Morse index and first Betti number for self-shrinkers in higher codimension

櫻井陽平氏（埼玉大学）との共同研究

國川慶太 (徳島大学)

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Introduction

Plan of the Talk

1. Minimal submanifolds and their Morse index estimates
2. Index estimates for self-shrinkers (our results)
3. Open Questions

Our Setting

We start the introduction of minimal submanifolds

Setting

- (M^m, g): complete Riem. mfd
- $\Sigma^n \subset M^m$: complete submfd with trivial normal bundle
- $m - n = 1$: trivial normal bundle = two-sided

Comment

- Most of my talk is forcused on closed (cpt w/o boundary) case
- Sometimes I show non-compact examples

First Variation and Minimal Submanifolds

First variation

- Σ_t: normal variation with cpt supported $V \in \Gamma(T^{\perp}\Sigma)$
- First variation formula is given by

$$\frac{d}{dt}\Big|_{t=0} \operatorname{vol}(\Sigma_t) = -\int_{\Sigma} \langle H, V \rangle \, dv$$

- H is the mean curvature vector field of Σ

Minimal submanifolds : $\Longleftrightarrow$ critical points of vol

- $H = 0$

Second Variation Formula

Second Variation

- $\Sigma \subset M$ minimal submanifold ($H = 0$)
- Second variation formula is given by

$$\left.\frac{d^2}{dt^2}\right|_{t=0} \mathrm{vol}(\Sigma_t) = \int_\Sigma |\nabla^\perp V|^2 - \langle \mathcal{B}(V), V\rangle - \langle \mathrm{Ric}^M(V), V\rangle \, dv$$
$$= -\int_\Sigma \langle V, \mathcal{J}_\Sigma V\rangle \, dv$$

for $V \in \Gamma(T^\perp \Sigma)$

- $\mathcal{B}$ is the so-called Simons' operator (defined by 2nd f.f. B)
- $\mathcal{J}_\Sigma := \Delta_\Sigma^\perp + \mathcal{B} + \mathrm{Ric}^M$ (Jacobi operator acting on $\Gamma(T^\perp\Sigma)$)

Morse Index of minimal submanifolds

Morse Index

- $\Sigma \subset M$ cpt minimal submanifold
- Morse Index is the number of negative eigenvalues of $\mathcal{J}_\Sigma$ (counted with multiplicity):

$$\mathcal{J}_\Sigma V + \lambda V = 0, \qquad V \in \Gamma(T^\perp \Sigma)$$

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots \to +\infty$$

Index Form

- $\mathcal{J}_\Sigma$ defines the quadratic form (Hessian of vol):

$$Q(V,V) := -\int \langle V, \mathcal{J}_\Sigma V \rangle \, dv$$

- Morse index of Σ $=$ Index of Σ as a critical point of vol

Stability

- Minimal submanifold Σ is stable if
$$Q(V,V) = \frac{d^2}{dt^2}\Big|_{t=0} \mathrm{vol}(\Sigma) \geq 0$$
for all cpt supported $V \in \Gamma(T^{\perp}\Sigma)$
- Q is positive semidefinite $\cdots$ Morse index is zero
$$0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \to +\infty$$

- If Σ is not stable, then we call it unstable
- Morse index measures the instability of Σ
$$\lambda_1 \leq \cdots \leq \lambda_k < 0 \leq \lambda_{k+1} \leq \cdots \to +\infty$$

Second Variation for Minimal Hypersurfaces

For minimal **hypersurface** case ($m - n = 1$)

- N: (globally defined) unit normal
- $V = \phi N$ with $\phi \in C^\infty(\Sigma)$
- Then the second variation formula becomes

$$Q(\phi,\phi) = \int_\Sigma |\nabla\phi|^2 - |B|^2\phi^2 - \mathrm{Ric}^M(N,N)\phi^2 \, dv$$
$$= -\int \phi \cdot J_\Sigma \phi \, dv$$

