1}{a}\frac{\partial}{\partial z},\ \ J_a(\frac{\partial}{\partial z}) = -a(\frac{\partial}{\partial y} + x\frac{\partial}{\partial z}).$$

$$P_m(M, J_a) = \begin{cases} 0, & a \notin \frac{4}{m}\pi\mathbb{Z} \\ 1, & a \in \frac{4}{m}\pi\mathbb{Z} \setminus \{0\} \end{cases} \qquad \kappa(M, J_a) = \begin{cases} -\infty, & a \notin \pi\mathbb{Q} \\ 0, & a \in \pi\mathbb{Q} \setminus \{0\} \end{cases}$$

Example 2.

$$M := \mathbb{T}^2 \times \Sigma,$$

$\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ with coordinate (x, y), Σ is a compact Riemannian surface with genus $g \geq 2$. The almost complex structure J is defined by

$$J(\frac{\partial}{\partial x}) = -h\frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \quad J(\frac{\partial}{\partial y}) = -(1+h^2)\frac{\partial}{\partial x} + h\frac{\partial}{\partial y},$$

h is a smooth real nonconstant function on Σ.

$$P_m(M, J) = (2m-1)(g-1) \text{ for } m > 1, \quad \kappa(M, J) = 1$$

Remark

Since we have $\kappa(M_1 \times M_2) = \kappa(M_1) + \kappa(M_2)$ for any two compact almost complex manifolds (M_1, J_1), (M_2, J_2).

2. Chern connection, torsion and curvature on almost Hermitian manifolds

Chern connection

Let (M^{2n}, J, ω) be a real $2n$-dimensional almost Hermitian manifold with the associated almost Hermitian metric g w.r.t. ω.

There exists a unique affine connection preserving g and J whose torsion has vanishing $(1,1)$-part, which is called the Chern connection denoted by ∇.

Choose a local $(1,0)$-frame $\{e_r\}$ w.r.t. the metric g.
Since ∇ preserves J, we can define the Christoffel symbols:

$$\nabla_{e_i} e_j = \Gamma_{ij}^k e_k, \quad \nabla_{e_i} e_{\bar{j}} = \Gamma_{i\bar{j}}^{\bar{k}} e_{\bar{k}}.$$

The structure coefficients of Lie bracket are defined by

$$[e_i, e_j] =: B_{ij}^r e_r + B_{ij}^{\bar{r}} e_{\bar{r}}, \quad [e_i, e_{\bar{j}}] =: B_{i\bar{j}}^r e_r + B_{i\bar{j}}^{\bar{r}} e_{\bar{r}}.$$

Torsion of the Chern connection

Since the torsion T of the Chern connection ∇ has no $(1,1)$-part, the only non-vanishing components are as follows:

$$(T' =) T_{ij}^{s} = \Gamma_{ij}^{s} - \Gamma_{ji}^{s} - B_{ij}^{s},$$

$$(T'' =) T_{ij}^{\bar{s}} = -B_{ij}^{\bar{s}},$$

which tells us that T splits in $T = T' + T''$, where
$T' \in \Gamma(\Lambda^{2,0}(M) \otimes T^{1,0}M)$ and $T'' \in \Gamma(\Lambda^{0,2}(M) \otimes T^{1,0}M)$.
Note that T'' depends only on J and it can be regarded as the Nijenhuis tensor of J, that is,

J is integrable $\Leftrightarrow$ T'' vanishes.

Curvature of the Chern connection

Let Ω^g denote the curvature of the Chern connection ∇ w.r.t. g. The Chern curvature $\Omega^g \in \Gamma(\Lambda^2(M) \otimes \mathrm{End}(T^{1,0}M))$ splits in

$$\Omega^g = H^g + R^g + \bar{H}^g,$$

where $R^g \in \Gamma(\Lambda^{1,1}(M) \otimes \mathrm{End}(T^{1,0}M))$,
$H^g \in \Gamma(\Lambda^{2,0}(M) \otimes \mathrm{End}(T^{1,0}M))$, $\bar{H}^g \in \Gamma(\Lambda^{0,2}(M) \otimes \mathrm{End}(T^{1,0}M))$.
We define the Chern scalar curvature s_ω and the Riemannian type scalar curvature $\hat{s}_\omega$ of the almost Hermitian metric g w.r.t. ∇:

$$s_\omega := g^{i\bar{j}} g^{k\bar{l}} R^g_{i\bar{j}k\bar{l}}, \quad \hat{s}_\omega := g^{i\bar{l}} g^{k\bar{j}} R^g_{i\bar{j}k\bar{l}}.$$

The holomorphic sectional curvature

For a point $p \in M$ and a non-zero $(1,0)$-vector $\xi \in T_p^{1,0}M$, the holomorphic sectional curvature $\mathcal{H}^g$ of ω at the point p and the direction ξ is define by

$$\mathcal{H}^g_p(\xi) := R^g(\xi, \bar{\xi}, \xi, \bar{\xi})|_p = R^g_{i\bar{j}k\bar{l}}|_p \xi^i \xi^{\bar{j}} \xi^k \xi^{\bar{l}}.$$

We write $\mathrm{HSC}(\omega) > 0$ when we have that $\mathcal{H}^g_p(\xi) > 0$ for any point $p \in M$ and any non-zero $(1,0)$-vector $\xi \in T_p^{1,0}M$.

