Totally Complex Submanifolds and *R*-spaces

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Plan of this talk

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0. Preliminaries

Let

$$\varphi : M^m \longrightarrow N^n$$

be a smooth immersion of a smooth manifold M into a Riemannian manifold (N, g_N).

- $g_M := \varphi^* g_N$: induced Riemannian metric on M by φ
- ∇^M : Levi-Civita connection of TM with respect to g_M
- ∇^N : Levi-Civita connection of TN with respect to g_N
- $E = \varphi^{-1} TN$: pull-back vector bundle on M by φ from TN
- ∇^φ : pull-back connection from ∇^N by φ on $\varphi^{-1} TN$

$$\begin{array}{ccc} \varphi^{-1}TN & \xrightarrow{\tilde{\varphi}} & TN \\ \pi \downarrow (\nabla^\varphi, \varphi^{-1} g_N) & & \pi \downarrow (\nabla^N, g_N) \\ M^m & \xrightarrow{\varphi} & N^n \end{array}$$

The Gauss and Weingarten formulas are

$$\nabla^{\varphi}_{X}(d\varphi(Y)) = d\varphi(\nabla^{M}_{X}Y) + \alpha^{M}(X, Y)$$
$$\nabla^{\varphi}_{X}\xi = -d\varphi(A^{M}_{\xi}(X)) + \nabla^{\perp}_{X}\xi$$

for each $X, Y \in \mathfrak{X}(M)$ and each $\xi \in C^{\infty}(T^{\perp}M)$.
Here

α^{M} : the second fundamental form of $\varphi : M \to N$,

A^{M} : the shape operator of $\varphi : M \to N$,

$\nabla^{\perp}$: the normal connection on $T^{\perp}M$ for $\varphi : M \to N$.

Note that

$$g_M(A^{M}_{\xi}(X), Y) = g_N(\alpha^{M}(X, Y), \xi).$$

The mean curvature vector field $\mathcal{H}$ for $\varphi : M \to N$ is defined by

$$\mathcal{H} := \mathrm{tr}_{g_M}\alpha^{M} = \sum_{i=1}^{m} \alpha^{M}(e_i, e_i) \in C^{\infty}(T^{\perp}M),$$

where $\{e_i\}$ is an orthonornal frame on (M, g_M).

A smooth immersion $\varphi : M \to N$ is called a minimal immersion or a minimal submanifold immersed in N if

$$\mathcal{H} \equiv 0.$$

Definition

The covariant derivative of the second fundamental form α^M of $\varphi : M \to N$ with respect to the Levi-Civita connection ∇^M is defined by

$$(\nabla^*_X \alpha^M)(Y, Z) := \nabla^\perp_X(\alpha^M(Y, Z)) - \alpha^M(\nabla^M_X Y, Z) - \alpha^M(Y, \nabla^M_X Z)$$

for each $X, Y, Z \in \mathcal{X}(M)$.

$\varphi : M \to N$ is said to have parallel second fundamental form (with respect to the Levi-Civita connection) if

$$\nabla^* \alpha^M = 0.$$

1. *R*-spaces and their standard imbeddings

Suppose

$$(G, K, \theta)\text{: compact Riemannian symmetric pair.}$$

Here G: connected compact Lie group.
$\langle\ ,\ \rangle$: $\mathrm{Ad}(G)$- and θ-inv. inner product of $\mathfrak{g}$.

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$$

: the canonical decomp. of $\mathfrak{g}$ as a symm. Lie alg.
$\mathfrak{a}$: a maximal abelian subspace of $\mathfrak{p}$.
For each $E \in \mathfrak{a}$, the compact homogeneous space

$$K/K_E \cong \mathrm{Ad}_{\mathfrak{p}}(K)E \subset \mathfrak{p}$$

is called an ***R**-space* (an orbit of ***s***-representation), where

$$K_E := \{a \in K \mid \mathrm{Ad}_{\mathfrak{p}}(a)(E) = E\} \text{ (isot. subgp. of } K \text{ at } E).$$

It has the *standard imbedding* into the Euclidean space $\mathfrak{p}$:

$$\varphi_E : K/K_E \ni aK_E \longmapsto \mathrm{Ad}_{\mathfrak{p}}(a)(E) \in \mathfrak{p}.$$

Notice that an $\boldsymbol{R}$-space is not a symmetric space in general.

An $\boldsymbol{R}$-space K/K_E is called a *symmetric $\boldsymbol{R}$-space* if (K, K_E) is a symmetric pair.