- $J_\Sigma = \Delta_\Sigma + |B|^2 + \mathrm{Ric}^M(N,N)$ on $C^\infty(\Sigma)$
- Our eigenvalue problem

$$J_\Sigma \phi + \lambda\phi = 0 \quad \text{on} \quad C^\infty(\Sigma)$$

Some Known Results for Stability

Simons and Schoen–Yau

As for general ambient space, we know

Theorem (Simons)

$\mathrm{Ric}^M \geq 0$ and $\Sigma^n \subset M^{n+1}$ closed stable minimal

$\Rightarrow \quad \Sigma$ is totally geodesic

Theorem (Schoen–Yau)

$\mathrm{Scal}_M > 0$ and $\Sigma^2 \subset M^3$ closed stable minimal

$\Rightarrow \quad \Sigma$ is topologically $\mathbb{S}^2$ or $\mathbb{RP}^2$

Fischer–Corbrie–Schoen, Do Carmo–Peng

Note In $\mathbb{R}^m$, $\nexists$ closed minimal submanifold

In 1980's Fischer-Corbrie–Schoen and Do Carmo–Peng independently proved generalized Bernstein type theorem:

Theorem

$\Sigma^2 \subset \mathbb{R}^3$ complete stable minimal $\Rightarrow$ plane

- Stable $\Rightarrow$ (topological) simplicity of Σ

Chodosh–Li and Catino–Mastrolia–Roncoroni

Recently, in $\mathbb{R}^4$ Chodosh–Li solved a long standing conjecture:

Theorem (Chodosh–Li, 2021)

$\Sigma^3 \subset \mathbb{R}^4$ complete stable minimal $\Rightarrow$ hyperplane

- Catino–Mastrolia–Roncoroni (2023) gave another proof

Higher dimensions

- Higher dimension case, not much is known for stability
- "Stable $\Rightarrow$ simplicity" is still true

Theorem (Cao–Shen–Zhu, 1997)

$\Sigma^n \subset \mathbb{R}^{n+1}$ $(n \geq 3)$ complete stable minimal

$\Rightarrow$ Σ has only one end

Palmer

Theorem (Palmer, 1991)

$\Sigma^n \subset \mathbb{R}^{n+1}$ $(n \geq 2)$ complete stable minimal

$\Rightarrow$ $\nexists$ codim 1 cycle $\gamma \subset \Sigma$ s.t.

$$\Sigma \setminus \gamma \quad \text{is connected}$$

- Miyaoka(1993): In non-negatively curved ambient spaces
- K.–Saito(2019): For stable translating solitons

Palmer's argument

Palmer's argument is proof by contradiction:

Assume: $\exists$ codim 1 cycle $\gamma \subset \Sigma$ with $\Sigma \setminus \gamma$ is connected

$\Rightarrow \quad \mathcal{H}^1_{L^2}(\Sigma) \neq \{0\}$ (by Doziuk)

$\Rightarrow \quad \exists$ nontrivial L^2-harmonic 1-form ω on Σ

$\Rightarrow \quad \omega$ gives a cpt supported variation with $Q < 0$

This contradicts the stability of Σ $\square$

Observation

- Harmonic 1-forms $\rightsquigarrow$ Volume decreasing variation

Index Estimates for Minimal Hypersurfaces by Betti Numbers

Ros' index estimate in a Flat Torus

Theorem (Ros, 2006)

Let $\Sigma^2 \subset T^3$ be a non-flat closed minimal surface in a flat 3-torus. Then

$$\text{index}(\Sigma) \geq \frac{2\text{genus}(\Sigma) - 3}{3} = \frac{b_1(\Sigma) - 3}{3}$$

- $b_1(\Sigma)$ is the first Betti number, i.e., dimension of

$$\mathcal{H}^1(\Sigma) \cong H^1_{\text{dR}}(\Sigma) \cong H^1(\Sigma, \mathbb{R})$$

- Topological complexity $\Rightarrow$ highly unstable

Savo' index estimate in a Round Sphere

Theorem (Savo, 2010)