3. Conditions for the negative Kodaira dimension

In 1992, S.-T. Yau proposed "100 open problems in geometry" and the following question is Problem 67.

Question (S.-T. Yau 1992)

If (M, J, ω) is a compact Kähler manifold with $\mathrm{HSC}(\omega) > 0$, does M have negative Kodaira dimension, i.e., $\kappa(M) = -\infty$?

X. Yang has given a answer for Yau's question in a general setting.

Theorem (X. Yang 2016)

Let (M, J, ω) be a compact Hermitian manifold with $\mathrm{HSC}(\omega) > 0$. Then, $\kappa(M) = -\infty$.

At this point, we ask the following more general question.

Question

What about the almost Hermitian case?

The almost Kählerity

Definition

An almost Hermitian metric ω is called almost Kähler if

$$d\omega = 0.$$

Remark

The almost Kählerity is equivalent to

$$T_{ij}^{k} = 0 \text{ and } T_{ij}^{\bar{k}} + T_{ki}^{\bar{j}} + T_{jk}^{\bar{i}} = 0 \quad \text{for } \forall i, j, k = 1, \ldots, n.$$

The almost Hermitian case

Theorem (K.)

Let (M^{2n}, J, ω) be a compact almost Kähler manifold with $n \geq 3$, $\mathrm{HSC}(\omega) > 0$. Then, $\kappa(M, J) = -\infty$.

Theorem (K.)

Let (M^4, J, ω) be a real 4-dimensional compact almost Hermitian manifold with $\mathrm{HSC}(\omega) > 0$. Then, $\kappa(M^4, J) = -\infty$.

A condition for the negative Kodaira dimension

Definition

An almost Hermitian metric ω is called Gauduchon if

$$\partial\bar{\partial}\omega^{n-1} = 0.$$

Theorem (P. Gauduchon 1977)

Let (M^{2n}, J, ω) be a compact almost Hermitian manifold with $n \geq 2$. Then $\exists u \in C^\infty(M)$, unique up to addition of a constant, s.t. the conformal almost Hermitian metric $e^u\omega$ is Gauduchon.

Theorem (H. Chen, W. Zhang 2023)

Let (M^{2n}, J) be a compact almost complex manifold with $n \geq 2$. If M admits a Gauduchon metric ω_0 with $\int_M s_{\omega_0}\omega_0^n > 0$, then $\kappa(M, J) = -\infty$.

It suffices to show that a conformal metric ω_0 has $\int_M s_{\omega_0}\omega_0^n > 0$.

The key formula

Lemma (K.)

Let (M^{2n}, J, ω) be compact almost Hermitian manifold with $n \geq 2$. One has that

$$s_\omega - \hat{s}_\omega = \langle \bar\partial \bar\partial^* \omega, \omega \rangle + T^{\bar r}_{ij} T^{i}_{\bar r \bar j}, \tag{1}$$

where $\partial^* = - * \bar\partial *$ is the adjoint operator, $\langle \bar\partial \bar\partial^* \omega, \omega \rangle = g^{i\bar j} g^{p\bar q} (\bar\partial \bar\partial^* \omega)_{i\bar q} \overline{\omega_{j\bar p}} = -g^{i\bar j} \nabla_{\bar j} w_i$, $w_i := g^{k\bar l} T_{ik\bar l}$. Note that $T^{\bar r}_{ij} T^{i}_{\bar r \bar j} := g^{j\bar k} g^{i\bar s} g^{p\bar q} T_{ijp} T_{\bar q \bar k \bar s}$, where $T_{ijp} = T^{\bar l}_{ij} g_{p\bar l}$, $T_{\bar q \bar k \bar s} = T^{r}_{\bar q \bar k} g_{r\bar s}$.

Lemma (P. Li 2021)

$$\mathrm{HSC}(\omega) > 0 \implies s_\omega + \hat{s}_\omega > 0.$$

The key lemmas

Let (M^{2n}, J, ω) be a real $2n$-dimensional compact almost Hermitian manifold with $n \geq 2$. Define the set of the conformal class of ω:

$$\{\omega\} := \{e^u \omega | u \in C^\infty(M; \mathbb{R})\}.$$

We may take a Gauduchon metric ω_0 in the conformal class of ω such that $\omega_0 = f_0^{\frac{1}{n-1}} \omega \in \{\omega\}$, f_0 is a positive smooth function.

