K3曲面の崩壊に付随する写像のエネルギーについて

服部広大　@慶應大

Def $(X^4, \omega = (\omega_1, \omega_2, \omega_3))$

: (4-dim.) hyper-Kähler manifold

nowhere vanishing

$$:\iff \begin{cases} \bullet\ \omega_i \wedge \omega_j = 2\delta_{ij} \cdot \mathrm{vol} \quad ({}^{\exists}\mathrm{vol} \in \Omega^4(X)) \\ \bullet\ g(u,v) = \dfrac{\iota_u \omega_1 \wedge \iota_v \omega_2 \wedge \omega_3}{\mathrm{vol}} > 0 \\ \bullet\ d\omega_i = 0 \end{cases}$$

Def $(X^4, \omega = (\omega_1, \omega_2, \omega_3))$

: (4-dim.) hyper-Kähler manifold

$$\Rightarrow \begin{cases} I_i : \quad \omega_i = g(I_i \cdot, \cdot) \Rightarrow \begin{cases} I_1 I_2 = I_3 \\ I_2 I_3 = I_1 \\ I_3 I_1 = I_2 \\ I_i^2 = -1 \end{cases} \\ \cdot \ g \ : \text{hyper-Kähler metric} \\ \qquad \Rightarrow (g, I_1, I_2, I_3) \end{cases}$$

Def $(X^4, \underline{\omega = (\omega_1, \omega_2, \omega_3)})$

: (4-dim.) hyper-Kähler manifold

hyper-Kähler structure

on X.

g : hyperkähler metric

$$\Rightarrow \begin{cases} \cdot\ \mathrm{Ric}_g \equiv 0 \\ \cdot\ \mathrm{Hol}(g) \subset SU(2) \quad (\subset SO(4)) \end{cases}$$

X : 4-dim. compact manifold

$\exists\ \omega$: hyper-Kähler structure

$\Longleftrightarrow$ X : cpt Kähler surface w/ $K_X \cong \mathcal{O}_X$

Yau's Thm.

$$\Longrightarrow X \underset{\text{diffeo}}{\cong} \begin{cases} \cdot\ T^4 \\ \cdot\ \text{K3 surface} \end{cases}$$

X : K3 surface

$$\mathcal{M}_{hk} := \frac{\{\omega : \text{hK-str on } X\}}{\text{isomorphism}} \curvearrowleft SO(3)$$

hyper-Kähler rotation

$$(\omega_i)_{i=1,2,3} \mapsto \left(\sum_{j=1}^{3} a_i{}^j \omega_j\right)_{i=1,2,3}$$

$(a_i{}^j) \in SO(3)$

X : K3 surface

$$\left\{ \omega : \text{hK-str on } X \right\} \Big/ \text{isomorphism} \; \Big/ \; SO(3) \times \mathbb{R}_+$$

$$\cong \left\{ g : \text{hK-metrics on } X \text{ w/ diam}=1 \right\} \Big/ \text{isometry}$$

X : K3 surface

※ $\Gamma = \mathrm{Aut}(H^2(X,\mathbb{Z}), \text{intersection})$

$$\Gamma \backslash O(3,19) / O(3) \times O(19)$$

$\cup$ open, dense ... global Torelli Theorem

$$\mathcal{M}_X := \{ g : \text{hK-metrics on } X \text{ w/ } \mathrm{diam} = 1 \} / \text{isometry}$$

$$\mathcal{M}_X := \left\{ g : \text{hK-metrics on } X_{K3} \text{ w/ diam} = 1 \right\} \Big/ \text{isometry}$$

Q. What is $\overline{\mathcal{M}_X}$?

$$\mathcal{M}_X := \{ g : \text{hK-metrics on } X_{K3} \text{ w/ diam}=1 \} \big/ \text{isometry}$$

$$\mathcal{M}_X \subset \mathcal{M}^{GH} := \{ (M, d) : \text{cpt metric sp.} \} \big/ \text{isom.}$$

$$\begin{array}{ccc} \cup\!\!\!\!\shortmid & & \cup\!\!\!\!\shortmid \\ (X, g) & \mapsto & (X, d_g) \end{array}$$

d_g : Riemannian distance of g

$(M_n, d_n), (M_\infty, d_\infty)$: cpt metric spaces

$(M_n, d_n) \xrightarrow{n\to\infty} (M_\infty, d_\infty)$ (Gromov-Hausdorff convergence)