Let $\Sigma^n \subset \mathbb{S}^{n+1}(1) \subset \mathbb{R}^{n+2}$ be a closed minimal surface. Then

$$\operatorname{index}(\Sigma) \geq \frac{2b_1(\Sigma)}{(n+2)(n+1)}$$

Moreover, if Σ^n is non-totally geodesic and $n \geq 3$, then

$$\operatorname{index}(\Sigma) \geq \frac{2b_1(\Sigma)}{(n+2)(n+1)} + n + 2$$

- For non-totally geodesic minimal surface $\Sigma^2 \subset \mathbb{S}^3$

$$\operatorname{index}(\Sigma) \geq 5$$

- Clifford torus $T^2 \subset \mathbb{S}^3$ has $\operatorname{index}(\Sigma) = 5$ (Urbano, 1990)

Marques–Neves, Schoen Conjecture

Conjecture (Marques–Neves, Schoen)

- (M^{n+1}, g) closed Riem mfd with $\mathrm{Ric} > 0$
- $\exists\, C = C(n, g) > 0$ s.t. for $\forall\, \Sigma^n \subset M^{n+1}$ closed minimal,

$$\mathrm{index}(\Sigma) \geq C b_1(\Sigma)$$

Ambrozio–Carlott–Sharp Conjecture

Conjecture (Ambrozio–Carlott–Sharp)

- Under the same assumption as MNS conjecture
- $\exists\, C = C(n, g) > 0$ s.t. for $\forall\, \Sigma^n \subset M^{n+1}$ closed minimal,

$$\mathrm{index}(\Sigma) \geq C \sum_{p=1}^{n} b_p(\Sigma)$$

Song's Result

Theorem (Song, 2023)

Let:

- (M^{n+1}, g) closed Riem mfd with $3 \leq n \leq 7$
- $A > 0$

Then:

$\exists\, C = C(n, g, A) > 0$ s.t. $\forall\, \Sigma^n \subset M^{n+1}$ with $\mathrm{vol}(\Sigma) < A$,

$$\mathrm{index}(\Sigma) \geq C \sum_{p=0}^{n} b_p(\Sigma) - 1$$

Note: We do not need to impose curvature assumption on M

Ambrozio–Carlott–Sharp's Index Estimate

Theorem (Ambrozio–Carlott–Sharp, 2016)

Let:

- $(M^{n+1}, g) \hookrightarrow \mathbb{R}^d$ closed Riem mfd
- $\Sigma^n \subset M^{n+1}$ closed minimal hypersurface

Assume some curvature condition for $M^{n+1} \subset \mathbb{R}^d$:

$$\int_\Sigma \{|\mathrm{II}(\cdot, X)|^2 - |\mathrm{II}(X, N)|^2 + (|\mathrm{II}(\cdot, N)|^2 - |\mathrm{II}(N, N)|^2)|X|^2 - \mathrm{tr}_\Sigma(\mathrm{Rm}^M(\cdot, X, \cdot, X)) - \mathrm{Ric}^M(N, N)|X|^2\}\, dv < 0$$

for every nonzero $\forall X \in \Gamma(T\Sigma)$.

Then:

$$\mathrm{index}(\Sigma) \geq \frac{2}{d(d-1)}\, b_1(\Sigma)$$

Application for ACS

- We need to check the curvature condition (ACS condition)

$$\int_\Sigma \mathrm{ACS}(X,X)\,dv$$
$$:= \int_\Sigma \{|\mathrm{II}(\cdot,X)|^2 - |\mathrm{II}(X,N)|^2 + (|\mathrm{II}(\cdot,N)|^2 - |\mathrm{II}(N,N)|^2)|X|^2$$
$$- \mathrm{tr}_\Sigma(\mathrm{Rm}^M(\cdot,X,\cdot,X)) - \mathrm{Ric}^M(N,N)|X|^2\}\,dv < 0$$