Theorem (D. Ferus, 1974)

Let N be a connected submanifold of the Euclidean space $\mathbb{R}^\ell$. Then the following two conditions are equivalent each other:

(a) *N is a parallel submanifold, that is,*

$$\nabla^* \alpha^N = 0. \tag{1}$$

(b) *N is congruent to an open piece of a standardly imbedded symmetric $\boldsymbol{R}$-space.*

Olmos-Sánchez's characterization on R-spaces

In general, (N, g_N): a Riemannian manifold,
∇^g: the Levi-Civita connection of g_N.
An affine connection $\tilde{\nabla}$ on N is called a metric connection with respect to g_N if

$$\tilde{\nabla} g_N = 0.$$

$$\iff \qquad g_N(D_X Y, Z) + g_N(Y, D_X Z) = 0 \quad (\forall Y, Z \in TN),$$

where $D := \nabla^N - \tilde{\nabla}$: a tensor field on N of type $(1, 2)$.

A metric connection ∇^c on N is called a canonical connection on (N, g_N) if ∇^c satisfies the condition

$$\nabla^c D^c = 0, \tag{2}$$

where $D^c := \nabla^N - \nabla^c$: a tensor field on N of type $(1, 2)$.
The Levi-Civita connection is a trivial example ($D^c = 0$) of a canonical connection on (N, g_N).

Suppose that N is a submanifold immersed in a Riemannian manifold.

Definition

The covariant derivative of the second fundamental form α^N of N with respect to the canonical connection ∇^c and the normal connection $\nabla^\perp$ is defied by

$$(\nabla^c_X \alpha^N)(Y, Z) := \nabla^\perp_X(\alpha^N(Y, Z)) - \alpha^N(\nabla^c_X Y, Z) - \alpha^N(Y, \nabla^c_X Z)$$

for each $X, Y, Z \in \mathfrak{X}(N)$.

Theorem (Olmos-Sánchez, 1991)

Let N *be a connected compact submanifold fully embedded in the Euclidean space* $\mathbb{R}^\ell$*. Then the following three conditions are equivalent each other:*

(1) *There is a canonical connection* ∇^c *on* N *such that*

$$\nabla^c \alpha^N = 0. \tag{3}$$

(2) N *is a homogeneous submanifold with constant principal curvatures.*

(3) N *is an orbit of an* s*-representation, that is, a standardly imbedded* R*-space.*

Note that the argument of Olmos-Sánchez 1991 also works to have the local version of this theorem, by the extension theorem of isoparametric submanifolds (C.-L. Terng, Olmos).

On (1) ⇒ (2): $\nabla^c \alpha^N = 0 \Rightarrow \nabla^c R^c = 0$ & $\nabla^c T^c = 0$
(Homog. str. of N by Kobayashi-Nomizu I, Tricerri-Vanhecke).

The proof of (2) ⇒ (3) uses

[Palais-Terng1987] R. S. Palais and C. L. Terng, *A general theory of canonical forms*, Trans. Amer. Math. Soc. **300** no. 2 , (1987), 771–789.

[HOT1991] E. Heintze, C. Olmos and G. Thorbergsson, *Submanifolds with constant principal curvatures and normal holonomy groups*. Internat. J. Math. **2** (1991), 167–175.

and the classification of polar representations by Dadok (1985),

[DadokTAMS1985] J. Dadok, *Polar coordinates induced by actions of compact Lie groups*, Trans. Amer. Math. Soc. **288**, no. 1 (1985), 125–137.

which clams that any orthogonal polar representation is orbit-equivalent to an s-representation.
Its geometric proof is investigated by Eschenburg-Heintze (1999), Bergmann (2001).

1. Kähler submanifolds and R-spaces

Theorem (Nakagawa-Takagi, 1976)

M is a parallel Kähler submanifold fully immersed in $\mathbb{C}P^n$ if and only if M is congruent to an open piece of the following seven Kähler submanifolds:

$$M_1 = \mathbb{C}P^m(4) \subset \mathbb{C}P^m(4) \text{ totally geodesic}$$

$$M_2 = \mathbb{C}P^m(2) \subset \mathbb{C}P^{m+\frac{1}{2}m(m+1)}(4)$$

$$M_3 = \mathbb{C}P^{m-s}(4) \times \mathbb{C}P^s(4) \longrightarrow \mathbb{C}P^{m+s(m-s)}(4)$$

$$M_4 = Q_m(\mathbb{C}) \longrightarrow \mathbb{C}P^{m+1}(4) \ (m \geq 3)$$

$$M_5 = SU(s+2)/S(U(2) \times U(s)) \longrightarrow \mathbb{C}P^{s+\frac{1}{2}s(s+1)}(4) \ (s \geq 3)$$

$$M_6 = SO(10)/U(5) \longrightarrow \mathbb{C}P^{15}(4)$$

$$M_7 = E_6/((U(1) \times Spin(10))/\mathbb{Z}_4) \longrightarrow \mathbb{C}P^{26}(4)$$

They are HSS of at most rank 2 and $\frac{1}{2}$-pinched hol. sect. curv.