$:\Longleftrightarrow \exists\, \varepsilon_n \downarrow 0, \exists\, \phi_n : M_n \to M_\infty$ (approximation map)

s.t. $\begin{cases} \cdot \left| d_\infty(\phi_n(x), \phi_n(y)) - d_n(x,y) \right| < \varepsilon_n \\ \qquad (\forall x, y \in M_n) \\ \cdot\ M_\infty \subset B(\phi_n(M_n), \varepsilon_n) \end{cases}$

$$\mathcal{M}_X := \{ g : \text{hK-metrics on } X_{K3} \text{ w/ } \mathrm{diam} = 1 \} / \text{isometry}$$

Fact $\mathcal{M}_X$ is precompact in $\mathcal{M}^{GH}$

$(\because)$ • $\forall (X, g) \in \mathcal{M}_X$, $\begin{cases} \mathrm{diam} = 1 \\ Ric_g = 0 \end{cases}$

• Gromov's precompactness theorem //

$$\mathcal{M}_X := \frac{\{ g : \text{hK-metrics on } X_{K3} \text{ w/ diam} = 1 \}}{\text{isometry}}$$

<u>Def</u>. · $\overline{\mathcal{M}_X} :=$ closure of $\mathcal{M}_X$ in $\mathcal{M}^{GH}$

· $\partial \mathcal{M}_X := \overline{\mathcal{M}_X} \setminus \mathcal{M}_X$

$(M, d) \in \partial \mathcal{M}_X$

- volume non-collapsing $(\dim M = 4)$

  $\Rightarrow$ (M, d) : cpt hyper-Kähler orbifold

  w/ ADE-type singularities

- Collapsing $\Rightarrow$ Classified by Sun-Zhang

$(M, d) \in \partial \mathcal{M}_X$

- Collapsing

- $(T^3/\mathbb{Z}_2,$ flat$)$ ⋯ Foscolo
- $(S^2,$ singular metric$)$ ⋯ Gross-Wilson
- $[0,1]$ ⋯ Hein-Sun-Viaclovski-Zhang

$\partial \mathcal{M}_X$ (Further research)

• Odaka-Oshima

Conj $\overline{\mathcal{M}_X}^{Sat} \longrightarrow \overline{\mathcal{M}_X}^{GH}$ is continuous

Gromov-Hausdorff compactification

Satake compactification

• Honda-Sun-Zhang

measured Gromov-Hausdorff compactification

Classify $(M, d, m) \in \partial \overline{\mathcal{M}_X}^{mGH}$ for $(M, d) = [0,1]$

$S^2 \in \partial \mathcal{M}_X$ (Gross-Wilson)

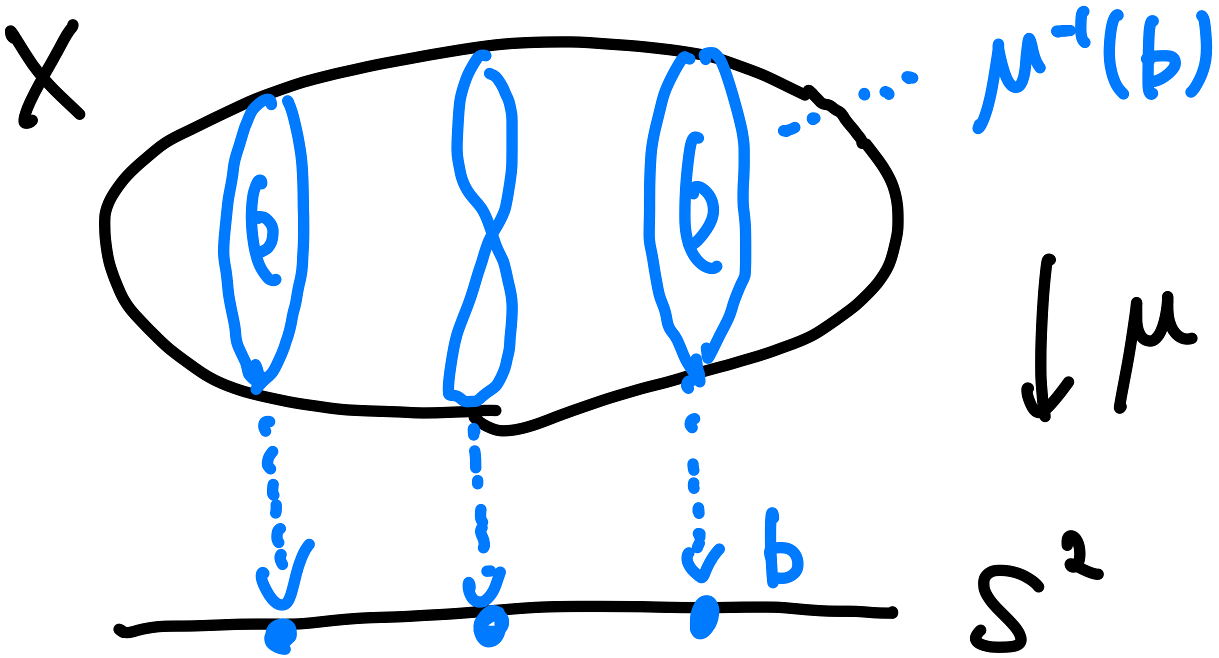
$\mu : X \longrightarrow S^2$ elliptic K3 surface