Corollary (Ambrozio–Carlott–Sharp, 2016)

M^{n+1} is CROSS $\Rightarrow$ ACS is OK $\Rightarrow$ Index est

- CROSS = **C**ompact **R**ank **O**ne **S**ymmetric **S**paces:

$$\mathbb{S}^{n+1}, \quad \mathbb{RP}^{n+1}, \quad \mathbb{CP}^m, \quad \mathbb{HP}^l, \quad \mathrm{Ca}\mathbb{P}^2 \quad \hookrightarrow \quad \mathbb{R}^d$$

Rough Skech of the Proof of ACS

Preparation

- $\{v_A\}_{A=1}^d$ ONB of $\mathbb{R}^d$ $\rightsquigarrow$ $\{v_A \wedge v_B\}_{A<B}$ ONB of $\mathbb{R}^{\binom{d}{2}}$
- For a harmonic 1-form $\omega \in \mathcal{H}^1(\Sigma)$,

$$u_{AB} := \langle N \wedge \omega^\sharp, v_A \wedge v_B \rangle, \quad A < B$$

- These are coordinates of $N \wedge \omega^\sharp$ in $\mathbb{R}^{\binom{d}{2}}$

Trace (average) of the Hessian

- ACS condition $\Rightarrow \sum_{A<B} Q(u_{AB}, u_{AB}) < 0$
- ACS condition makes the trace of Q negative
- $\omega \in \mathcal{H}^1(\Sigma) \rightsquigarrow u_{AB}$: vol decreasing variation on average

Rough Skech of the Proof of ACS

Goal

$$\mathrm{index}(\Sigma) \geq \frac{2}{d(d-1)}\, b_1(\Sigma) = \frac{b_1(\Sigma)}{\binom{d}{2}}$$

Assume

$$\mathrm{index}(\Sigma) \times \binom{d}{2} < b_1(\Sigma) = \mathcal{H}^1(\Sigma)$$

$\Rightarrow \quad \exists\, \omega \in \mathcal{H}^1(\Sigma) \setminus \{0\}$ s.t. for all $A < B$,

$$u_{AB} \perp (\underbrace{\text{negative eigenspaces of } J_\Sigma}_{\dim = \mathrm{index}(\Sigma) = k})$$

$$\Rightarrow \quad Q(u_{AB}, u_{AB}) \geq \lambda_{k+1} \int_\Sigma |u_{AB}|^2\, dv \geq 0 \qquad \text{(min-max)}$$

$$\Rightarrow \quad \sum_{A<B} Q(u_{AB}, u_{AB}) = \lambda_{k+1} \int_\Sigma |\omega|^2\, dv \geq 0 \quad \textbf{contradiction} \quad \square$$

Higher Codimension: Adauto–Batista in Sphere

Theorem (Adauto–Batista, 2022)

Assume:

- $\Sigma^n \subset \mathbb{S}^{n+m}(1) \subset \mathbb{R}^{n+m+1}$ closed min submfd ($m \geq 0$)
- $m \cdot \mathrm{Ric}^{\Sigma} > -(n-1)$

Then: index estimate

$$\mathrm{index}(\Sigma) \geq \frac{b_1(\Sigma)}{\binom{n+m+2}{2}}$$

- This is a higher codimensional generalization of Savo's result
- However, we addionally need curvature condition for Σ
- Maybe it is possible: $\mathrm{index}(\Sigma) \geq b_1(\Sigma)$

Higher Codimension: Adauto–Batista in Riemann

Theorem (Adauto–Batista, 2022)

Assume:

