Quandles from the viewpoint of symmetric spaces

田丸 博士
(Hiroshi TAMARU)

Osaka Metropolitan University / OCAMI

部分多様体幾何とリー群作用2023
2023/11/21

Abstract

Framework

- "quandle" is an algebraic system motivated by knot theory;
- A symmetric space is a quandle;
- We study quandles from the viewpoint of "discrete symmetric spaces" (symmetric spaces without using manifold/topology).

Def. (Joyce (1982), Matveev (1982))

Let Q be a set, and $s : Q \to \mathrm{Map}(Q, Q)$. Then (Q, s) is a **quandle** if

$:\Leftrightarrow$ (S1) $\forall x \in Q$, $s_x(x) = x$.
(S2) $\forall x \in Q$, s_x is bijective.
(S3) $\forall x, y \in Q$, $s_x \circ s_y = s_{s_x(y)} \circ s_x$.

Introduction - (1/3)

Example

The following are quandles:

- Symmetric spaces, k-symmetric spaces;
- The set of n-equal dividing points on S^1 with the usual symmetry (**dihedral quandle**);
- regular tetrahedron with "left $2\pi/3$-rotation".

Note

Originated in knot theory:

- quandle coloring is a map $\{\text{arcs}\} \to Q$ compatible with crossing.
- $\#\{\text{quandle coloring}\}$ is a knot-invariant.

Introduction - (2/3)

Note

It is interesting to have nice finite quandles (or nice finite symmetric spaces).

Thm. (analogy of symmetric pairs)

(G, K, σ) : **quandle triplet**
(G group, K subgroup, $\sigma \in \mathrm{Aut}(G)$, $K \subset \mathrm{Fix}(\sigma, G)$)
$\Rightarrow$ G/K is a quandle by $s_{[g]}([h]) := [g\sigma(g^{-1}h)]$.

Note

- $\forall$ homogeneous quandle is obtained in this way;
- **homogeneous** $:\Leftrightarrow \mathrm{Aut}(Q, s) \curvearrowright Q$ transitive;
- $f : (X, s^X) \to (Y, s^Y)$: **homomorphism** $:\Leftrightarrow f \circ s_x^X = s_{f(x)}^Y \circ f \ (\forall x \in X)$.

Introduction - (3/3)

Example

G group, $\sigma \in \mathrm{Aut}(G)$

$\Rightarrow$ $(G, \{e\}, \sigma)$ is always a quandle triplet
(it gives a **generalized Alexander quandle**)

Note

The classification of finite quandles is hopeless
(even if we pose "simple" in some sense...)

Our Fundamental Question

- What are the most basic classes of (finite) quandles?

Two-point homogeneity - (1/3)

Note

In Riemannian symmetric spaces, **two-point homogeneous spaces** would be most fundamental.

Recall

- A Riemannian manifold is **two-point homogeneous** if any pairs of equidistant points can be mapped by an isometry;
- iff rank-one symmetric spaces or $\mathbb{R}^n$;
- iff the isotropy representation acts transitively on the unit sphere.

Def.

$\mathrm{Inn}(Q, s) := \langle \{ s_x \mid x \in Q \} \rangle$ is called the **inner automorphism group**.

Def.

(Q, s) is **two-point homogeneous** if $\mathrm{Inn}(Q, s)$ acts doubly transitively on Q
(i.e., $\mathrm{Inn}(Q, s) \curvearrowright ((Q \times Q) - \mathrm{diag}\, Q)$ is transitive)

Two-point homogeneity - (2/3)

Thm. (T., Wada, Vendramin ('13–'17))

For finite quandle (Q, s) with $\#Q > 2$, TFAE:

- (Q, s) is two-point homogeneous;
- $(Q, s) \cong Q(\mathbb{F}_q, \{0\}, L_a)$, where $\mathbb{F}_q$ is a finite field, and a is a primitive element.

Cor.

- $\forall q$ (> 2, prime power), $\exists(Q, s)$: two-point homogeneous, $\#Q = q$.

Idea of Proof

- a ($\in \mathbb{F}_q$) is called a primitive element if $\{a, a^2, \ldots, a^{q-1}\} = \mathbb{F}_q - \{0\}$;
- Thus the "isotropy rep" $\mathrm{Inn}(Q, s)_0 \curvearrowright \mathbb{F}_q - \{0\}$ is very big;
- This derives the two-point homogeneity.

Two-point homogeneity - (3/3)

Example

- $(\mathbb{F}_3 = \mathbb{Z}_3, L_2)$ gives dihedral;
- $(\mathbb{F}_4 = \mathbb{Z}_2[t]/(t^2 + t + 1), L_t)$ gives tetrahedron.

Open Problems

- All Two-point homogeneous quandles are obtained from finite fields, by classification. Any theoretical reasons?
- Classify two-point homogeneous quandles with respect to the automorphism group $\mathrm{Aut}(Q, s)$.

Flatness - (1/3)

Note

In Riemannian symmetric spaces, flat ones would be more fundamental ($\mathbb{R}^n$ or torus).

Recall

- A Riemannian symmetric space (Q, s) satisfies $R \equiv 0$ iff $G^0(Q, s)$ is abelian.
- $G^0(Q, s) := \langle \{ s_x \circ s_y \mid x, y \in Q \} \rangle$.

Def. (Ishihara-T. 2016)

A quandle (Q, s) is **flat**
$:\Leftrightarrow \quad G^0(Q, s) := \langle \{ s_x \circ s_y \mid x, y \in Q \} \rangle$ is abelian.

Flatness - (2/3)

Def.

