Maximal antipodal sets of classical compact symmetric spaces and their classification

（古典型コンパクト対称空間の極大対蹠集合とその分類）

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Introduction and preliminaries

M: a Riemannian symmetric space

$A \subset M$: a subset

A: an **antipodal** set $:\Leftrightarrow \forall x, y \in A,\ s_x(y) = y$,

where s_x: the symmetry at x.

An antipodal set is discrete since x is an isolated fixed point of s_x.

E.g. $M = S^n\,(\subset \mathbb{R}^{n+1})$, $\forall x \in S^n, \{x, -x\}$: an antipodal set.

$M = \mathbb{R}P^n$, $\forall x \in \mathbb{R}P^n$, $\forall y \in \mathbb{R}P^n, y \subset x^{\perp}, \{x, y\}$: an antipodal set.

A: a **maximal** antipodal set $:\Leftrightarrow$

$A' \subset M$: an antipodal set, $A \subset A' \Rightarrow\ A = A'$.

If M: connected, $\ |A| < \infty$.

$\exists \max\{|A| : A \subset M : \mathrm{antpodal}\} =: \#_2 M$: the **2-number** of M

A: a **great** antipodal set $:\Leftrightarrow |A| = \#_2 M$.

Remark. A great antipodal set $\Rightarrow$ a maximal antipodal set.

A maximal antipodal set $\not\Rightarrow$ a great antipodal set.

E.g. $\#_2 S^n = 2,\ \#_2 \mathbb{R}P^n = n+1,\ \#_2 \mathbb{R}^n = 1,\ \#_2 \mathbb{R}H^n = 1$.

$u_1, \ldots, u_{n+1}$: an o.n.b. of $\mathbb{R}^{n+1}$, $\{\mathbb{R}u_1, \ldots, \mathbb{R}u_{n+1}\}$: a great antipodal set of $\mathbb{R}P^n$.

G: a compact Lie group

$r_2(G)$: the **2-rank** of G, i.e., the maximal integer t satisfying $\exists G'$: a subgroup of G, $G' \cong (\mathbb{Z}_2)^t$.

G is a Riemannian symmetric space w.r.t. a bi-invariant metric.

$\#_2 G = 2^{r_2(G)}$.

Chen and Nagano (Trans. Amer. Math. Soc. 1988 [1]) studied $\#_2 M$ of a compact Riemannian symmetric space M.

Antipodal sets are "good" finite subsets of compact Riemannian symmetric spaces. To classify maximal antipodal sets is a fundamental problem for studies of antipodal sets.

Aims: (1) To classify maximal antipodal sets of a compact Riemannian symmetric space up to congruence and give an explicit description of a representative of each congruent class.

(2) To determine the maximum of the cardinalities of maximal antipodal sets as well as to determine antipodal sets whose cardinalities attain the maximum. (Explicit descriptions of maximal antipodal sets make us possible to calculate their cardinalities. This gives an alternative proof of Chen-Nagano's result of the determination of $\#_2 M$.)

Tasaki and T. showed that if M is a symmetric R-space, any antipodal sets of M is included in a great antipodal sets, and furthermore, any two great antipodal sets of M are congruent (Osaka J. Math. 2013).

Under the collaboration with Tasaki we classified maximal antipodal sets of:

・some compact classical Lie groups. $(U(n), SU(n), Sp(n), O(n), SO(n)$, and their quotient groups) (J. Lie Theory 2017 [2]).

・some compact classical symmetric spaces. ($G_k(\mathbb{K}^n)$, $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, $Sp(n)/U(n)$, $SO(2n)/U(n)$ and their quotient spaces) (Differ. Geom. Appl. 2020 [3]).

・some compact classical symmetric spaces. ($U(n)/O(n)$, $U(2n)/Sp(n)$, $SU(n)/SO(n)$, $SU(2n)/Sp(n)$ and their quotient spaces) (in preparation).

Maximal antipodal subgroups of compact Lie groups

G: a compact Lie group with a bi-invariant Riemannian metric

$x \in G$, $s_x(y) = xy^{-1}x$ ($y \in G$).

e: the identity element, $s_e(y) = y^{-1}$ ($y \in G$).