Nakagawa-Takagi's theorem was reproved (secondly) and generalized by Takeuchi (1978) to homogeneous Kähler submanifolds by more sophisticated use of unitary representation theory.
The third proof of Nakagawa-Takagi,'s theorem was given by Takeuchi (1984) by the algebraic method of Jordan triple systems, based on the correspondence between positive definite Hermitian Jordan triple systems and irreducible symmetric bounded domains. Particularly his work contains

Theorem (Takeuchi,1984)

Any parallel Kähler submanifold of $\mathbb{C}P^n$ can be obtained by the projection of an R-space obtained as an orbit of the isotropy representation of an irreducible Hermitian symmetric space, under the Hopf fibration $\pi : S^{2n+1}(1) \to \mathbb{C}P^n$.

The inverse images of complex Kähler submanifolds

Suppose that M^m is a complex m-dimensional complex submanifold immersed in $\mathbb{C}P^n$. The inverse image of the submfd. M under the Hopf fib. $\pi : S^{2n+1}(1) \to \mathbb{C}P^n$ is defined as

$$\begin{aligned}\hat{M} =& \pi^{-1}(M)\\ =& \{(p, x) \in M \times S^{2n+1}(1) \mid p \in M, x \in \pi^{-1}(\varphi(p)) \subset S^{2n+1}(1)\}.\end{aligned}$$

$$\begin{array}{ccc} & & \mathbb{C}^{n+1} \\ & & \cup \\ \hat{M} = \pi^{-1}(M) & \xrightarrow{\hat{\varphi}} & S^{2n+1}(1) \\ \pi \big\downarrow S^1 & & \pi \big\downarrow S^1 \\ M & \xrightarrow{\varphi} & CP^n \end{array}$$

$\hat{M}$ is a real $2m+1$-dimensional submanifold immersed in $S^{2n+1}(1) \subset \mathbb{C}^{n+1} \cong \mathbb{R}^{2n+2}$ (an invariant submanifold of a Sasakian manifold).

Homogeneous structures on the inverse images of parallel Kähler submanifolds

Then our main result here is as follows:

Theorem (OhnitaCONM2021)

There exists a canonical connection ∇^c *on* $T\hat{M}$*, which is not the Levi-Civita connection, such that*

$$\nabla^c \alpha^{\hat{M}} = 0 \quad \Longleftrightarrow \quad \nabla^* \alpha^{M} = 0 \,.$$

By use of this result we can give the fourth proof of Takeuchi's theorem and Nakagawa-Takagi's theorem.

The canonical connection on the inverse images of Kähler submanifolds

Such a canonical connection $\nabla^c = \nabla^{\hat{M}} - D$ can be explicitly constructed from the tensor field D of type $(1, 2)$ on $\hat{M} = \pi^{-1}(M) \subset S^{2n+1}(1)$ defined by

$$\begin{cases} D_{\tilde{X}}(\tilde{Y}) := -\langle \sqrt{-1}\tilde{X}, \tilde{Y}\rangle \sqrt{-1}\mathbf{x} \in \mathcal{V}_{\mathbf{x}}\hat{M}, \\ D_{\tilde{X}}(V) := \sqrt{-1}\tilde{X} = \widetilde{JX} \in \mathcal{H}_{\mathbf{x}}\hat{M}, \\ D_V(\tilde{X}) := \dfrac{1}{2}\sqrt{-1}\tilde{X} = \dfrac{1}{2}\widetilde{JX} \in \mathcal{H}_{\mathbf{x}}\hat{M}, \\ D_V(V) := 0 \end{cases} \tag{4}$$

for each horizontal vectors $\tilde{X}, \tilde{Y}$ and the vertical vector $V = \sqrt{-1}\mathbf{x}$ at a point $\mathbf{x} \in \hat{M}$ on $\hat{M}$.

Remark. More recently we knew that this canonical connection coincides with Masafumi Okumura's connection on a Sasakian manifold with $r = -\dfrac{1}{2}$ (see [Ohnita23rdProc2021]).

Suppose that M is a parallel Kähler submanifold of $\mathbb{C}P^n$.
Then by Olmos-Sánchez's theorem our result implies

補題

$\hat{M}$ is an R-space obtained as an orbit of the isotropy representation of a compact Riemannian symmetric pair (G, K).

Moreover by discussing symmetries and irreducibility of the submanifolds $\hat{M}$ we can show

補題

(G, K) can be taken as a Hermitian symmetric pair.

and

補題

(G, K) must be an irreducible Hermitian symmetric pair of compact type.

Therefore we obtain Takeuchi's theorem.