$$\begin{cases} \bullet\ \mu \text{ is holomorphic \& surjective} \\ \bullet\ \mu^{-1}(b) \cong T^2 \text{ (elliptic curve)} \\ \qquad \text{for generic } b \in S^2 \\ \bullet\ {}^{\exists}24\text{-singlar fibers of Kodaira type } I_1. \end{cases}$$

$S^2 \in \partial \mathcal{M}_X$ (Gross-Wilson)

$\mu : X \longrightarrow S^2$ elliptic K3 surface

- $\exists\, \omega_t$: hyper-Kähler structures $(t>0)$



area of $\mu^{-1}(b)$ w.r.t g_t $= O(t)$

$S^2 \in \partial \mathcal{M}_X$ (Gross-Wilson)

$\mu : X \longrightarrow S^2$ elliptic K3 surface

- μ is an approximation map of

$$(X, g_t) \longrightarrow (S^2, {}^{\exists}\text{sing. met.})$$

- μ is $\begin{cases} \text{holomorphic.} \\ \text{Special Lagrangian fibration} \end{cases}$ hyperKähler rotation

Holomorphic maps are energy minimizing

$(X^m, \omega), (Y^n, \eta)$: cpt Kähler mfds

$f : X \longrightarrow Y$ $\quad C^\infty$-map

$$I_{\omega,\eta}(f) := \frac{2}{(m-1)!} \int_X \omega^{m-1} \wedge f^*\eta$$

$$\mathcal{E}(f) := \int_X |df|^2 \, d\mu_g$$

Holomorphic maps are energy minimizing

Thm (Lichnerowicz)

(1) $I_{\omega,\eta}(f)$ is determined by $[f]$

(2) $$\left.\begin{aligned} I_{\omega,\eta}(f) &= 2\int_X \left(|\partial f|^2 - |\bar\partial f|^2\right) d\mu_g \\ \mathcal{E}(f) &= 2\int_X \left(|\partial f|^2 + |\bar\partial f|^2\right) d\mu_g \end{aligned}\right\} \therefore I_{\omega,\eta}(f) \le \mathcal{E}(f)$$

(3) $I_{\omega,\eta}(f) = \mathcal{E}(f) \Leftrightarrow f$: holomorphic

Gross-Wilson & Lichnerowicz

When $(X_{K3}, g_t) \xrightarrow{GH} S^2$, then

energy minimizing maps appear as

approximation maps.

3-dim'l collapsing?

What to do

- Review Foscolo's constructions of collapsing $(X, g_t) \to (T^3/\mathbb{Z}_2, \text{flat})$
- approximation map $X \xrightarrow{\mu_t} T^3/\mathbb{Z}_2$
- The lower bound of $\mathcal{E}(\mu_t)$