A: an antipodal set, $e \in A$

$x, y \in A \Rightarrow x^2 = y^2 = e, \ xy = yx.$

If A is maximal, A is a subgroup $\cong (\mathbb{Z}_2)^t$, where $t = r_2(G)$.

We call A a **maximal antipodal subgroup** (MAS).

$\underline{G = O(n), U(n), Sp(n)}$

$$\Delta_n := \left\{ \begin{bmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{bmatrix} \right\} \subset O(n)$$

Δ_n is a unique MAS of G up to conjugation (simultaneous diagonalization). $\#_2 O(n) = \#_2 U(n) = \#_2 Sp(n) = 2^n$.

$G = SO(n), SU(n)$

$\Delta_n^+ := \{d \in \Delta_n \mid \det(d) = 1\}$ is a unique MAS of G up to conjugation.
$\#_2 SO(n) = \#_2 SU(n) = 2^{n-1}$.

Quotient groups of $U(n)$

"The center of $U(n)$" $= \{\alpha 1_n \mid \alpha \in \mathbb{C}, |\alpha| = 1\} \supset \mathbb{Z}_\mu = \{\alpha 1_n \mid \alpha^\mu = 1\}$

(μ: a natural number)

$\pi_n : U(n) \to U(n)/\mathbb{Z}_\mu$: the natural projection

$$I_1 := \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad J_1 := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad K_1 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$D[4] := \{\pm 1_2, \pm I_1, \pm J_1, \pm K_1\}$

$n = 2^k \cdot l,\ \ l$: an odd number

$$s \in \{0, \ldots, k\}, \quad D(s, n) := \underbrace{D[4] \otimes \cdots \otimes D[4]}_{s} \otimes \Delta_{n/2^s} \subset O(n)$$

$$D(0, n) = \Delta_n$$

$$D(s, n) = \{d_1 \otimes \cdots \otimes d_s \otimes d_0 \mid d_1, \ldots, d_s \in D[4], d_0 \in \Delta_{n/2^s}\} \ (1 \leq s \leq k)$$

$$A = [a_{ij}], \quad A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{bmatrix}$$

E.g. $I_1 \otimes J_1 = \begin{bmatrix} -J_1 & 0 \\ 0 & J_1 \end{bmatrix}, \ J_1 \otimes I_1 = \begin{bmatrix} 0 & -I_1 \\ I_1 & 0 \end{bmatrix}$

Theorem 1 ([2])

$\pi_n : U(n) \to U(n)/\mathbb{Z}_\mu$: the natural projection

θ: a primitive 2μ-th root of 1

$n = 2^k \cdot l$, l: odd

Any MAS of $U(n)/\mathbb{Z}_\mu$ is conjugate to one of the following:

(1) μ: odd $\Rightarrow$ $\pi_n(\{1, \theta\}\Delta_n)$

(2) μ: even

(2-1) $k = 0$ $\Rightarrow$ $\pi_n(\{1, \theta\}\Delta_n)$

(2-2) $k \geq 1$ $\Rightarrow$ $\pi_n(\{1, \theta\}D(s, n))$ $(0 \leq s \leq k)$, where the case of $s = k - 1, n = 2^k$ is excluded.

Remark. $\Delta_2 \subsetneq D[4]$ implies $D(k-1, 2^k) \subsetneq D(k, 2^k)$.

Remark. In $SU(8)/\mathbb{Z}_\mu$ ($\mu = 2, 4, 8$), there are two great antipodal subgroups which are not conjugate:

$\pi_8(\{1, \theta\}\Delta_8^+)$, $\pi_8(\{1, \theta\}D(3, 8))$, their cardinalities $= 2^7$.

Sketch of the proof of Theorem 1

A : a maximal antipodal subgroup of $U(n)/\mathbb{Z}_\mu$

$B := \pi_n^{-1}(A)$

B : commutative $\rightsquigarrow$ $A \overset{\text{conj}}{\sim} \pi_n(\{1, \theta\}\Delta_n)$

B : not commutative, i.e., $\exists a, b \in B$ s.t. $ab \neq ba$.