Lemma (A. Balas 1985)

$$\int_M (s_{\omega_0} + \hat{s}_{\omega_0})\omega_0^n = \int_M f_0(s_\omega + \hat{s}_\omega)\omega^n.$$

$$\mathrm{HSC}(\omega) > 0 \implies \int_M f_0(s_\omega + \hat{s}_\omega)\omega^n > 0.$$

The proof for the almost Hermitian case

Choose a Gauduchon metric ω_0 in the conformal class of ω such that $\omega_0 = f_0^{\frac{1}{n-1}}\omega \in \{\omega\}$, where f_0 is a positive smooth function. Then, by integrating the formula (1) for ω_0:
$s_{\omega_0} - \hat{s}_{\omega_0} = \langle \bar{\partial}\bar{\partial}^*\omega_0, \omega_0\rangle + T^{\bar{r}}_{si} T^{s}_{\bar{r}\bar{i}}$, and assuming $T^{\bar{r}}_{ij} T^{i}_{\bar{r}\bar{j}} \geq 0$,

$$\begin{aligned}\int_M (s_{\omega_0} - \hat{s}_{\omega_0})\omega_0^n &= \int_M \langle \bar{\partial}\bar{\partial}^*\omega_0, \omega_0\rangle \omega_0^n + \int_M T^{\bar{r}}_{ij} T^{i}_{\bar{r}\bar{j}}\omega_0^n \\ &= \int_M |\bar{\partial}^*\omega_0|^2\omega_0^n + \int_M T^{\bar{r}}_{ij} T^{i}_{\bar{r}\bar{j}}\omega_0^n \geq 0.\end{aligned}$$

Under the assumption $\mathrm{HSC}(\omega) > 0$,
since $\mathrm{HSC}(\omega) > 0$ implies that $s_\omega + \hat{s}_\omega > 0$, we obtain that

$$\begin{aligned}\int_M s_{\omega_0}\omega_0^n &= \frac{1}{2}\int_M (s_{\omega_0} + \hat{s}_{\omega_0})\omega_0^n + \frac{1}{2}\int_M (s_{\omega_0} - \hat{s}_{\omega_0})\omega_0^n \\ &\geq \frac{1}{2}\int_M (s_{\omega_0} + \hat{s}_{\omega_0})\omega_0^n = \frac{1}{2}\int_M f_0(s_\omega + \hat{s}_\omega)\omega^n > 0.\end{aligned}$$

The almost Hermitian case

Proposition (K.)

Let (M^{2n}, J, ω) be a compact almost Hermitian manifold with $n \geq 2$. Assume that $T^{\bar{r}}_{ij} T^{i}_{\bar{r}\bar{j}} \geq 0$ and $\mathrm{HSC}(\omega) > 0$. Then, $\kappa(M, J) = -\infty$.

Some conditions for $T_{ij}^{\bar{r}} T_{\bar{r}\bar{j}}^{i} \geq 0$

Lemma (K.)

Let (M^{2n}, J, ω) be an almost Kähler manifold. Then we have that

$$T_{ij}^{\bar{r}} T_{\bar{r}\bar{j}}^{i} \geq 0.$$

The equality $T_{ij}^{\bar{r}} T_{\bar{r}\bar{j}}^{i} = 0$ holds if and only if the almost Kähler manifold is Kähler.

By applying $T_{ij}^{\bar{k}} + T_{ki}^{\bar{j}} + T_{jk}^{\bar{i}} = 0$, we compute that

$$\begin{aligned} T_{ij}^{\bar{r}} T_{\bar{r}\bar{j}}^{i} &= -T_{ij}^{\bar{r}} T_{\bar{i}\bar{r}}^{j} - T_{ij}^{\bar{r}} T_{\bar{j}\bar{i}}^{r} \\ &= -T_{ji}^{\bar{r}} T_{\bar{r}\bar{i}}^{j} + T_{ij}^{\bar{r}} T_{\bar{i}\bar{j}}^{r} \\ &= -T_{ij}^{\bar{r}} T_{\bar{r}\bar{j}}^{i} + |T''|_g^2 \Leftrightarrow T_{ij}^{\bar{r}} T_{\bar{r}\bar{j}}^{i} = \frac{1}{2}|T''|_g^2 (\geq 0), \end{aligned}$$

where $|T''|_g^2 := g^{j\bar{k}} g^{i\bar{s}} g_{r\bar{l}} T_{ij}^{\bar{l}} T_{\bar{s}\bar{k}}^{r}$.

In the case of $n = 2$

On a real 4-dimensional almost Hermitian manifold (M^4, J, ω),

$$\begin{aligned}
T^{\bar{r}}_{ij} T^{i}_{\bar{r}\bar{j}} &= g^{j\bar{k}} \delta_{qr} \delta_{il} T^{\bar{r}}_{ij} T^{l}_{\bar{q}\bar{k}} \\
&= g^{j\bar{k}} (T^{\bar{1}}_{1j} T^{1}_{\bar{1}\bar{k}} + T^{\bar{2}}_{1j} T^{1}_{\bar{2}\bar{k}} + T^{\bar{1}}_{2j} T^{2}_{\bar{1}\bar{k}} + T^{\bar{2}}_{2j} T^{2}_{\bar{2}\bar{k}}) \\
&= T^{\bar{1}}_{12} T^{1}_{\bar{1}\bar{2}} + T^{\bar{2}}_{12} T^{1}_{\bar{2}\bar{1}} + T^{\bar{1}}_{21} T^{2}_{\bar{1}\bar{2}} + T^{\bar{2}}_{21} T^{2}_{\bar{2}\bar{1}} \\
&= T^{\bar{1}}_{12} T^{1}_{\bar{1}\bar{2}} - T^{\bar{2}}_{12} T^{1}_{\bar{1}\bar{2}} - T^{\bar{1}}_{12} T^{2}_{\bar{1}\bar{2}} + T^{\bar{2}}_{12} T^{2}_{\bar{1}\bar{2}} \\
&= (T^{\bar{1}}_{12} - T^{\bar{2}}_{12}) \overline{(T^{\bar{1}}_{12} - T^{\bar{2}}_{12})} \geq 0.
\end{aligned}$$

The equality $T^{\bar{r}}_{ij} T^{i}_{\bar{r}\bar{j}} = 0$ holds if and only if $T^{\bar{1}}_{12} = T^{\bar{2}}_{12}$.