By Lie algebraic argument on R-spaces associated with irreducible Hermitian symmetric pairs of compact type (cf. Hyunjung Song, Tsukuba J. Math. 2001), we can explicitly determine parallel Kähler submanifolds $M_E = K/K_{\pi(E)} = K/C_K(\tilde{\alpha})$ as follows:

(G, K)	M_E
$(SU(m+2), S(U(m+1) \times U(1)))$	$\mathbb{C}P^m(4)$
$(Sp(m+1), U(m+1))$	$\mathbb{C}P^m(2)$
$(SU(m+2), S(U(s+1) \times U(m-s+1)))$	$\mathbb{C}P^s(1) \times \mathbb{C}P^{m-s}(1)$
$(SO(m+4), SO(m+2) \times SO(2))$	$Q_m(\mathbb{C})$
$(SO(2(s+2)), U(s+2))$	$\dfrac{SU(s+2)}{S(U(2) \times U(s))}$
$(E_6, (Spin(10) \times U(1))/\mathbb{Z}_4)$	$\dfrac{SO(10)}{U(5)}$
$(E_7, (E_6 \times U(1))/\mathbb{Z}_3)$	$\dfrac{E_6}{(Spin(10) \times U(1))/\mathbb{Z}_4}$

Therefore we obtain Nakagawa-Takagi's theorem.

References

[Nakagawa-Takagi1976] H. Nakagawa and R. Takagi, *On locally symmetric Kaehler submanifolds in a complex projective space*. J. Math. Soc. Japan **28** (1976), no. 4, 638–667.

[Takeuchi1978] M. Takeuchi, *Homogeneous Kähler submanifolds in complex projective spaces*. Japan. J. Math. (N.S.) **4** (1978), no. 1, 171–219.

[OhnitaCONM2022] Y. Ohnita, *Parallel Kähler submanifolds and* $\boldsymbol{R}$*-spaces*, Differential Geometry and Global Analysis: In Honor of Tadashi Nagano, Contemporary Mathematics **777** (2022), 163–184.

[Ohnita23rdProc2021] Y. Ohnita, *Canonical connections of a Sasakian manifold and invariant submanifolds with parallel second fundamental form*, Proceedings of The 23rd International Differential Geometry Workshop on Submanifolds in Homogeneous Spaces and Related Topics **23** (2021), 31–40, KNU, RIRCM, OCAMI, NRF, JSPS, Pukyong Univ.

3. Totally complex submanifolds and R-spaces

Let (M, g, Q) be a quaternionic Kähler manifold, that is, a Riemannian manifold with a Riemannian metric g and a quaternionic Kähler structure Q (see [ShigeruIshihara1974]). Then $\dim M = 4n$. Denote by $\nabla = \nabla^M$ Levi-Civita conn. of g.

The quaternionic Kähler structure Q is a rank 3 vector subbundle of the skew-symmetric endomorphism bundle $\mathrm{End}(TM)$ over (M, g) invariant under the parallel displacements with respect to the Levi-Civita connection ∇, which has a local field of bases $\{I, J, K\}$ in a nbd. U of each point $p \in M$ satisfying the quaternionic ralations:

$$I \circ I = -\mathrm{Id}, \; J \circ J = -\mathrm{Id}, \; K \circ K = -\mathrm{Id},$$
$$I \circ J = -J \circ I = K, \; J \circ K = -K \circ J = I, \; K \circ I = -I \circ K = J.$$

We call such $\{I, J, K\}$ a local canonical basis of the quaternionic Kähler structure Q.

Totally complex submanifolds of q. K. mfds.

A smooth immersion $\varphi : N \to M$ of a smooth manifold N into a quaternionic Kähler manifold (M, g, Q) with $\tau \neq 0$ is called a totally complex immersion if N has an open covering

$$N = \bigcup_{\lambda \in \Lambda} U_\lambda$$

such that there exists a smooth section J^λ defined on each U_λ for the pull-backed quaternionic Kähler structure $\varphi^{-1}Q$ over N satisfying

$$J^\lambda \circ J^\lambda = -\mathrm{Id}_{T_pM}, \tag{5}$$

$$J^\lambda((d\varphi)_p T_p N) = (d\varphi)_p T_p N \tag{6}$$

and

$$\nabla^\varphi_X J^\lambda = 0 \tag{7}$$

for each $X \in T_pN$ and each $p \in U_\lambda$. Then N is called a totally complex submanifold immesed in M.

Then note that there is a local canonical basis $\{I^\lambda, J^\lambda, K^\lambda\}$ of $\varphi^{-1}Q$ such that

$$\begin{aligned} I^\lambda((d\varphi)_p(T_pN)) \perp (d\varphi)_p(T_pN), \\ K^\lambda((d\varphi)_p(T_pN)) \perp (d\varphi)_p(T_pN) \\ (\forall p \in U_\lambda \subset N) \end{aligned} \tag{8}$$

(cf. [KTsukada1985]).

Thus it must be $\dim N = 2\ell$ for some $1 \leq \ell \leq n$. In the case when $\ell = n$, N is called a maximal dimensional totally complex submanifold.