$\rightsquigarrow$ $ab = -ba$

$\rightsquigarrow$
- $\mathrm{tr}(a) = \mathrm{tr}(b) = 0$
- n, μ : even
- $\{a, b\} \overset{\text{conj}}{\sim} \{I_1 \otimes 1_{n'},\ K_1 \otimes 1_{n'}\}$ ($n' = n/2$)

$\rightsquigarrow\ \langle a, b\rangle \cong D[4] \otimes 1_{n'}$

$\rightsquigarrow\ B \overset{\text{conj}}{\sim}$ a subgroup of $D[4] \otimes U(n')$

$A = \pi_n(B) \overset{\text{conj}}{\sim}$ a subgroup of $\pi_n(D[4] \otimes U(n'))$

Furthermore, $\exists A'$: a maximal antipodal subgroup of $U(n')/\mathbb{Z}_\mu$ s.t. $A \overset{\text{conj}}{\sim} \pi_n(D[4] \otimes \pi_{n'}^{-1}(A'))$. Conversely, if C is a maximal antipodal subgroup of $U(n')/\mathbb{Z}_\mu$, $\pi_n(D[4] \otimes \pi_{n'}^{-1}(C))$ is a maximal antipodal subgroup of $U(n)/\mathbb{Z}_\mu$.

Induction on k $\rightsquigarrow$ Theorem 1.

In [2] we classified MAS of the quotient groups of $O(n)$, $Sp(n)$, $SO(n)$ in similar ways and determined great antipodal subgroups and their cardinalities. To classify MAS of the quotient groups of $SU(n)$ we used the following: $\forall A$: a MAS of $SU(n)/\mathbb{Z}_\mu$, $\exists \tilde{A}$: a MAS of $U(n)/\mathbb{Z}_\mu$ satisfying $A = \tilde{A} \cap SU(n)/\mathbb{Z}_\mu$, and vise versa.

Maximal antipodal subsets of compact symmetric spaces

M : a compact connected Riemannian symmetric space, not a Lie group

Strategy: To Use the realization of M as a totally geodesic submanifold, called a polar, of a compact Lie group G and to apply classification results of MAS of G.

M : a compact Riemannian symmetric space, $o \in M$

A connected component of $F(s_o, M) := \{x \in M \mid s_o(x) = x\}$ is called a **polar** of M w.r.t. o.

A polar M^+ $(\dim M^+ > 0)$ is a totally geodesic submanifold.

$\rightsquigarrow \forall x \in M^+,\ s_x(M^+) = M^+$, hence M^+ is a Riemannian symmetric space w.r.t. the induced metric.

Polars of a compact Lie group

G: a compact Lie group, G_0: the identity component of G

e: the identity element of G

We simply refer to a polar of G w.r.t. e as a polar of G.

$g \in G$, $\tau_g : G \to G$, $\tau_g(x) := gxg^{-1}$ $(x \in G)$ (an inner automorphism)

Proposition 2 ([3])

Let M be a polar of G. Then $M = \{\tau_g(x_0) \mid g \in G_0\}$ for $x_0 \in M$ and $\mathrm{Iso}(M)_0 = \{\tau_g|_M \mid g \in G_0\}$.

Basic principle:

M: a polar of G

A: a maximal antipodal set of $M \rightsquigarrow A \cup \{e\}$: an antipodal set of G

$(\because\ A \subset M \subset F(s_e, G))$

$\exists \tilde{A}$: a maximal antipodal subgroup of G satisfying $A \cup \{e\} \subset \tilde{A}$

$\rightsquigarrow \;\; A = M \cap \tilde{A} \quad (\because \; A\text{: maximal})$

$[B_0], \ldots, [B_k]$: all G_0-conjugacy classes of maximal antipodal subgroups of G, where $B_0, \ldots, B_k$ denotes representatives.

$\rightsquigarrow \;\; \exists g \in G_0, \; \exists s \in \{0, \ldots, k\} \;$ s.t. $\; \tilde{A} = \tau_g(B_s)$.

$A = M \cap \tilde{A} = M \cap \tau_g(B_s) = \tau_g(M \cap B_s) \quad (\because \; \tau_g(M) = M)$

i.e., A is $\mathrm{Iso}(M)_0$-congruent to $M \cap B_s$.

$\rightsquigarrow$ Any representative of $\mathrm{Iso}(M)_0$-congruent class of maximal antipodal sets of M is one of $M \cap B_0, \ldots, M \cap B_k$.

Maximal antipodal sets of Grassmann manifolds

$\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$

$G_{\mathbb{K}} := O(n) \; (\mathbb{K} = \mathbb{R}), \; U(n) \; (\mathbb{K} = \mathbb{C}), \; Sp(n) \; (\mathbb{K} = \mathbb{H})$

$G_k(\mathbb{K}^n)$ is regarded as a polar of $G_{\mathbb{K}}$ by the correspondence $x \mapsto \pi_{x^\perp} - \pi_x$.

$\pi_{x^\perp}, \pi_x$: the orthogonal projections on $x^\perp, x$

Theorem 3 ([3])

A maximal antipodal set of $G_k(\mathbb{K}^n)$ is $G_{\mathbb{K}}$-congruent to $\{\langle e_{i_1}, \ldots, e_{i_k}\rangle_{\mathbb{K}} \mid 1 \le i_1 < \cdots < i_k \le n\}$, where $e_1, \ldots, e_n$ is the standard orthonormal basis of $\mathbb{K}^n$. This is a great antipodal set and $\#_2 G_k(\mathbb{K}^n) = \binom{n}{k}$.

Quotient space of $G_m(\mathbb{K}^{2m})$

$\gamma : G_m(\mathbb{K}^{2m}) \to G_m(\mathbb{K}^{2m}),\ \gamma(x) := x^{\perp} \quad (x \in G_m(\mathbb{K}^{2m}))$

$\langle\gamma\rangle$: the subgroup of $\mathrm{Iso}(G_m(\mathbb{K}^{2m}))$ generated by γ

$G_m(\mathbb{K}^{2m})^* := G_m(\mathbb{K}^{2m})/\langle\gamma\rangle$ is a compact Riemannian symmetric space, doubly covered by $G_m(\mathbb{K}^{2m})$.

We regard $G_m(\mathbb{K}^{2m})$ as a polar of $G_{\mathbb{K}}$ under the correspondence $x \mapsto \pi_{x^{\perp}} - \pi_x$.

Since $\gamma(x) \mapsto \pi_{\gamma(x)^\perp} - \pi_{\gamma(x)} = -(\pi_{x^\perp} - \pi_x)$, γ is regarded as $-\mathrm{id}$.

$\rightsquigarrow \ G_m(\mathbb{K}^{2m})^* \subset G_{\mathbb{K}}^* := G_{\mathbb{K}}/\{\pm 1_{2m}\}$.

$\pi : G_{\mathbb{K}} \to G_{\mathbb{K}}^*$: the natural projection

$\pi \circ s_x = s_{\pi(x)} \circ \pi \quad (x \in G_{\mathbb{K}})$

$x \in G_m(\mathbb{K}^{2m}),\ \ s_{\pi(e)}(\pi(x)) = \pi(s_e(x)) = \pi(x)$

$\rightsquigarrow \ G_m(\mathbb{K}^{2m})^*$ is a polar of $G_{\mathbb{K}}^*$

By using the classification of maximal antipodal subgroups of $G_{\mathbb{K}}^*$ we obtained the classification of maximal antipodal sets of $G_m(\mathbb{K}^{2m})^*$.

Remark. A compact connected Riemannina symmetric space is not necessarily realized as a polar of a connected compact Lie group. For example, $U(n)/O(n), U(2n)/Sp(n)$, $SU(n)/SO(n), SU(2n)/Sp(n)$, so-called outer symmetric spaces, are realized as polars of disconnected compact Lie groups. They are not realized as polars of connected compact Lie groups, since if so, they should be inner symmetric spaces.

Maximal antipodal sets of $AI(n) = SU(n)/SO(n)$

$\sigma_I : SU(n) \to SU(n),\ \sigma_I(x) := \bar{x}$ (the complex conjugation)

$\sigma_I \in \mathrm{Aut}(SU(n)),\ \sigma_I^2 = 1$

$\langle \sigma_I \rangle = \{1, \sigma_I\}$: the subgroup of $\mathrm{Aut}(SU(n))$ generated by σ_I

$G := SU(n) \rtimes \langle \sigma_I \rangle$: the semidirect product

$G = (SU(n), 1) \cup (SU(n), \sigma_I)$: the direct sum of the connected components

$F(s_e, G)$
$= \{(g, 1) \in (SU(n), 1) \mid s_e(g, 1) = (g, 1)\} \cup \{(g, \sigma_I) \in (SU(n), \sigma_I) \mid s_e(g, \sigma_I) = (g, \sigma_I)\}$
$= \{(g, 1) \in (SU(n), 1) \mid (g^{-1}, 1) = (g, 1)\} \cup \{(g, \sigma_I) \in (SU(n), \sigma_I) \mid (\sigma_I(g)^{-1}, \sigma_I) = (g, \sigma_I)\}$

Hence $F(s_e, G) = (F(s_{1_n}, SU(n)), 1) \cup (\{g \in SU(n) \mid \sigma_I(g) = g^{-1}\}, \sigma_I)$.

$AI(n) := \{g \in SU(n) \mid \sigma_I(g) = g^{-1}\} = \rho_{\sigma_I}(SU(n))(1_n) \cong SU(n)/SO(n)$,

where $\rho_{\sigma_I}(g)(x) := g\, x\, \sigma_I(g)^{-1} \quad (g, x \in SU(n))$.

$AI(n)$ is a compact connected Riemannian symmetric space.

$(AI(n), \sigma_I)$ is a polar of $G = SU(n) \rtimes \langle \sigma_I \rangle$.

Theorem 4

Any maximal antipodal subgroup of $SU(n) \rtimes \langle \sigma_I \rangle$ *is conjugate to* $\Delta_n^+ \rtimes \langle \sigma_I \rangle$ *by an element of* $(SU(n), 1)$.

By this, we obtain the following:

Theorem 5

Any maximal antipodal set of $AI(n)$ *is congruent to* Δ_n^+.

Corollary 6

Δ_n^+ *is a unique great antipodal set of* $AI(n)$ *up to* $SU(n)$*-congruence. We have* $\#_2 AI(n) = 2^{n-1}$.

Quotient spaces of $AI(n)$

$\mathbb{Z}_\mu := \{z1_n \mid z^\mu = 1\}$, μ : a natural number divides n

$\mathbb{Z}_\mu \subset$ the center of $SU(n)$

$\mathbb{Z}_\mu AI(n) \subset AI(n)$, where $AI(n) = \{g \in SU(n) \mid \sigma_I(g) = g^{-1}\}$.

$\rightsquigarrow$ the quotient space $AI(n)/\mathbb{Z}_\mu$ is defined.

$\sigma_I(\mathbb{Z}_\mu) = \mathbb{Z}_\mu$

$\rightsquigarrow$ σ_I induces an involutive autom. of $SU(n)/\mathbb{Z}_\mu$, also denoted by σ_I.

$AI(n)/\mathbb{Z}_\mu \subset M := \{x \in SU(n)/\mathbb{Z}_\mu \mid \sigma_I(x) = x^{-1}\}$

Note. M is not necessarily connected. $AI(n)/\mathbb{Z}_\mu$ is the connected component containing the identity element.

$(AI(n)/\mathbb{Z}_\mu, \sigma_I)$ is a polar of $SU(n)/\mathbb{Z}_\mu \rtimes \langle \sigma_I \rangle$.

$(\mathbb{Z}_\mu, 1)$ is a normal subgroup of $SU(n) \rtimes \langle \sigma_I \rangle$.