(Q, s) is **connected**

$:\Leftrightarrow \quad \mathrm{Inn}(Q, s) \curvearrowright Q$ is transitive.

Thm. (Ishihara-T. 2016)

For a finite connected quandle, TFAE:

- (Q, s) is flat;
- $(Q, s) \cong R_{n_1} \times \cdots \times R_{n_k}$ with n_i all odd (R_n : the dihedral quandle of cardinality n).

Note

- $R_{n_1} \times \cdots \times R_{n_k}$ is a "discrete torus";
- A discrete torus is connected iff n_i all odd;
- This is a discrete version of "a cpt connected flat Riemannian symmetric space is a torus".

Flatness - (3/3)

Note

- "flat + connected" is very restrictive.
- connected $\Rightarrow$ homogeneous.

Open Problem (ongoing)

- Classify homogeneous flat finite quandles. (There must be many.)

Thm. (Furuki-T.)

- For every vertex-transitive graph $G = (V, E)$, one can construct a homogeneous disconnected flat quandle $(Q = V \times \mathbb{Z}_2, s)$.

Particular subsets - (1/3)

Note

In Riemannian symmetric spaces, the **rank** (dimension of a maximal flat) is important.

Def.

A subset A in a quandle (Q, s) is

- **antipodal** if $s_x(y) = y$ $(\forall x, y \in A)$;
- ***s*-commutative** if $s_x \circ s_y = s_y \circ s_x$ $(\forall x, y \in A)$

Note

- antipodal $\Rightarrow$ s-commutative;
- The maximal possible cardinalities of these subsets are invariants of (Q, s);
- The maximal possible cardinality of antipodal subsets in a Riemannian symmetric space is called the **2-number** (Chen-Nagano 1988).

Particular subsets - (2/3)

Def.

The real oriented Grassmannian $G_k(\mathbb{R}^n)^\sim$:

- consisting of (V, σ) with $V \in G_k(\mathbb{R}^n)$, and σ orientation of V;
- for $(i, j) := \mathrm{span}\{e_i, e_j\}$ with orientation $\pm$,
  $s_{(1,2)}(1, 3) = (1, -3) = -(1, 3)$;
  $s_{(1,2)}(3, 4) = (-3, -4) = (3, 4)$;
- $G_1(\mathbb{R}^n)^\sim = S^{n-1}$ (sphere).

Thm. (Kubo-Nagashiki-Okuda-T.)

Except for the case "$n = 2k$ and k even", TFAE:

- A is maximal s-commutative in $G_k(\mathbb{R}^n)^\sim$;
- A is congruent to $\{\pm(i_1, \ldots, i_k)\}$.

Note

- In this case, a maximal s-commutative subset is unique up to congruence.
- But this is not true in general...

Particular subsets - (3/3)

Note

- $\{\pm(i_1, \ldots, i_k)\}$ is a subquandle in $G_k(\mathbb{R}^n)^\sim$;
- it is homogeneous, disconnected, and flat.

Open Problems

- Classify maximal antipodal (or s-commutative) subsets in Riemannian symmetric spaces.
- One can show that maximal s-commutative subsets are subquandles. In a Riemannian symmetric space, is it always homogeneous?
- For two-point homogeneous quandles (Q, s), we know $\#(\text{max. } s\text{-comm. subset}) = 1$. This number measures some complexity?

Ongoing Studies - (1/2)

Note

In compact homogeneous spaces $M = G/K$, the Euler number $\chi(M)$ can be calculated as

- $\chi(M) = \#\mathrm{Fix}(T; M)$, with T max. torus in G;
- In other word,
  $\chi(M) = \#\mathrm{Fix}(g; M)$, with $g \in G$ generic.

Def. (Kai-T.)

The **Euler number** $\chi(Q, s)$ of a quandle (Q, s) is

- $\chi(Q, s) := \min \#\{\mathrm{Fix}(g, Q) \mid g \in \mathrm{Dis}(Q, s)\}$,
- where $\mathrm{Dis}(Q, s) := \langle \{s_x \circ s_y^{-1} \mid x, y \in Q\} \rangle$.

Ongoing Studies - (2/2)

Note

Recall that

- $D(S^n) := \{\pm e_1, \ldots, \pm e_{n+1}\}$

is a max. s-commutative subset in $S^n = G_1(\mathbb{R}^{n+1})^\sim$.

Thm. (Kai-T.)

$$\chi(D(S^n)) = \chi(S^n) = \begin{cases} 0 \ (n \text{ odd}), \\ 2 \ (n \text{ even}). \end{cases}$$

Ongoing Studies

- General properties? Other examples?
- By calculations, most of connected quandles satisfy $\chi = 0$...

References

Survey:

- H. Tamaru; *Discrete symmetric spaces and quandles* (in Japanese), Sugaku (数学), to appear

Two-point homogeneity:

- H. Tamaru, *Two-point homogeneous quandles with prime cardinality*, J. Math. Soc. Japan (2013)
- S. Kamada, H. Tamaru, K. Wada, *On classification of quandles of cyclic type*, Tokyo J. Math. (2016)

Flatness:

- Y. Ishihara, H. Tamaru, *Flat connected finite quandles*, Proc. Amer. Math. Soc. (2016)
- K. Furuki, H. Tamaru, *Flat homogeneous quandles and vertex-transitive graphs*, preprint

Subsets:

- A. Kubo, M. Nagashiki, T. Okuda, H. Tamaru, *A commutativity condition for subsets in quandles — a generalization of antipodal subsets*, In: Differential Geometry and Global Analysis: In Honor of Tadashi Nagano. Contemp. Math. (2022)

Thank you very much!