ヒルベルト空間への極作用における例外軌道の非存在

Non-existence of exceptional orbits under polar actions on Hilbert spaces

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Sec. 1 - Polar actions on Riemannian manifolds (1/4)

Recall

smooth left action

$$\underset{\text{Lie group}}{G} \curvearrowright \underset{\text{smooth mfd}}{M}$$

- An action of G on M is **proper**

$\underset{\text{def}}{\iff}$ the map $G \times M \to M \times M$, $(g, p) \mapsto (g \cdot p, p)$ is proper

$\implies$ • the orbit space M/G is a Hausdorff space;

• the orbit $G \cdot p = \{g \cdot p \mid g \in G\}$ through $p \in M$ is a properly embedded submanifold of M;

• the isotropy subgroup $G_p = \{g \in G \mid g \cdot p = p\}$ is compact.

Note

G: compact $\implies$ the action of G on M is proper.

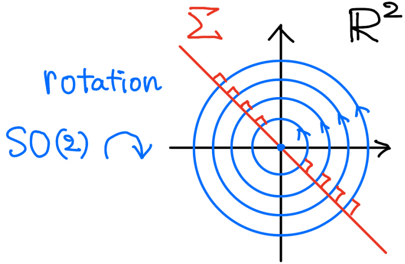
Sec. 1 - Polar actions on Riemannian manifolds (2/4)

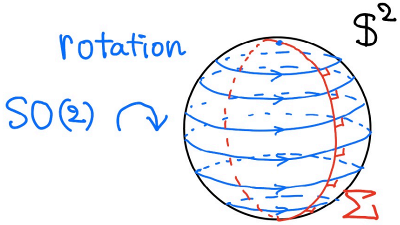
Def (Hermann 1962, see also Bott-Samelson 1958)

$$\underset{\text{Lie group}}{G} \overset{\text{proper, isometric}}{\curvearrowright} \underset{\text{complete Riem mfd}}{M}$$

- the action of G on M is **polar**

$\overset{\text{def}}{\Longleftrightarrow}$ $\exists\, \Sigma$: connected complete submfd of M which meets every G-orbit orthogonally (Here Σ is called a **section** of the action)





Sec. 1 - Polar actions on Riemannian manifolds (3/4)

Note

- It follows that section Σ is **totally geodesic** in M.
- Moreover, in many examples, **Σ is flat** in the induced metric.

  A polar action with flat section is called **hyperpolar**.

Example

The action of $O(n)$ on $\mathrm{Sym}(n, \mathbb{R})$ by $A \cdot X := AXA^{-1}$ is hyperpolar, where $\Sigma = \mathrm{diag}(n)$. (Inner product: $\langle X, Y \rangle := \mathrm{tr}(XY)$)

Example (adjoint action)

G: connected compact Lie group with bi-inv metric. Then the action $G \curvearrowright G$ by $g \cdot h := ghg^{-1}$ is hyperpolar (Σ: maximal torus)

Sec. 1 - Polar actions on Riemannian manifolds (4/4)

Example (isotropy action)

G/K: compact symmetric space. Then the action $K \curvearrowright G/K$ by $g \cdot hK := (gh)K$ is hyperpolar (Σ: maximal flat)

Example (Hermann action)

$G/K, G/H$: compact symmetric spaces. Then the actions $H \curvearrowright G/K$ and $K \curvearrowright G/H$ are hyperpolar.

Note

- If $M = \mathbb{E}^n$, then "polar $\Longleftrightarrow$ hyperpolar".
- If $M \neq \mathbb{E}^n$, then polar action is *not* necessarily hyperpolar.

  Such examples are known when
  - M: compact symmetric sp of rank 1,
  - M: non-compact symmetric sp (not necessarily of rank 1).

Sec. 2 - Exceptional orbits (1/4)

Recall

Left smooth action

G (Lie group) $\curvearrowright$ M (smooth mfd)

- **The orbit $G \cdot p$ is principal**

  $\overset{}{\underset{\text{def}}{\iff}}$ $\forall q \in M, \exists g \in G$ **s.t.** $G_p \subset g G_q g^{-1}$.

  $\implies$ $G \cdot p \ (\cong G/G_p)$ **is maximal dimensional.**

- **The orbit $G \cdot p$ is exceptional**

  $\underset{\text{def}}{\iff}$ $G \cdot p$ **is maximal dimensional, and *not* principal.**

- **The orbit $G \cdot p$ is singular**

  $\underset{\text{def}}{\iff}$ $G \cdot p$ **is *not* maximal dimensional.**

Sec. 2 - Exceptional orbits (2/4)

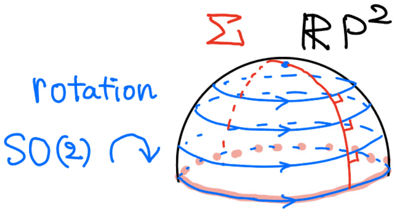
Examples

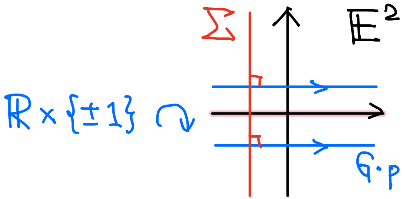
- **The rotational action $SO(2) \curvearrowright \mathbb{R}P^2$ is hyperpolar and has an exceptional orbit (*equator*).**

  (Note: the manifold is not simply connected)

- **The action $\mathbb{R} \times \{\pm 1\} \curvearrowright \mathbb{E}^2$ defined by $(a, \pm 1) \cdot (x, y) := (x + a, \pm y)$ is hyperpolar and has an exceptional orbit (x-*axis*).**

  (Note: the group is not connected)





Sec. 2 - Exceptional orbits (3/4)

Theorem (Alexandrino-Töben 2006, Alexandrino 2011, Grove-Ziller 2012)

G: connected Lie group,
M: simply connected complete Riemannian manifold.
$\Longrightarrow$ Polar action of G on M has no exceptional orbits.

The proof is *not* easy.

However, in the Euclidean case $M = \mathbb{E}^n$,
a simple geometric proof is known (next page):

Sec. 2 - Exceptional orbits (4/4)

Recall (Terng 1985)

- A submanifold N of $\mathbb{E}^n$ is isoparametric

$\underset{\text{def}}{\Longleftrightarrow}$ (1) the normal bundle $T^\perp N$ is flat;

(2) the principal curvatures in the direction of any parallel normal vector field are constant.

$\Longrightarrow$ $T^\perp N$ is globally flat (: non-trivial result).

- $G \curvearrowright \mathbb{E}^n$: polar, N: principal orbit $\Longrightarrow$ N: isoparam submfd of $\mathbb{E}^n$

Proof in the Euclidean case (due to Berndt-Console-Olmos)

Suppose $G \curvearrowright \mathbb{E}^n$: polar. There are two steps:

(1) N: maximal dimensional orbit $\Longrightarrow$ N: isoparametric submfd of $\mathbb{E}^n$

(2) $T^\perp N$: globally flat $\Longrightarrow$ N: principal orbit.

My question

Can we generalize their theorem to the case of Hilbert space?

Sec. 3 - Polar actions on Hilbert space (1/4)

Definition (Palais-Terng 1988)

smooth left action

$\mathcal{L}$ $\curvearrowright$ V

Hilbert Lie group (separable) Hilbert sp

(Hilbert mfd with smooth group str)

- The action of $\mathcal{L}$ on V is Fredholm

$\overset{\Longleftrightarrow}{\text{def}}$ For each $p \in M$, the map $\omega_p : \mathcal{L} \to V$, $g \mapsto g \cdot p$ is Fredholm.

$\overset{\Longleftrightarrow}{\text{def}}$ For each $p \in M$, the differential

$$d\omega_p : T_e\mathcal{L} \to T_pV$$

is a Fredholm operator (i.e. Ker and Coker are finite dim).

$\Longrightarrow$ • the isotropy subgroup $\mathcal{L}_p$ has finite dimension.

• every orbit $\mathcal{L} \cdot p$ has finite codimension.

Sec. 3 - Polar actions on Hilbert space (2/4)

Definition (Palais-Terng 1988)

$$\begin{array}{ccc} & \text{isometric} & \\ & \text{proper, Fredholm (PF)} & \\ \mathcal{L} & \curvearrowright & V \\ \text{Hilbert Lie group} & & \text{(separable) Hilbert sp} \end{array}$$

- The action of $\mathcal{L}$ on V is **polar**

$\underset{\text{def}}{\iff}$ $\exists\, \Sigma$: closed affine subspace of V

s.t. Σ meets every $\mathcal{L}$-orbit orthogonally.

Sec. 3 - Polar actions on Hilbert space (3/4)

Example (Palais-Terng 1989, Pinkall-Thorbergsson 1990, Terng 1995)

G: connected compact Lie grp with a bi-inv metric, $\mathfrak{g}$: its Lie algebra.

- path group (pointwise multiplication $(gh)(t) = g(t)h(t)$)

  $\mathcal{G} := H^1([0,1], G) = \{g : [0,1] \to G : \text{Sobolev } H^1\text{-map}\}$ $(\subset C([0,1], G))$

- Hilbert space

$$V_{\mathfrak{g}} := L^2([0,1], \mathfrak{g}) = \{u : [0,1] \to \mathfrak{g} : \ L^2\text{-map}\}$$

- The isometric PF action of $\mathcal{G}$ on $V_{\mathfrak{g}}$ is defined by:

$$g * u := gug^{-1} - g'g^{-1}$$

(⇒ transitive)

- For any closed subgroup U of $G \times G$, the subgroup is defined by

$$P(G, U) := \{g \in \mathcal{G} \mid (g(0), g(1)) \in U\}.$$

- The action of $P(G, U)$ on $V_{\mathfrak{g}}$ is polar if and only if the action of U on G by $(b, c) \cdot a := bac^{-1}$ is hyperpolar.

Sec. 3 - Polar actions on Hilbert space (4/4)

In particular:

Hyperpolar action on compact symmetric space G/K can be lifted:

$$
\begin{array}{ccccccc}
\mathcal{G} & \supset & P(G, H \times K) & \curvearrowright & V_{\mathfrak{g}} & : & \text{polar} \\
 & & & & & & \Updownarrow \\
G \times G & \supset & U = H \times K & \curvearrowright & G & : & \text{hyperpolar} \\
 & & & & \pi \downarrow & & \Updownarrow \\
G & \supset & H & \curvearrowright & G/K & : & \text{hyperpolar}
\end{array}
$$

Sec. 4 - Results (1/2)

Remark

In the case of PF actions:

- Principal orbits are defined similarly to the finite dim case.
- Principal orbits have minimal codimension.

Definition

In the case of PF actions:

- An exceptional orbit is an orbit of minimal codimension which is not principal.
- A singular orbit is an orbit which is not of minimal codimension.

Sec. 4 - Results (2/2)

Theorem (M., preprint)

$\mathcal{L}$: connected Hilbert Lie group, V: Hilbert space.
$\Longrightarrow$ Polar action of $\mathcal{L}$ on V has no exceptional orbits.

Note

- The strategy of proof is similar to that in the Euclidean case.
  - An isoparametric PF submanifold N of a Hilbert space V is defined similarly to the finite dim case (Terng 1989).
  - N: isoparametric submanifold in $V \Longrightarrow T^{\perp}N$: globally flat. (Heintze-Liu-Olmos 2006)
- My proof is complete and valid also in the Euclidean case.

Corollary

G/K: simply connected compact Riemannian symmetric space.
H: closed connected subgroup of G.
$\Longrightarrow$ Hyperpolar action of H on G/K has no exceptional orbits.

Sec. 5 - Toward the generalization (1/2)

Difficulty 1: The lift of polar action?

$$\begin{array}{ccccccc}
P(G, H \times K) & \curvearrowright & V_{\mathfrak{g}} & : & \text{polar} & & ?? \\
 & & \Phi \downarrow & & \Updownarrow & & \Updownarrow \\
H \times K & \curvearrowright & G & : & \text{hyperpolar} & & ?? \\
 & & \pi \downarrow & & \Updownarrow & & \Updownarrow \\
H & \curvearrowright & G/K & : & \text{hyperpolar} & & \text{polar}
\end{array}$$

Here $\Phi : V_{\mathfrak{g}} \to G$ denotes the parallel transport map (natural Riemannian submersion).

The lifted action has low copolarity?
The lifted action is taut?

Related problem: Polar actions on G/K of higher rank.

Sec. 5 - Toward the generalization (2/2)

Difficulty 2: Extension to the non-compact case?

The problem is the lack of bi-invariant Riemannian metric on G.
($\Rightarrow$ the lifted action (gauge transformation) is not isometric)

$$
\begin{array}{ccc}
P(G, H \times K) & \curvearrowright & (V_{\mathfrak{g}},\ \langle\cdot,\cdot\rangle_{L^2}) \\
 & & \Phi \downarrow \\
H \times K & \curvearrowright & (G,\ g_G) \\
 & & \pi \downarrow \\
H & \curvearrowright & (G/K,\ g_{G/K})
\end{array}
$$

The relation of the affine connections?
The relation of the topology?

Appendix: Method of Gorodski-Thorbergsson

Definition

N: submanifold of $\mathbb{E}^n$.

N is taut $\underset{\text{det}}{\Leftrightarrow}$ every non-degenerate squared distance function is a perfect Morse function.

A representation is taut $\underset{\text{det}}{\Leftrightarrow}$ their orbits are taut submanifolds.

Theorem (Gorodski-Thorbergsson (2003))

$\Longrightarrow$ Taut representation on $\mathbb{E}^n$ has no exceptional orbits.

Remark

Their method is also valid in the case of Hilbert spaces

- The squared distance function to a PF submanifold satisfies the Palais-Smale condition $\Longrightarrow$ Morse theory can be applied.
- Polar actions on Hilbert spaces are taut.

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Thank you for your attention !