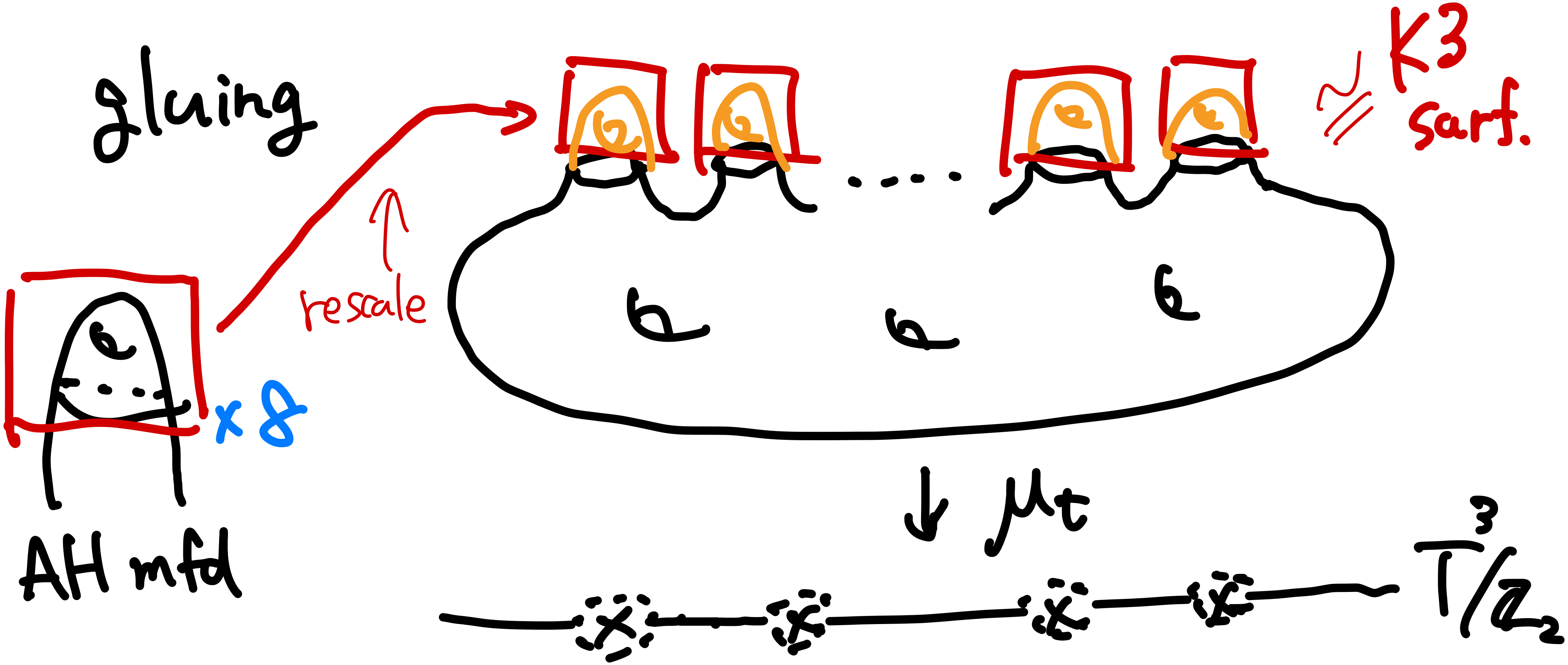
Collapsing to $T^3/\mathbb{Z}_2$ (Foscolo)

- Construct (X, ω_t) $(t>0)$ by

gluing $\begin{cases} \bullet \text{ punctured Gibbons-Hawking metrics} \\ \bullet \text{ Atiyah-Hitchin mfds} \times S \end{cases}$

Collapsing to $T^3/\mathbb{Z}_2$ (Foscolo)

- Construct (X, ω_t) $(t>0)$ by gluing



Gibbons-Hawking ansatz

- (Y, g_θ) : flat 3-mfd

$$\begin{cases} \theta = (\theta_1, \theta_2, \theta_3) \in \Omega^1(Y)^{\oplus 3} \\ \theta_i : \text{closed} \\ g_\theta = \theta_1^2 + \theta_2^2 + \theta_3^2 \end{cases}$$

- $h : Y \longrightarrow \mathbb{R}_+$: positive harmonic fct.

$\boxed{\text{Assume}}$ $[*dh] \in H^2(Y, 2\pi\mathbb{Z})$

$\exists\ \check{X} \xrightarrow{\check{\mu}} Y$: principal $U(1)$-bdle

$\exists\ \alpha \in \Omega^1(\check{X})$ s.t

$d\alpha = *dh$

Gibbons-Hawking ansatz

- $(Y, g_\theta = \theta_1^2 + \theta_2^2 + \theta_3^2)$
- (h, α)
- $\check{\mu} : \check{X} \to Y$

$$\Longrightarrow \begin{cases} \omega_1 := \check{\mu}^* \theta_1 \wedge \alpha + \check{\mu}^* (h \cdot \theta_2 \wedge \theta_3) \\ \omega_2 := \check{\mu}^* \theta_2 \wedge \alpha + \check{\mu}^* (h \cdot \theta_3 \wedge \theta_1) \\ \omega_3 := \check{\mu}^* \theta_3 \wedge \alpha + \check{\mu}^* (h \cdot \theta_1 \wedge \theta_2) \end{cases}$$

Gibbons-Hawking metric

$\vdots$

$\omega = (\omega_1, \omega_2, \omega_3)$

hyperkähler structure

$\Uparrow$ on $\check{X}$.