- $\Sigma^n \subset M^{n+m}(1) \hookrightarrow \mathbb{R}^d$ closed min submfd ($m \geq 0$)
- Curvature conditions M and Σ

$$\int_\Sigma \mathrm{ACS}(X, X)\, dv < 2m \int \mathrm{Ric}^\Sigma(X, X)\, dv$$

for any nonzero $X \in \Gamma(T\Sigma)$

Then: index estimate

$$\mathrm{index}(\Sigma) \geq \frac{2b_1(\Sigma)}{d(d-1)}$$

Index Estimates for Self-Shrinkers

Self-Shrinker

- Self-shrinkers $x : \Sigma^n \to \mathbb{R}^d$ are known as singularity models of mean curvature flow
- They are critical points of the Gaussian weighted volume

  $$\mathrm{vol}_f(\Sigma) = \int_\Sigma e^{-f}\, dv$$

  with $f = |x^2|/4$
- They can be considered as weighted minimal submanifolds

First variation

- Σ_t: normal variation with cpt supported $V \in \Gamma(T^\perp \Sigma)$
- First variation formula is given by

$$\frac{d}{dt}\Big|_{t=0} \mathrm{vol}_f\,(\Sigma_t) = -\int_\Sigma \langle H_f, V \rangle e^{-f}\, dv$$

- $H_f = H + \frac{x^\perp}{2}$ is the weighted mean curvature
- $x : \Sigma^n \to \mathbb{R}^d$ is a self-shrinker $\iff H = -\frac{x^\perp}{2}$

Second Variation Formula for self-shrinkers

Second Variation

- $\Sigma^n \subset \mathbb{R}^d$ self-shrinker with $f = |x|^2/4$ $(H_f = 0)$
- Second variation formula is given by

$$\frac{d^2}{dt^2}\Big|_{t=0} \mathrm{vol}_f(\Sigma_t) = -\int \langle V, LV \rangle e^{-f}\, dv$$

- $L := \Delta_\Sigma^\perp - \frac{1}{2}\nabla_{x^\top}^\perp + \mathcal{B} + \frac{1}{2}$ (Jacobi op acting on $\Gamma(T^\perp\Sigma)$)
- Eigenvalue problem

$$LV + \lambda V = 0$$

- **Morse index** and **stability** are defined in the same way as usual minimal submanifolds

Stability for self-shrinkers

Stability

- Every self-shrinkers $\Sigma^n \subset \mathbb{R}^d$ are unstable:

$$LH = H, \quad Lv^\perp = \frac{1}{2}v^\perp \quad \text{for} \quad v \in \mathbb{R}^d$$

are unstable directions (and produce index)

F-stability

- Introduced by Colding–Minicozzi
- Stability which do not consider the trivial directions H and $v^\perp$
- Classification for closed self-shrinker $\Sigma^n \subset \mathbb{R}^{n+1}$

$$\text{F-stable} \quad \Rightarrow \quad \Sigma^n = \mathbb{S}^n(\sqrt{2n}) \subset \mathbb{R}^{n+1}$$

- Partial classification resuls for higher codimension are found in the paper by Andrews–Li–Wei and Lee–Lue.

Weighted harmonic forms and Betti numbers

Harmonic 1-forms

- $\Sigma^n \subset \mathbb{R}^d$ closed self-shrinker with $f = |x|^2/4$
- Space of weighted harmonic 1-forms

$$\mathcal{H}_f^1(\Sigma) = \{\omega \in \Omega^1(\Sigma) \mid (\delta_f d + d\delta_f)\omega = 0\}$$

- The Hodge decomposition still holds in the weithed case:

$$\mathcal{H}_f^1(\Sigma) \cong H_{\mathrm{dR}}^1(\Sigma)$$

- First Betti number $b_1(\Sigma) = \dim \mathcal{H}_f^1(\Sigma)$

Index Estimate for Self-Shrinking Hypersurfaces

Theorem (Impera–Rimoldi–Savo, 2020)

Let $x : \Sigma^n \to \mathbb{R}^{n+1}$ be a closed self-shrinker. Then

$$\text{index}(\Sigma) \geq \frac{2}{n(n+1)} b_1(\Sigma) + n + 1$$

In particular if $n = 2$, we have

$$\text{index}(\Sigma) \geq \frac{2}{3} \text{genus}(\Sigma) + 3$$

- No curvature condition for Σ is needed
- Similar to closed minimal hypersurfaces in the sphere (Savo)
- Possible: general weighted setting with ACS condition