We denote the quotient group $(SU(n) \rtimes \langle \sigma_I \rangle)/(\mathbb{Z}_\mu, 1)$ by $(SU(n) \rtimes \langle \sigma_I \rangle)/\mathbb{Z}_\mu$. By the equality

$$(SU(n) \rtimes \langle \sigma_I \rangle)/\mathbb{Z}_\mu \ \ni \ (g, \alpha)(\mathbb{Z}_\mu, 1) = (g\mathbb{Z}_\mu, \alpha) \ \in \ SU(n)/\mathbb{Z}_\mu \rtimes \langle \sigma_I \rangle,$$

we identify $SU(n)/\mathbb{Z}_\mu \rtimes \langle \sigma_I \rangle$ with $(SU(n) \rtimes \langle \sigma_I \rangle)/\mathbb{Z}_\mu$.

$D[4]^\pm := \{d \in D[4] \mid \det(d) = \pm 1\}$

$D[4]^+ = \{\pm 1_2, \pm J_1\},\ D[4]^- = \{\pm I_1, \pm K_1\}$

Theorem 7

$\pi_n : SU(n) \rtimes \langle \sigma_I \rangle \to (SU(n) \rtimes \langle \sigma_I \rangle)/\mathbb{Z}_\mu$: *the natural projection*

$n' := n/\mu$, $n = 2^k \cdot l$, l: *an odd number*, θ: *a primitive* 2μ*-th root of* 1

Any maximal antipodal subgroup of $(SU(n) \rtimes \langle \sigma_I \rangle)/\mathbb{Z}_\mu$ *is* $\pi_n((SU(n), 1))$*-conjugate to one of the following.*

(1) If μ *is odd,* $\pi_n(\Delta_n^+ \rtimes \langle \sigma_I \rangle)$.

(2) If μ *is even, (2-1)* n'*: odd* $\Rightarrow \pi_n((\Delta_n^+ \cup \theta \Delta_n^-) \rtimes \langle \sigma_I \rangle)$, $\pi_n(D(s, n) \rtimes \langle \sigma_I \rangle)$ $(1 \leq s \leq k)$, *where the case of* $(s, n) = (k-1, 2^k)$ *is excluded. Here when* $k = 1$, $\pi_n(D(s, n) \rtimes \langle \sigma_I \rangle)$ *is replaced by* $\pi_n(((D[4]^+ \cup \theta D[4]^-) \otimes \Delta_l) \rtimes \langle \sigma_I \rangle)$.

(2-2) n'*: even* $\Rightarrow \pi_n(\{1, \theta\} \Delta_n^+ \rtimes \langle \sigma_I \rangle)$, $\pi_n(\{1, \theta\} D(s, n) \rtimes \langle \sigma_I \rangle)$ $(1 \leq s \leq k)$, *where the case of* $(s, n) = (k-1, 2^k)$ *and* $\pi_4(\{1, \theta\} \Delta_4^+ \rtimes \langle \sigma_I \rangle)$ *are excluded.*

Remark. $\Delta_4^+ = \Delta_2 \otimes \Delta_2 \subsetneq D[4] \otimes D[4] = D(2,4)$.

In general, the following holds.

Proposition 8

G, G' : compact Lie groups

$\pi : G \to G'$: a covering homomorphism whose covering degree is odd

(1) If A is a maximal antipodal subgroup of G, $\pi(A)$ is a maximal antipodal subgroup of G' and $\pi(A)$ is isomorphic to A by π. (2) If A' is a maximal antipodal subgroup of G', there exists a maximal antipodal subgroup A of G such that A is isomorphic to A' by π.

$d \in D(s,n) \rightsquigarrow d^2 = \pm 1_n$

$PD(s,n) := \{d \in D(s,n) \mid d^2 = 1_n\}$

Theorem 9

$\pi_n : SU(n) \to SU(n)/\mathbb{Z}_\mu$: *the natural projection*

Any maximal antipodal set of $AI(n)/\mathbb{Z}_\mu$ *is* $SU(n)/\mathbb{Z}_\mu$*-congruent to one of the following.*

(1) If μ *is odd,* $\pi_n(\Delta_n^+)$.