Foscolo's setting

- $q_1, \cdots, q_8 \in T^3$: fixed pts of $\mathbb{Z}_2$-action
- $p_1, \cdots, p_{16}, -p_1, \cdots, -p_{16} \in T^3 \setminus \{q_j\}$: mutually distinct
- $Y = T^3 \setminus \bigcup_{j=1}^{8} U_t(q_j)$ small ball
- $h : Y \to \mathbb{R}$ harmonic w/

$$h(x) \simeq \begin{cases} \dfrac{1}{2|x-p_i|} & \left(\begin{smallmatrix}\text{around}\\ p_i\end{smallmatrix}\right) \\ \dfrac{1}{2|x+p_i|} & \left(\begin{smallmatrix}\text{around}\\ -p_i\end{smallmatrix}\right) \\ -\dfrac{2}{|x-q_j|} & \left(\begin{smallmatrix}\text{around}\\ q_j\end{smallmatrix}\right) \end{cases}$$

- α, $\check{\mu}_t : \check{X}^t \to Y$

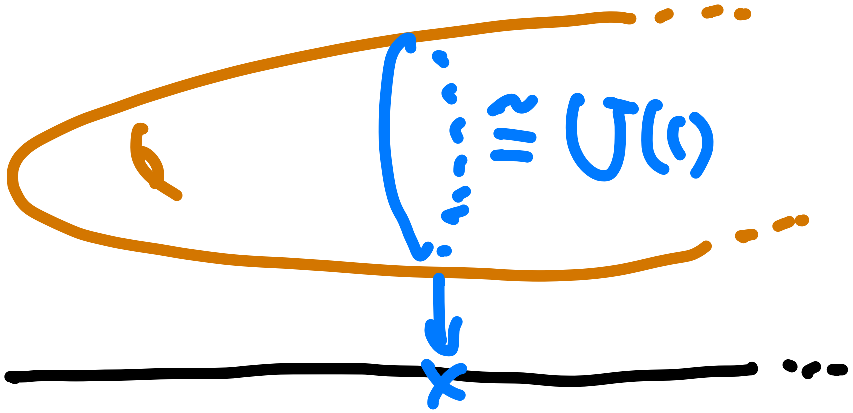
Foscolo's setting

- $q_1, \dots, q_8 \in T^3$
- $p_1, \dots, p_{16}, -p_1, \dots, -p_{16} \in T^3 \setminus \{q_j\}$
- $t^{-1} + h : Y \to \mathbb{R}$
- α, $\check{\mu}_t : \check{X}^t \to Y$

$$\xRightarrow{\text{G-H ansatz}} \quad \begin{array}{c} (\check{X}^t/\mathbb{Z}_2, \check{\omega}^t) \\ \downarrow \check{\mu}_t \\ Y/\mathbb{Z}_2 \end{array}$$