Index Estimate for Self-Shrinkers in High Codimension

Theorem (K.–Sakurai, 2023)

Assume:

- $x : \Sigma^n \to \mathbb{R}^d$ closed self-shrinker
- $\mathrm{Ric}_f^\Sigma > -\dfrac{1}{2(d-n-1)}$

Then:

$$\mathrm{index}(\Sigma) \geq \frac{2}{d(d-1)} b_1(\Sigma)$$

- Compare to the result by Adauto–Batista ($\Sigma^n \subset \mathbb{S}^{n+m}$: min)
- Ric_f^Σ condition is very restrictive ($\leftarrow$ used for ACS argument)

Examples: Spherical Self-Shrinkers

Spherical Self-Shrinkers

- Self-shrinker $x : \Sigma^n \to \mathbb{R}^d$ is called **spherical** if

$$\Sigma^n \subset \mathbb{S}^m \subset \mathbb{R}^d$$

- Equivalently, $\Sigma^n \subset \mathbb{S}^m$ is minimal
- $\mathrm{Ric}_f^\Sigma = \mathrm{Ric}^\Sigma$

Clifford torus

- Clifford torus $T^2 \subset \mathbb{S}^3 \subset \mathbb{R}^4$ is a spherical self-shrinker
- $\mathrm{Ric}_f^{T^2} = \mathrm{Ric}^{T^2} = 0$

$$\mathrm{index}(\Sigma) \geq \frac{1}{3} \qquad \text{useless...}$$

Open Questions

Open Questions

- Index estimate by p-th Betti numbers $b_p(\Sigma)$
- Index estimate for CMC hypersurfaces
- Index estimate for λ-hypersurfaces

Index estimate for min hypersurf by p-th Betti numbers

- Remark in the ACS paper:

  "Partially analogous computations for harmonic p-forms also yield formula similar to ACS condition. However the ACS condition contains terms that depend on the second fundamental form B of $\Sigma^n \subset M^{n+1}$. "

- We need additional curvature condition w.r.t. B for the index estimate by p-th Betti number

- In their paper, no computations for harmonic p-forms...

Index estimate by the Second Betti Number

Proposition (ACS)

Assume:

- $\Sigma^n \subset M^{n+1} \hookrightarrow \mathbb{R}^d$: closed minimal
- $|B|^2 < \dfrac{3n-5}{n}$

Then:

$$\text{index}(\Sigma) \geq \frac{2}{d(d-1)} b_2(\Sigma)$$

- If $n = 4$, we use the argument by Tanno to improve the curvature assumption (K.–Sakurai):

$$|B|^2 < 7$$

Index Estimate for CMC surfaces

Theorem (Aiex–Hong, 2021)

Assume:

- $\Sigma^2 \subset M^3 \hookrightarrow \mathbb{R}^d$ CMC surface
- $M^3 \hookrightarrow \mathbb{R}^d$ satisfies ACS type condition

Then:

$$\text{index} \geq \frac{\text{genus}(\Sigma)}{d}$$

- The idea is basically the same as ACS
- Check: the coordinates of harmonic 1-form ω is admissible in the CMC sense (restricted to vol-preserving variations)
- Use: ω is a harmonic 1-form $\Rightarrow \star\omega$ is also a harmonic 1-form

λ-Hypersurfaces and Index Estimates

λ-hypersurfaces

- Cheng–Wei (2014) introduced the notion of λ-hypersurface
- Weighted version of CMC hypersurfaces:

$$H_f = \lambda \, (\text{constant})$$

Index estimate?

- We expect that the similar technique is also available for λ-hypersurfaces
- However, in this case, duality for harmonicity does not hold:

$$\omega \text{ is } f\text{-harmonic} \quad \rightsquigarrow \quad \star_f \omega \text{ may not be } f\text{-harmonic}$$