Remark

A totally complex submanifold N is not necessary a complex manifold, but if necessary N can be doubly covered by a complex manifold $\widetilde{N}$. A totally complex submanifold N is a minimal submanifold of M and a totally complex immersion $\varphi : \widetilde{N} \to M$ is a pluriharmonic map.

Theorem (Tsukada, 1985)

A maximal dimensional totally complex submanifold $\widetilde{N}^{2n}$ of $\mathbb{H}P^n$ with parallel second fundamental form is locally congruent to one of the following immersions (Tsukada immersions):

(0) $\mathbb{C}P^1 \longrightarrow \mathbb{R}P^2 \subset S^4 = \mathbb{H}P^1$ (*Veronese minimal surface*)

(1) $\mathbb{C}P^n \subset \mathbb{H}P^n$ (*totally geodesic*)

(2) $Sp(3)/U(3) \longrightarrow \mathbb{H}P^6$

(3) $SU(6)/S(U(3) \times U(3)) \longrightarrow \mathbb{H}P^9$

(4) $SO(12)/U(6) \longrightarrow \mathbb{H}P^{15}$

(5) $E_7/((U(1) \times E_6)/\mathbb{Z}_3) \longrightarrow \mathbb{H}P^{27}$

(6) $\mathbb{C}P^1(\tilde{c}) \times \mathbb{C}P^1(\tilde{c}/2) \longrightarrow \mathbb{H}P^2$

(7) $\mathbb{C}P^1(\tilde{c}) \times \mathbb{C}P^1(\tilde{c}) \times \mathbb{C}P^1(\tilde{c}) \longrightarrow \mathbb{H}P^3$

(8) $\mathbb{C}P^1(\tilde{c}) \times \dfrac{SO(n+1)}{SO(2) \times SO(n-1)} \longrightarrow \mathbb{H}P^n \quad (n \geq 4)$

In order to get this theorem, Professor Kazumi Tsukada used the classification theory of homogeneous Kähler submanifolds in complex projective spaces based on the representation theory (Hisao Nakagawa and Ryoichi Takagi 1976, Masaru Takeuchi 1978).

So our question is as follows:

Question

Why do those submanifolds appear there?

The canonical connection on the inverse images of totally complex submanifolds in $\mathbb{H}P^n$

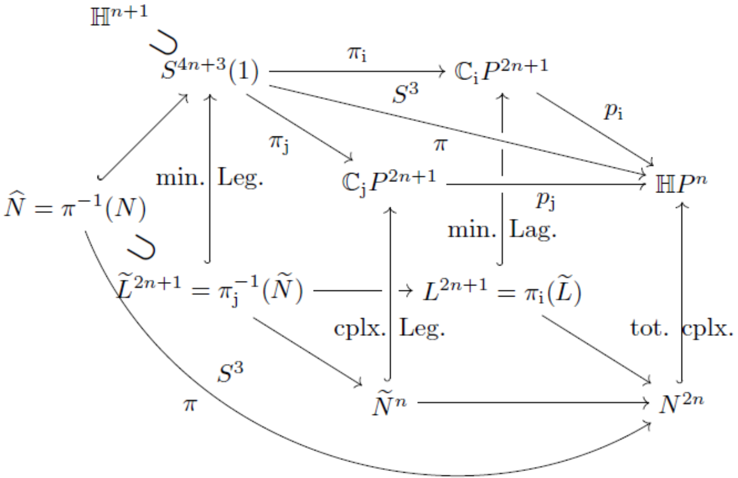
Suppose that $N^{2\ell}$ is a 2ℓ-dimensional totally complex submanifold immersed in $\mathbb{H}P^n$. The inverse image of N under the Hopf fibration $\pi : S^{4n+3}(1) \longrightarrow \mathbb{H}P^n$ is defined as

$$\begin{aligned} \widehat{N} &= \pi^{-1}(N) \\ &= \{(p, x) \in N \times S^{4n+3}(1) \mid p \in N, \pi(x) = \varphi(p)\}. \end{aligned}$$

$$\begin{array}{ccc} \widehat{N} = \pi^{-1}(N) & \xrightarrow{\ \hat{\varphi}\ } & S^{4n+3}(1) \subset \mathbb{H}^{n+1} \\ \pi \downarrow S^3 & & \pi \downarrow Sp(1) = S^3 \\ N & \xrightarrow{\ \varphi\ } & \mathbb{H}P^n \end{array}$$

$\widehat{N}$ is a $(2\ell + 3)$-dimensional minimal submanifold immersed in $S^{4n+3}(1) \subset \mathbb{H}^{n+1} \cong \mathbb{R}^{4n+4}$.

Such maximal dim. totally complex submfds. produce minimal submfds. in other spaces such as complex Legendre submfds. in $\mathbb{C}_j P^{2n+1}$, minimal Legendrian submfds. in $S^{4n+3}(1)$ and minimal Lagrangian submfds. in $\mathbb{C}_i P^{2n+1}$ as follows.