(2) If μ *is even, (2-1)* n'*:odd* $\Rightarrow$ $\pi_n(\Delta_n^+ \cup \theta\Delta_n^-)$, $\pi_n(PD(s,n))$ $(1 \le s \le k)$, *where the case of* $(s,n) = (k-1, 2^k)$ *is excluded. Here when* $k = 1$, $\pi_n(PD(s,n))$ *is replaced by* $\pi_n(\{\pm 1_2, \pm\theta I_1, \pm\theta K_1\} \otimes \Delta_l)$.

(2-2) n'*: even* $\Rightarrow$ $\pi_n(\{1,\theta\}\Delta_n^+)$, $\pi_n(\{1,\theta\}PD(s,n))$ $(1 \le s \le k)$, *where the case of* $(s,n) = (k-1, 2^k)$ *and* $\pi_4(\{1,\theta\}\Delta_4^+)$ *are excluded.*

When μ is odd, $|\pi_n(\Delta_n^+)| = 2^{n-1}$

When μ is even, $|\pi_n(\Delta_n^+ \cup \theta\Delta_n^-)| = |\pi_n(\{1, \theta\}\Delta_n^+)| = 2^{n-1}$,
$|\pi_n(\{1, \theta\}PD(s, n))| = (2^s + 1) \cdot 2^{s-1+2^{k-s} \cdot l} = 2|\pi_n(PD(s, n))|$.

Theorem 10

Great antipodal sets of $AI(n)/\mathbb{Z}_\mu$ and their cardinalities are as follows:

(1) μ: odd $\Rightarrow \pi_n(\Delta_n^+)$*: a unique great antipodal set,* $|\pi_n(\Delta_n^+)| = 2^{n-1}$*.*

(2) μ: even

(2-1) n':odd, $n = 2, \mu = 2 \Rightarrow \pi_2(\{\pm 1_2, \pm\theta I_1, \pm\theta K_1\})$*: a unique great antipodal set,* $|\pi_2(\{\pm 1_2, \pm\theta I_1, \pm\theta K_1\})| = 3$*.* $n = 4, \mu = 4 \Rightarrow \pi_4(PD(2,4))$*: a unique great antipodal set,* $|\pi_4(PD(2,4))| = 10$*. Otherwise,* $\pi_n(\Delta_n^+ \cup \theta\Delta_n^-)$*: a great antipodal set,* $|\pi_n(\Delta_n^+ \cup \theta\Delta_n^-)| = 2^{n-1}$*,*

(2-2) n'; even, $n = 4, \mu = 2 \Rightarrow \pi_4(\{1, \theta\}PD(2,4))$*: a unique great antipodal set,* $|\pi_4(\{1, \theta\}PD(2,4))| = 20$*. Otherwise,* $\pi_n(\{1, \theta\}\Delta_n^+)$*: a unique great antipodal set,* $|\pi_n(\{1, \theta\}\Delta_n^+)| = 2^{n-1}$*.*

Corollary 11 (cf. [1])

$\#_2 AI(n)/\mathbb{Z}_\mu$ is as follows.

When $n = 2, \mu = 2$, $\#_2 AI(2)/\mathbb{Z}_2 = 3$.

When $n = 4, \mu = 2$, $\#_2 AI(4)/\mathbb{Z}_2 = 20$.

When $n = 4, \mu = 4$, $\#_2 AI(4)/\mathbb{Z}_4 = 10$.

Otherwise, $\#_2 AI(n)/\mathbb{Z}_\mu = 2^{n-1}$.

Remark. $AI(2) \cong S^2$, $AI(2)/\mathbb{Z}_2 \cong \mathbb{R}P^2$. $AI(4) \cong \tilde{G}_3(\mathbb{R}^6)$, $AI(4)\mathbb{Z}_2 \cong G_3(\mathbb{R}^6)$, $AI(4)/\mathbb{Z}_4 \cong G_3(\mathbb{R}^6)^*$.

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