Remark

Since on a nearly Kähler manifold ($(D_X J)X = 0$ for any tangent vector field X, $DJ \neq 0$, D is the Levi-Civita connection), we have $T^{\bar{r}}_{ij} T^{i}_{\bar{r}\bar{j}} = -|T''|^2_g \leq 0$, we obtain $T^{\bar{r}}_{ij} T^{i}_{\bar{r}\bar{j}} = 0$ (then $T'' \equiv 0$) on a real 4-dimensional nearly Kähler manifold and it must be Kähler.

Another condition for the negative Kodaira dimension

Theorem (H. Chen, W. Zhang 2023)

Let (M^{2n}, J) be a compact almost complex manifold with $n \geq 2$. If M admits an almost Hermitian metric ω with $s_\omega > 0$, then $\kappa(M, J) = -\infty$.

Question

What about the other scalar curvature?

Quasi-Kählerity

Definition

An almost Hermitian metric ω is called quasi-Kähler if

$$\bar{\partial}\omega = 0,$$

which is equivalent to $T_{ij}^{k} = 0$ for $\forall i, j, k = 1, \ldots, n$.

$$\stackrel{(1)}{\Rightarrow} s_\omega = \hat{s}_\omega + T_{ri}^{\bar{k}} T_{\bar{k}\bar{i}}^{r}$$

Real 4-dimensional quasi-Kähler case

Since we have $T^{\bar{k}}_{ri} T^{r}_{\bar{k}\bar{i}} \geq 0$ on a real 4-dimensional almost Hermitian manifold, we have the following.

Corollary (K.)

If $\hat{s}_\omega > 0$ on a real 4-dimensional compact quasi-Kähler manifold (M^4, J, ω), then $\kappa(M, J) = -\infty$.

Since on a real 4-dimensional quasi-Kähler manifold, (J. Fu, X. Zhou 2022)

$$\hat{s}_\omega = \frac{1}{2}s + \frac{1}{32}|N|^2 \geq \frac{1}{2}s, \quad s_\omega = \frac{1}{2}s + \frac{1}{16}|N|^2 \geq \frac{1}{2}s$$

where s is the Riemannian scalar curvature w.r.t. D, and N is the Nijenhuis tensor of J, we have the following.

Corollary (K.)

If $s > 0$ on a real 4-dimensional compact quasi-Kähler manifold (M^4, J, ω), then $\kappa(M, J) = -\infty$.

Almost Kähler case

Since we have $T^{\bar{k}}_{ri} T^{r}_{\bar{k}\bar{i}} \geq 0$ on an almost Kähler manifold, we have the following.

Corollary (K.)

If $\hat{s}_\omega > 0$ on a compact almost Kähler manifold (M^{2n}, J, ω) with $n \geq 2$, then $\kappa(M, J) = -\infty$.

Since on an almost Kähler manifold, (J. Fu, X. Zhau 2022)

$$\hat{s}_\omega = \frac{1}{2}s + \frac{1}{32}|N^0|^2 \geq \frac{1}{2}s, \quad s_\omega = \frac{1}{2}s + \frac{1}{16}|N^0|^2 \geq \frac{1}{2}s$$

where $N^0 := N - \mathfrak{b}N$, $\mathfrak{b}N$ is the skew-symmetric part of N, we have the following.

Corollary (K.)

If $s > 0$ on a compact almost Kähler manifold (M^{2n}, J, ω) with $n \geq 2$, then $\kappa(M, J) = -\infty$.

4. The t-Gauduchon connection

The t-Gauduchon connection

Let (M, J, g) be an almost Hermitian manifold.
Let ∇ be the Chern (second canonical) connection, and let D be the Levi-Civita connection and let ${}^LD$ denote the restriction to $T^{1,0}M$, which is called the Lichnerowicz (first canonical) connection:

$$\begin{array}{ccc} \Gamma(M, T^{\mathbb{C}}M) & \xrightarrow{D} & \Gamma(M, (T^{\mathbb{C}}M)^* \otimes T^{\mathbb{C}}M) \\ \cup & & \\ \Gamma(M, T^{1,0}M) & \xrightarrow{{}^LD = D|_{T^{1,0}M}} & \Gamma(M, (T^{\mathbb{C}}M)^* \otimes T^{1,0}M), \end{array}$$

$${}^LD_X Y = D_X Y - \frac{1}{2} J(D_X J) Y \text{ for } X, Y \in \Gamma(TM).$$

We define the t-Gauduchon connection for $t \in \mathbb{R}$ on (M, J, g) by

$${}^t\nabla := t\nabla + (1-t)\,{}^LD.$$

${}^t\nabla$ is reduced to ${}^LD$ when the manifold is quasi-Kähler.

Scalar curvature

$$ {}^t s_\omega := g^{i\bar{j}} g^{k\bar{l}}\, {}^t R^g_{i\bar{j}k\bar{l}}, \quad {}^t \hat{s}_\omega := g^{i\bar{l}} g^{k\bar{j}}\, {}^t R^g_{i\bar{j}k\bar{l}}. $$

Note that ${}^1 s_\omega = s_\omega$, ${}^1 \hat{s}_\omega = \hat{s}_\omega$.

$$ {}^t s_\omega = t s_\omega + (1-t)(\hat{s}_\omega + T^{\bar{r}}_{ki} T^{k}_{\bar{r}\bar{i}}), $$

$$ {}^t \hat{s}_\omega = t \hat{s}_\omega + (1-t)(s_\omega - T^{\bar{r}}_{ki} T^{k}_{\bar{r}\bar{i}}) - \left(\frac{1-t}{2}\right)^2 (|T'|^2 + |w|^2), $$

where $w_i = g^{r\bar{s}} T_{ir\bar{s}}$.

Theorem (K.)

Let (M^{2n}, J, ω) be a compact almost Hermitian manifold with $n \geq 2$. Suppose that one of the following conditions holds:

1. ${}^t s_\omega + (t-1)(\hat{s}_\omega + T^{\bar{k}}_{ri} T^{r}_{\bar{k}\bar{i}}) > 0$ for some $t > 0$,
2. ${}^t s_\omega + (t-1)(\hat{s}_\omega + T^{\bar{k}}_{ri} T^{r}_{\bar{k}\bar{i}}) < 0$ for some $t < 0$,
3. ${}^t s_\omega + (1-t)\langle \bar{\partial}\bar{\partial}^* \omega, \omega \rangle > 0$ for some $t \in \mathbb{R}$.

Then, $\kappa(M, J) = -\infty$.

Semi-Kähler case

Definition

An almost Hermitian metric ω is called semi-Kähler if

$$d\omega^{n-1} = 0.$$

$$\overset{(1)}{\Rightarrow} s_\omega = \hat{s}_\omega + T^{\bar{k}}_{ri} T^{r}_{k\bar{i}}$$

Theorem (K.)

Let (M^{2n}, J, ω) be a compact semi-Kähler manifold with $n \geq 2$. Then, we have that for any $t \in \mathbb{R}$,

$$^t s_\omega = s_\omega.$$

Corollary (K.)

If $^t s_\omega > 0$ for some $t \in \mathbb{R}$ on a compact semi-Kähler manifold (M^{2n}, J, ω) with $n \geq 2$, then $\kappa(M, J) = -\infty$.

5. Positive Hermitian curvature flow (Positive HCF)

The positive HCF with non-negative Griffiths curvature

We consider the following type of the HCF, called the positive HCF or Ustinovskiy's flow, on a compact Hermitian manifold (M, J, g_0):

$$\mathrm{HCF}_+ \quad \begin{cases} \frac{\partial}{\partial t} g(t) = -S^{g(t)} - Q^{g(t)}, \\ g(0) = g_0, \end{cases}$$

where $S^g_{i\bar{j}} = g^{k\bar{l}} R^g_{k\bar{l}i\bar{j}}$ is the second Chern-Ricci curvature with the specific torsion quadratic term:

$$Q^g_{i\bar{j}} := \frac{1}{2} g^{p\bar{s}} g^{m\bar{n}} T_{\bar{s}\bar{n}i} T_{pm\bar{j}}.$$

Preservation of Griffiths positivity along the HCF_+

Theorem (Y. Ustinovskiy 2019)

On a compact Hermitian manifold (M, J, g_0), let $g(t)$, $t \in [0, \tau)$ be the solution of the HCF_+ with $g(0) = g_0$ for any $\tau < \tau_{\max} < \infty$, where $\tau_{\max}$ is the finite explosion time of the HCF_+. Assume that the Chern curvature R^{g_0} is Griffiths nonnegative (resp. Griffiths positive) on M, i.e., for any $x \in M$ and any non-zero $\xi, \eta \in T^{1,0}_x M$:

$$R^{g_0}_x(\xi, \bar{\xi}, \eta, \bar{\eta}) \geq 0 \quad (\text{resp.} > 0).$$

Then for $t \in [0, \tau)$, the Chern curvature $R^{g(t)}$ remains Griffiths nonnegative (resp. Griffiths positive) on M. If, moreover, the Chern scalar curvature R^{g_0} is Griffiths positive at least at one point, then for any $t \in (0, \tau)$ the Chern curvature is Griffiths positive everywhere on M.

Uniformization theorem

Theorem (Y. Ustinovskiy 2019)

Let (M, J, g_0) be a compact complex n-dimensional Hermitian manifold such that

1. its Chern curvature R^{g_0} is Griffiths nonnegative on M;
2. R^{g_0} is Griffiths positive at least at one point.

Then M is biholonorphic to the complex projective space $\mathbb{CP}^n$.

6. Parabolic flows on almost Hermitian manifolds

Almost Hermitian flow

On a compact almost Hermitian manifold (M, J, ω_0), we define a parabolic flow starting at the almost Hermitian metric ω_0.

AHF
$$\begin{cases} \frac{\partial}{\partial t}\omega(t) = \partial\partial^*_{g(t)}\omega(t) + \bar{\partial}\bar{\partial}^*_{g(t)}\omega(t) - P^{g(t)}, \\ \omega(0) = \omega_0, \end{cases}$$

where $\partial^*_{g(t)}$, $\bar{\partial}^*_{g(t)}$ are the L^2-adjoint operators w.r.t. $g(t)$, and $P^g_{i\bar{j}} = g^{k\bar{l}} R^g_{i\bar{j}k\bar{l}}$ denote the first Chern-Ricci curvature.

Short time existence and uniqueness of the AHF

Proposition (K. 2019)

For an almost Hermitian metric ω, the right-hand side of the AHF $\partial\partial_g^*\omega + \bar\partial\bar\partial_g^*\omega - P^g$ is strictly elliptic.

Let $\{e_r\}$ be a local $(1,0)$-frame w.r.t. g around an arbitrary chosen point.

1. $(\partial\partial_g^*\omega)_{i\bar j} + (\bar\partial\bar\partial_g^*\omega)_{i\bar j}$ involves the second derivatives of g; $g^{k\bar l}e_k e_{\bar l}(g_{i\bar j}) - g^{k\bar l}e_{\bar j}e_i(g_{k\bar l})$,
2. $P^g_{i\bar j}$ involves the second derivatives of g; $-g^{k\bar l}e_{\bar j}e_i(g_{k\bar l})$.
3. the right-hand side of the AHF involves the second derivative of g; $g^{k\bar l}e_k e_{\bar l}(g_{i\bar j}) = g^{k\bar l}\partial_k\partial_{\bar l}g_{i\bar j} + g^{k\bar l}B^{\bar s}_{k\bar l}e_{\bar s}(g_{i\bar j})$.

Therefore, the right-hand side of the AHF is strictly elliptic since g is positive definite, which implies that the AHF is strictly parabolic. From the standard parabolic theory, we obtain the short-time unique existence result since the manifold is supposed to be compact.

Short time existence and uniqueness of the AHF

Theorem (K. 2019)

$\exists! \omega(t)$ the short-time solution of the AHF starting at ω_0 on $M \times [0, \varepsilon)$ for some $\varepsilon > 0$.

Almost Hermitian curvature flow

We define another parabolic flow on a compact almost Hermitian manifold (M, J, g_0) :

AHCF
$$\begin{cases} \frac{\partial}{\partial t} g(t) = -S^{g(t)} - Q^7 - Q^8 + B * T' + \bar{e}(T'), \\ g(0) = g_0, \end{cases}$$

where $S^g_{i\bar{j}} = g^{k\bar{l}} R^g_{k\bar{l}i\bar{j}}$ is the second Chern-Ricci curvature,

$$Q^7_{i\bar{j}} := g^{r\bar{s}} g^{k\bar{l}} T_{irk} T_{\bar{s}\bar{l}\bar{j}}, \quad Q^8_{i\bar{j}} := g^{r\bar{s}} g^{k\bar{l}} T_{irk} T_{\bar{j}\bar{l}\bar{s}},$$

and $w_i = g^{r\bar{s}} T_{ir\bar{s}}$,

$$(B * T')_{i\bar{j}} := g^{r\bar{s}} g^{p\bar{q}} B^j_{\bar{s}p} T_{ir\bar{q}} + g^{p\bar{q}} B^r_{\bar{q}i} T_{pr\bar{j}} + g^{s\bar{r}} B^p_{\bar{r}s} T_{pi\bar{j}} + B^r_{\bar{j}i} w_r,$$

and

$$\bar{e}(T')_{i\bar{j}} := -g^{r\bar{l}} e_{\bar{l}}(T^s_{ri}) g_{s\bar{j}} + g^{r\bar{l}} e_{\bar{j}}(T^s_{ri}) g_{s\bar{l}}.$$

Relation between the AHCF and the HCF$_{Q^1}$

Definition

A Hermitian metric ω is called pluriclosed or SKT if

$$\partial\bar{\partial}\omega = 0.$$

Let g_0 be a pluriclosed metric on a compact Hermitian manifold.

$$\text{HCF}_{Q^1} \quad \begin{cases} \frac{\partial}{\partial t} g(t) = -S^{g(t)} + Q^1, \\ g(0) = g_0, \end{cases}$$

$$Q^1_{i\bar{j}} := g^{k\bar{l}} g^{r\bar{s}} T_{ik\bar{s}} T_{\bar{j}\bar{l}r}.$$

Notice that HCF$_{Q^1}$ preserves the pluriclosedness.

Proposition (K. 2019)

If J is integrable, the AHCF coincides with the HCF$_{Q^1}$.

PROOF.

Under our assumption that J is integrable, we have $Q^7 = Q^8 = B * T' = 0$ and we may choose a local $(1,0)$-frame $e_r = \frac{\partial}{\partial z_r} =: \partial_r$ for holomorphic local coordinates $\{z_1, \ldots, z_n\}$. Since we have $\partial_{\bar{r}} T_{ri\bar{j}} = \partial_{\bar{j}} T_{ri\bar{r}}$ for a pluriclosed metric on a Hermitian manifold,

$$
\begin{aligned}
\bar{e}(T')_{i\bar{j}} &= -g^{r\bar{l}}\partial_{\bar{l}}(T^s_{ri})g_{s\bar{j}} + g^{r\bar{l}}\partial_{\bar{j}}(T^s_{ri})g_{s\bar{l}} \\
&= -g^{r\bar{l}}\partial_{\bar{l}}T_{ri\bar{j}} + g^{r\bar{l}}T^s_{ri}\partial_{\bar{r}}g_{s\bar{j}} + g^{r\bar{l}}\partial_{\bar{j}}T_{ri\bar{l}} - g^{r\bar{l}}T^s_{ri}\partial_{\bar{j}}g_{s\bar{l}} \\
&= -g^{r\bar{l}}\partial_{\bar{j}}T_{ri\bar{l}} + g^{r\bar{l}}\partial_{\bar{j}}T_{ri\bar{l}} + g^{r\bar{l}}T^s_{ri}\Gamma^{\bar{k}}_{\bar{r}\bar{j}}g_{s\bar{k}} - g^{r\bar{l}}T^s_{ri}\Gamma^{\bar{k}}_{\bar{j}\bar{l}}g_{s\bar{k}} \\
&= g^{r\bar{l}}g^{k\bar{s}}T_{ir\bar{s}}T_{\bar{j}\bar{l}k} \\
&= Q^1_{i\bar{j}}.
\end{aligned}
$$

□

Difference between P^g and S^g

Lemma (L. Vezzoni 2011)

$$P^g - S^g = \mathrm{div}^\nabla T' - \nabla \bar{w} + Q^7 + Q^8$$

holds for any almost Hermitian metric g, where

$$\mathrm{div}^\nabla T'_{i\bar{j}} := g^{k\bar{l}} \nabla_{\bar{l}} T_{ki\bar{j}}, \quad (\nabla \bar{w})_{i\bar{j}} := g^{k\bar{l}} \nabla_i T_{\bar{j}\bar{l}k}.$$

Generalized version of PF and HCF$_{Q^1}$ equivalence

Since the AHF takes the expression

$$\frac{\partial}{\partial t}\omega = -\nabla \bar{w} - \bar{\nabla} w - P^g$$

and we have that

$$\begin{aligned} P^g &= \underline{\mathrm{div}^{\nabla} T'} + S^g - \nabla \bar{w} + Q^7 + Q^8 \\ &= \underline{-\bar{\nabla} w - B * T' - \bar{e}(T')} + S^g - \nabla \bar{w} + Q^7 + Q^8 \\ \Leftrightarrow &\quad -\nabla \bar{w} - \bar{\nabla} w - P^g = -S^g - Q^7 - Q^8 + B * T' + \bar{e}(T'). \end{aligned}$$

Generalized version of PF and HCF$_{Q^1}$ equivalence (K. 2019)

The AHF starting at ω_0 is equivalent to the AHCF starting at ω_0.

Regularity result

Theorem (K. 2020)

Let $(M^{2n}, J, g(t))$ be a solution to the AHCF starting at an almost Hermitian metric g_0 for a maximal time interval $[0, \tau_{\max})$ on a compact almost Hermitian manifold. Choose arbitrary $0 < \tau < \tau_{\max}$. Assume that, for a positive constants α with $\alpha/\tau > 1$, the following inequalities hold:

$$\sup_{M\times[0,\tau)} |R|_{g(t)} \leq \frac{\alpha}{\tau}, \quad \sup_{M\times[0,\tau)} |T'|^2_{g(t)} \leq \frac{\alpha}{\tau}, \quad \sup_{M\times[0,\tau)} |\nabla T'|_{g(t)} \leq \frac{\alpha}{\tau}.$$

Then, for any $m \in \mathbb{N}$, the following inequalities hold:

$$|\nabla^m R|_{g(t)} \leq \frac{C_{m,n,\alpha}}{\tau \cdot t^{\frac{m}{2}}}, \quad |\nabla^{m+1} T'|_{g(t)} \leq \frac{C_{m,n,\alpha}}{\tau \cdot t^{\frac{m}{2}}}$$

for any $t \in (0, \tau]$ on M, where $C_{m,n,\alpha}$ is some positive constant depending only on m, n and α.

Blow-up at the maximal time

Theorem (K. 2020)

If $\tau_{\mathrm{max}} < \infty$, then

$$\limsup_{t \to \tau_{\mathrm{max}}} \max \left\{ |R|_{C^0(g(t))}, |T'|^2_{C^0(g(t))}, |\nabla T'|_{C^0(g(t))} \right\} = \infty.$$

7. Preserved properties along the positive HCF on an almost Hermitian manifold

The HCF$_+$ on an almost Hermitian manifold

We consider the HCF$_+$ on a compact almost Hermitian manifold (M, J, g_0):

$$\text{HCF}_+ \quad \begin{cases} \frac{\partial}{\partial t} g(t) = -S^{g(t)} - Q^{g(t)}, \\ g(0) = g_0. \end{cases}$$

where S^g is the second Chern-Ricci curvature, Q^g is given by

$$Q^g_{i\bar{j}} = \frac{1}{2} g^{p\bar{s}} g^{m\bar{n}} T_{\bar{s}\bar{n}i} T_{pm\bar{j}}.$$

Since $-S^g_{i\bar{j}}$ involves the second derivatives of g; $g^{k\bar{l}} e_k e_{\bar{l}}(g_{i\bar{j}})$, and the metric g is positive definite, the right-hand side of the HCF$_+$ is strictly elliptic, which implies that the HCF$_+$ is strictly parabolic. From the standard parabolic theory, we obtain the short-time unique existence result since the manifold is supposed to be compact.

Preserved properties along the HCF_+

Theorem (K.)

Let $g(t)$, $t \in [0, \tau)$ be the solution of the HCF_+ on a compact almost Hermitian manifold (M, J, g_0) with $g(0) = g_0$ for any $\tau < \tau_{\max} < \infty$, where $\tau_{\max}$ is the finite explosion time of the HCF_+. Assume that the Chern curvature R^{g_0} is Griffiths nonnegative (resp. positive), i.e., for any $\xi, \eta \in T^{1,0}M$:

$$R^{g_0}(\xi, \bar{\xi}, \eta, \bar{\eta}) \geq 0 \quad (\text{resp. } > 0).$$

Then for $t \in [0, \tau)$, the Chern curvature $R^{g(t)}$ remains Griffiths nonnegative (resp. positive). If, moreover, the Chern curvature R^{g_0} is Griffiths positive at least at one point, then for any $t \in (0, \tau)$, the Chern curvature $R^{g(t)}$ is Griffiths positive everywhere on M.

Application

Definition

We write $\tilde{g} \in \mathcal{G}(M, J)$ if $\tilde{g}$ is an almost Hermitian metric on a compact almost complex manifold M whose Chern curvature $R^{\tilde{g}}$ is Griffiths nonnegative, and Griffiths positive at least at one point.

Corollary (K.)

Let (M, J) be a compact almost complex manifold.
If $\exists g \in \mathcal{G}(M, J)$, then $\kappa(M, J) = -\infty$.

In the case of the 6-sphere $\mathbb{S}^6$

Remark

Let e_i $(i = 1, \ldots, 7)$ be the standard basis of $\mathbb{R}^7$ and let e^i $(i = 1, \ldots, 7)$ be the dual basis. Denote $e^{ijk} := e^i \wedge e^j \wedge e^k$. Define

$$\Phi := e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}.$$

Φ induces the cross product $\times : \mathbb{R}^7 \times \mathbb{R}^7 \to \mathbb{R}^7$ by $(u \times v) \cdot w = \Phi(u, v, w)$, where $\cdot$ is the Euclidean metric on $\mathbb{R}^7$. We define the cross product operator of $u \in \mathbb{S}^6 = \{u \in \mathbb{R}^7 : u \cdot u = 1\}$ by $J_u := u \times \cdot$, which gives the standard almost complex structure $J_{\mathrm{std}} = \{J_u : u \in \mathbb{S}^6\}$. Then,

$$P_m(\mathbb{S}^6, J_{\mathrm{std}}) = 1 \text{ for } \forall m \in \mathbb{Z}_{\geq 1}, \text{ and } \kappa(\mathbb{S}^6, J_{\mathrm{std}}) = 0$$

(H. Chen, W. Zhang 2023), which implies that

$$\nexists g \in \mathcal{G}(\mathbb{S}^6, J_{\mathrm{std}}).$$

Thank you for your attention.

This work was partially supported by JSPS KAKENHI Grant Number JP19K14543

Sugiyama Jogakuen University
kawamura-m@sugiyama-u.ac.jp

Chen, H., Zhang, W. *Kodaira dimensions of almost complex manifolds I*, Amer. J. Math., **145**, (2) (2023), 477-514.

Chen, H., Zhang, W. *Kodaira dimensions of almost complex manifolds II*, Comm. Anal. Geom. (In Press)

Fu, J., Zhou, X. *Scalar curvatures in almost Hermitian geometry and some applications*, Sci. China Math. **65** no. 12, (2022), 2583-2600.

Streets, J., Tian, G. *A parabolic flow of pluriclosed metrics*, Int. Math. Res. Not. IMRN, (2010), no. 16, 3101-3133.

Streets, J., Tian, G. *Hermitian curvature flow*, J. Eur. Math. Soc. (JEMS), **13** (3) (2011), 601-634.

Ustinovskiy, Y. *The Hermitian curvature flow on manifolds with non-negative Griffiths curvature*, Amer. J. Math., **141** (6) (2019), 1751-1775.

Vezzoni, L. *On Hermitian curvature flow on almost complex manifolds*, Diff. Geom. and its Appl., **29** (2011), 709-722.

- Kawamura, M. *Parabolic flows on almost Hermitian manifolds*, Kyushu J. Math., **73** (1) (2019), 69-87.
- Kawamura, M. *Regularity results for the almost Hermitian curvature flow*, Tsukuba J. Math., **44** (1) (2020), 63-120.
- Kawamura, M. *Nonnegative holomorphic sectional curvature on almost Hermitian manifolds*, preprint.
- Kawamura, M. *On the t-Gauduchon connection on almost Hermitian manifolds*, preprint.
- Kawamura, M. *The positive Hermitian curvature flow on almost complex manifolds with the Griffiths nonnegative Chern curvature*, preprint.