Then we explicitly constructed a new canonical connection on the inverse image $\widehat{N} = \pi^{-1}(N)$ of any maximal dimensional totally complex submanifold N of $\mathbb{H}P^n$.

Theorem (Cho-Hasimoto-O. [CHO2023pp])

Assume that $\ell = n$. Then there exists a (non Levi-Civita) canonical connection $\nabla^c = \nabla^{\widehat{N}} - D$ on the tangent vector bundle $T\widehat{N}$ such that

$$\begin{aligned}(\nabla^c_X \alpha^{\widehat{N}})(Y, Z) &= (\nabla^c_{\mathcal{H}X} \alpha^{\widehat{N}})(\mathcal{H}Y, \mathcal{H}Z) \\ &= ((\nabla^*_{\pi_* X} \alpha^N)(\pi_* Y, \pi_* Z))^{\sim}\end{aligned} \tag{9}$$

for each $X, Y, Z \in T\widehat{N}$. In particular,

$$\nabla^c \alpha^{\widehat{N}} = 0 \quad \text{if and only if} \quad \nabla^* \alpha^N = 0.$$

Let $\varphi : N^{2\ell} \to \mathbb{H}P^n$ be a totally complex immersion.
Let $\{I^N, J^N, K^N\}$ be a local canonical basis of $\varphi^{-1}Q$ on each $U = U_\lambda \subset N$.
For each point $x \in \pi^{-1}(U) \subset \widehat{N}$, there are

$$\lambda_1(x), \lambda_2(x), \lambda_3(x) \in Sp(1)$$

such that

$$I^N_{\pi(x)}(x) = x\lambda_1(x),\ J^N_{\pi(x)}(x) = x\lambda_2(x),\ K^N_{\pi(x)}(x) = x\lambda_3(x).$$

We define the tensor field $\boldsymbol{D}$ of type $(1,2)$ on $\boldsymbol{U} \subset \widehat{\boldsymbol{N}}$ as follows:
At each point $\mathbf{x} \in \pi^{-1}(\boldsymbol{U}) \subset \widehat{\boldsymbol{N}}$, for each $\boldsymbol{X}$, $\boldsymbol{Y} \in T_{\pi(\mathbf{x})}\boldsymbol{N}$,
$\boldsymbol{V} \in \mathcal{V}_{\boldsymbol{x}}\widehat{\boldsymbol{N}}$, $v_1, v_2, v_3 \in \mathbb{R}$,

$$\bullet\, D_{\widetilde{X}}(\widetilde{Y}) = \langle(\widetilde{J^N_{\pi(\mathbf{x})}X}), \widetilde{Y}\rangle J^N_{\pi(\mathbf{x})}(\mathbf{x}) = g_N(J^N_{\pi(\mathbf{x})}X, Y)J^N_{\pi(\mathbf{x})}(\mathbf{x}) \in \mathcal{V}_{\mathbf{x}}\widehat{N},$$

$$\bullet\, D_{\widetilde{X}}(V) = v_2\widetilde{X}\lambda_2(\mathbf{x}) = -v_2(\widetilde{J^N_{\pi(\mathbf{x})}X}) \in \mathcal{H}_{\mathbf{x}}\widehat{N} \quad \left(\forall V = \mathbf{x}\left(\sum_{a=1}^{3} v_a\lambda_a(\mathbf{x})\right)\right),$$

$$\bullet\, D_V(\widetilde{X}) = \frac{v_2}{2}\widetilde{X}\lambda_2(\mathbf{x}) = -\frac{v_2}{2}(\widetilde{J^N_{\pi(\mathbf{x})}X}) \in \mathcal{H}_{\mathbf{x}}\widehat{N} \quad \left(\forall V = \mathbf{x}\left(\sum_{a=1}^{3} v_a\lambda_a(\mathbf{x})\right)\right),$$

$$\begin{aligned}\bullet\, D_U(V) &= \mathbf{x}\,\{(v_2u_3 + \frac{1}{2}v_3u_2)\lambda_1(\mathbf{x}) + (v_3u_1 - v_1u_3)\lambda_2(\mathbf{x}) \\ &\qquad + (-\frac{1}{2}v_1u_2 - v_2u_1)\lambda_3(\mathbf{x})\} \\ &= \mathbf{x}\begin{vmatrix} v_1 & v_2 & v_3 \\ u_1 & -\frac{1}{2}u_2 & u_3 \\ \lambda_1(\mathbf{x}) & \lambda_2(\mathbf{x}) & \lambda_3(\mathbf{x}) \end{vmatrix} \in \mathcal{V}_{\mathbf{x}}\widehat{N}.\end{aligned}$$

Then D is well-defined entirely on $\widehat{N}$ and it satisfies

Lemma

$\nabla^c = \nabla^{\widehat{N}} - D$ satisfies

$$\nabla^c g_{\widehat{N}} = 0$$

and

$$\nabla^c D = 0$$

and hence ∇^c is a non Levi-Civita canonical connection on $T\widehat{N}$.