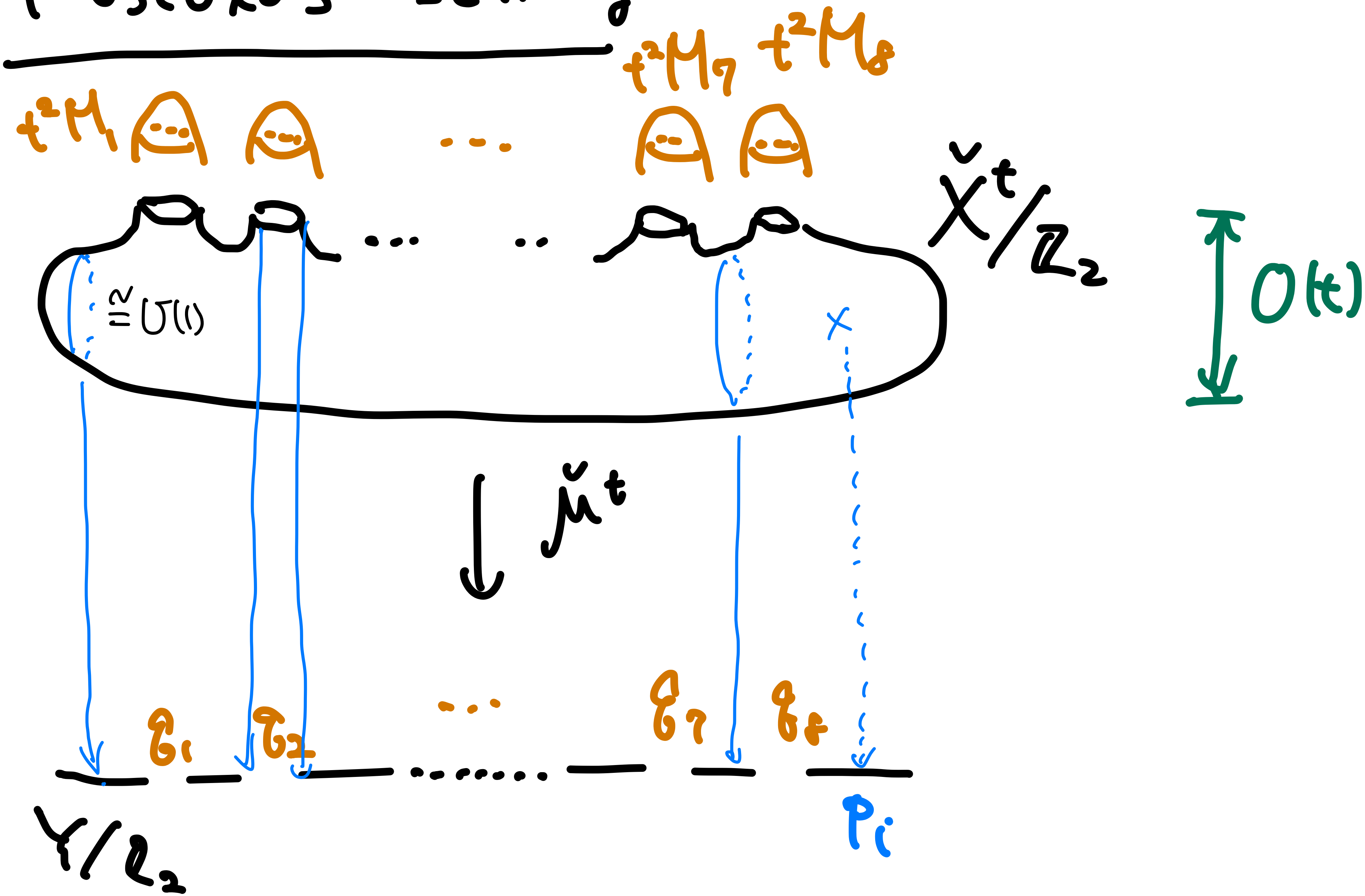
Atiyah-Hitchin manifold

- 4-dim. hyper-Kähler mfd. M.
- Asymptotically locally flat space of type D_0. (cubic volume growth)

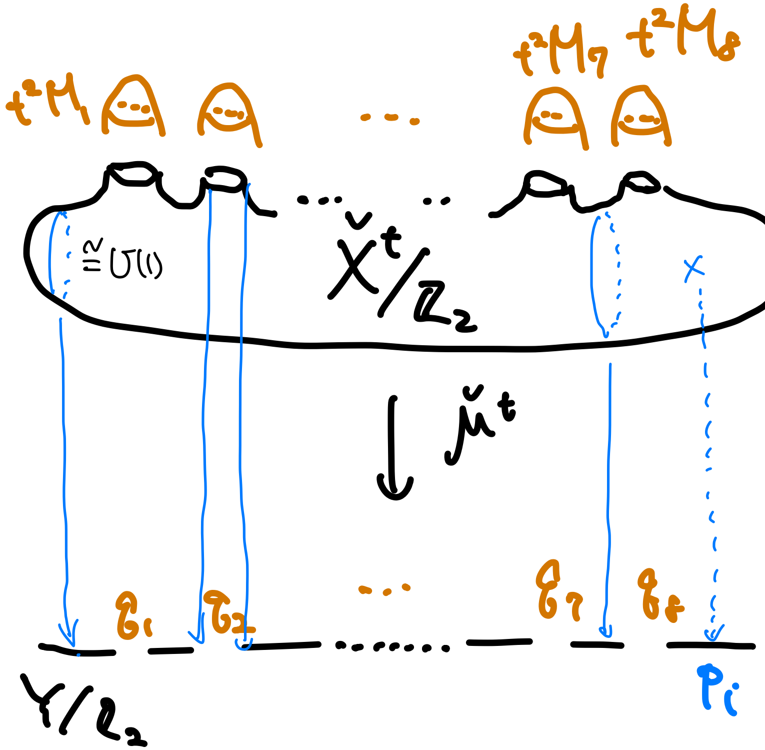


$$M \xrightarrow{\mu} \mathbb{R}^3/\mathbb{Z}_2$$

Foscolo's setting



Foscolo's setting



- $X = \check{X}^t/\mathbb{Z}_2 \cup t^2M_1 \cup \cdots \cup t^2M_8$

  diffeo

  $\cong K3$

- ω^t : hk-str on X

orbifold

Thm (H) $\check{\mu}^t$ extends smoothly to $\mu^t : X \to T^3/\mathbb{Z}^2$

Lower bound of the Energy

- $(X, \omega = (\omega_1, \omega_2, \omega_3))$: K3 surf. w/ hk-str.
- $(T^3/\mathbb{Z}_2, g_\theta = \theta_1^2 + \theta_2^2 + \theta_3^2)$: flat orbifold
- $\mu : X \to T^3/\mathbb{Z}_2$ smooth map

$$\boxed{\begin{aligned} *\theta_1 &= \theta_2 \wedge \theta_3 \\ *\theta_2 &= \theta_3 \wedge \theta_1 \\ *\theta_3 &= \theta_1 \wedge \theta_2 \end{aligned}}$$

- $I_{\omega,\theta}(\mu) := \sum_{i=1}^{3} \int_X \omega_i \wedge \mu^*(*\theta_i)$

Lower bound of the Energy

- $I_{\omega,\theta}(\mu) := \sum_{i=1}^{3} \int_X \omega_i \wedge \mu^*(*\theta_i)$

Thm (H)

(1) $\mu_0 \overset{\text{homotopic}}{\sim} \mu_1 \Rightarrow I_{\omega,\theta}(\mu_0) = I_{\omega,\theta}(\mu_1)$

(2) ${}^\forall \mu$, $I_{\omega,\theta}(\mu) \leq \mathcal{E}_{g,g_\theta}(\mu) = \int_X |d\mu|^2 \mathrm{vol}_g$

Lower bound of the Energy

- $I_{\omega,\theta}(\mu) := \sum_{i=1}^{3} \int_X \omega_i \wedge \mu^*(*\theta_i)$

Thm (H)

homotopic

(1) $\mu_0 \sim \mu_1 \Rightarrow I_{\omega,\theta}(\mu_0) = I_{\omega,\theta}(\mu_1)$

(2) $\mu \Rightarrow I_{\omega,\theta}(\mu) = \mathcal{E}_{g,g_\theta}(\mu)$

G-H ansatz

Thm.(H) X : K3 surface

ω^t : Foscolo's family of hk-strs

g^t : hk-metrics of ω^t

$\Rightarrow$ ${}^\exists \mu^t : X \to T^3/\mathbb{Z}_2$ smooth s.t.

- μ^t : approximation maps of $(X, g^t) \xrightarrow[t\to 0]{} (T^3/\mathbb{Z}_2, g_\theta)$: Gromov-Hausdorff convergence

- $I_{\omega^t,\theta}(\mu^t) > 0$

- $I_{\omega^t,\theta}(\mu^t) \leq \mathcal{E}_{g^t, g_\theta}(\mu^t)$

- $\lim_{t\to 0} \mathcal{E}_{g^t, g_\theta}(\mu^t) / I_{\omega^t,\theta}(\mu^t) = 1$

• Outline of proof

• Recall $X = \check{X}^t/\mathbb{Z}_2 \cup t^2 M_1 \cup \cdots \cup t^2 M_8$

Gibbons-Hawking

• On $\check{X}^t/\mathbb{Z}_2$, $I_{\omega^t,\theta}(\mu^t) \doteqdot \mathcal{E}_{g^t, g_\theta}(\mu^t)$

• On $t^2 M_j$, $I_{\omega^t,\theta}(\mu^t)$ & $\mathcal{E}_{g^t, g_\theta}(\mu^t)$

are small.