Lemma

If $\ell = n$, then

$$\begin{aligned}(\nabla^c_X \alpha^{\widehat{N}})(Y, Z) &= (\nabla^c_{\mathcal{H}X} \alpha^{\widehat{N}})(\mathcal{H}Y, \mathcal{H}Z) \\ &= ((\nabla^*_{\pi_* X} \alpha^N)(\pi_* Y, \pi_* Z))^{\sim}\end{aligned} \tag{10}$$

for each $X, Y, Z \in T\widehat{N}$. In particular,

$$\nabla^c \alpha^{\widehat{N}} = 0 \quad \text{if and only if} \quad \nabla^* \alpha^N = 0.$$

So by applying Theorem of [Olmos-Sanchez1991] to this canonical connection ∇^c and our observation of the $Sp(n+1) \times Sp(1)$-symmetry, we obtain

Theorem [CHO2023pp]

If N is a maximal dimensional totally complex submanifold of $\mathbb{H}P^n$ with $\nabla^* \alpha^N = 0$, then $\widehat{N}$ is obtained as a certain singular orbit of the isotropy representation (i.e. *standardly imbedded R-space*) of a quaternionic Kähler symmetric pair (G, K).

Moreover, we can concretely determine such R-spaces by root system computation involved to each quaternionic Kähler symmetric pair ([CHO2023pp]). Thus we obtain a classification of totally complex submanifolds of $\mathbb{H}P^n$ with $\nabla^* \alpha^N = 0$. This gives a geometric proof by a different method for the classification theorem of totally complex submanifolds with parallel second fundamental form due to Tsukada.

Q.K. Symm.Sp. G/K	$\tilde{N}$	$\Pi(G,K)$	n
$\frac{G_2}{(Sp(1)\times Sp(1))/\mathbb{Z}_2}$	$\frac{SU(2)}{S(U(1)\times U(1))}$	G_2	1
$\frac{SU(n+3)}{S(U(2)\times U(n+1))}$	$\frac{SU(n+1)}{S(U(1)\times U(n))}$	B_2	n
$\frac{F_4}{(Sp(1)\times Sp(3))/\mathbb{Z}_2}$	$\frac{Sp(3)}{U(3)}$	F_4	6
$\frac{E_6}{(Sp(1)\times SU(6))/\mathbb{Z}_2}$	$\frac{SU(6)}{S(U(3)\times U(3))}$	F_4	9
$\frac{E_7}{(SU(2)\times Spin(12))/\mathbb{Z}_2}$	$\frac{SO(12)}{U(6)}$	F_4	15
$\frac{E_8}{(SU(2)\times E_7)/\mathbb{Z}_2}$	$\frac{E_7}{(U(1)\times E_6)/\mathbb{Z}_3}$	F_4	27
$\frac{SO(7)}{SO(4)\times SO(3)}$	$\frac{SO(4)}{U(2)}\times\frac{SO(3)}{SO(2)}$	B_3	2
$\frac{SO(8)}{SO(4)\times SO(4)}$	$\frac{SO(4)}{U(2)}\times\frac{SO(4)}{SO(2)\times SO(2)}$	D_4	3
$\frac{SO(n+5)}{SO(4)\times SO(n+1)}$ $(n\geq 4)$	$\frac{SO(4)}{U(2)}\times\frac{SO(n+1)}{SO(2)\times SO(n-1)}$	B_4	n

Remark

In the case when $(G, K) = (G_2, SO(4))$,
a totally complex submanifold with $\nabla^* \alpha^N = 0$

$$N^2 \subset \mathbb{H}P^1 = S^4$$

is a Veronse minimal surface $\mathbb{R}P^2 \subset S^4$, and
a corresponding minimal Lagrangian submanifold

$$L^3 = \pi_{\mathbf{i}}(\widetilde{L}^3) \subset \mathbb{C}_{\mathbf{i}}P^3$$

is a minimal Lagrangian $SU(2)$-orbit which is well-known as the River Chiang Lagrangian ([Chiang2004]).

The relevant structures in this case were discussed in [Ohnita2007], [Ohnita2009], which motivates our present work.

Definition (Chiang2004)

The Chiang Lagrangian submanifold $L^3 \subset \mathbb{C}P^3$ *is defined by*

$$L^3 := \left\{ [z_0 : z_1 : z_2 : z_3] \in \mathbb{C}P^3 \;\middle|\; \begin{array}{c} 3|z_0|^2 + |z_1|^2 - |z_2|^2 - 3|z_3|^2 = 0 \\ z_0\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_3 = 0 \end{array} \right\}$$

(1) minimal Lagrangian $SU(2)$ orbit
$L^3 = \rho(SU(2))[1 : 0 : 0 : 1]$.

(2) conn. cpt. embedded minimal Lagranagian submanifold in $\mathbb{C}P^3$.

(3) L^3 does not possess parallel second fundamental forms.
$\nabla^* \alpha^N \neq 0$

(4) Homogeneous space but not symmetric space.

(5) Curvature characterization
(B. Y. Chen, Dillen, Verstraelen, Vrancken, Bolton, 1996)
Strictly Hamiltonian stable (Ohnita, Bedulli-Gori, 2007)

(6) Floer homology (Evans-Lekili, 2015), min.Maslov number of $L^3 = 2$

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Thank you very much for your kind attention!

Many Congratulations for the 60th birthday of Professor Naoyuki Koike-san!!

Classification of parallel Kähler submanifolds

Any irreducible Hermitian symmetric pair of compact type is given as follows (J. A. Wolf,1964):
G: cpt. conn. simple Lie gp. with Lie alg. $\mathfrak{g}$
$\mathfrak{h}$: maximal abelian subalgebra of $\mathfrak{g}$
$\Pi(G) = \{\alpha_1, \cdots, \alpha_l\}$: the fund. root syst. (Dynkin diagram) of $\mathfrak{g}$
$\{\Lambda_1, \cdots, \Lambda_l\}$: its corresponding fund. weight system
$\tilde{\alpha} = m_1\alpha_1 + \cdots + m_l\alpha_l$: the highest root of root syst. $\Delta^+(G)$.

For each vertex α_{i_0} with $m_{i_0} = 1$, set $K = C_G(H_{\Lambda_{i_0}})$.
Then the Dynkin diagram of $\mathfrak{k}$ is

$$\Pi(K) := \{\alpha_j \in \Pi(G) \mid j \neq i_0\}$$

and (G, K) is an irreducible Hermitian symmetric pair of compact type:

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}, \quad \mathfrak{h} \subset \mathfrak{k}$$

$$\mathfrak{k} = \mathfrak{h} + \sum_{\alpha \in \Delta_{\mathfrak{k}}^{+}} (\mathbb{R}(X_\alpha - X_{-\alpha}) + \mathbb{R}\sqrt{-1}(X_\alpha + X_{-\alpha})),$$

$$\mathfrak{p} = \sum_{\alpha \in \Delta_{\mathfrak{p}}^{+}} (\mathbb{R}(X_\alpha - X_{-\alpha}) + \mathbb{R}\sqrt{-1}(X_\alpha + X_{-\alpha})).$$

Note that $\tilde{\alpha} \in \Delta_{\mathfrak{p}}^{+}$. We can choose a maximal system

$$\{\gamma_1 = \tilde{\alpha}, \gamma_2, \cdots, \gamma_r\}$$

of strongly orthogonal roots in $\Delta_{\mathfrak{p}}^{+}$. Here $r = \mathbf{rank}(G, K)$. Then

$$\mathfrak{a} := \sum_{i=1}^{r} \mathbb{R}\sqrt{-1}(X_{\gamma_i} + X_{-\gamma_i}).$$

is a maximal abelian subspace of $\mathfrak{p}$.

For any $E \in \mathfrak{a} \cap S^{2n+1}(1) \subset \mathfrak{p} \cong \mathbb{C}^{n+1}$, set an R-space

$$\hat{M}_E := (\mathrm{Ad}_{\mathfrak{p}} K)E \subset S^{2n+1}(1) \subset \mathfrak{p}$$

and its projection

$$M_E := \pi((\mathrm{Ad}_{\mathfrak{p}} K)E) = K/K_{\pi(E)} \subset \mathbb{C}P^n.$$

Here

$$K_{\pi(E)} := \{a \in K \mid \mathrm{Ad}(a)\pi(E) = \pi(E)\}.$$

Theorem (Hyunjung Song, 2001)

M_E is a complex submanifold of $\mathbb{C}P^n$ if and only if $E \in \mathbb{R}\sqrt{-1}(X_{\gamma_1} + X_{-\gamma_1})$ after a suitable change of E on $\hat{M}_E$ under the action of K.

In this case M_E always has parallel second fundamental form.

Since $\gamma_1 = \tilde{\alpha}$, we have

$$E \in \mathbb{R}\sqrt{-1}(X_{\tilde{\alpha}} + X_{-\tilde{\alpha}}).$$

Then $(K, K_{\pi(E)})$ is a compact Hermitian symmetric pair,

$$K_{\pi(E)} = C_K(H_{\tilde{\alpha}})$$

and its Dynkin diagram is

$$\Pi(K_{\pi(E)}) = \{\alpha_j \in \Pi(K) \mid (\alpha_j, \tilde{\alpha}) = 0\}.$$

Hence by root system computations of each type we can explicitly determine parallel Kähler submanifolds

$$M_E = K/K_{\pi(E)} = K/C_K(\tilde{\alpha})$$

as follows: