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概アーベルリー群上のリッチ平坦左不変ローレンツ計量

部分多様体幾何とリー群作用2024

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Results

- A certain classification of Ricci-flat metrics.
- A certain generalization of Petrov's solution.

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1. Preparation
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This talk is based on a j. w. w/ Takanao Tsuyuki (Hokkaido Information Univ.) who is an expert on particle physics.

Assume that the dimension of a manifold is greater than three.

1. Preparation

(M, g_M) : an n-dim. Lorentzian mfd. ($\leftarrow$ we call it a spacetime)

The *Einstein equations* are

$$\mathrm{Ric} - \frac{\mathrm{Scal}}{2} g_M + \Lambda g_M = T.$$

In this talk, the *vacuum* Einstein equations are

$$\mathrm{Ric} - \frac{\mathrm{Scal}}{2} g_M = 0.$$

Namely, in case $T = 0, \ \Lambda = 0$.

If (M, g_M) satisfies the vacuum Einstein equations,
then we call it a *vacuum solution*.

(M, g_M) : vacuum solution $\Longleftrightarrow$ (M, g_M) : Ricci-flat $(\mathrm{Ric} = 0)$.

Physics	Mathematics
There are no preferred places in space	Spatially homogeneous
There are no preferred directions in space	Spatially isotropic

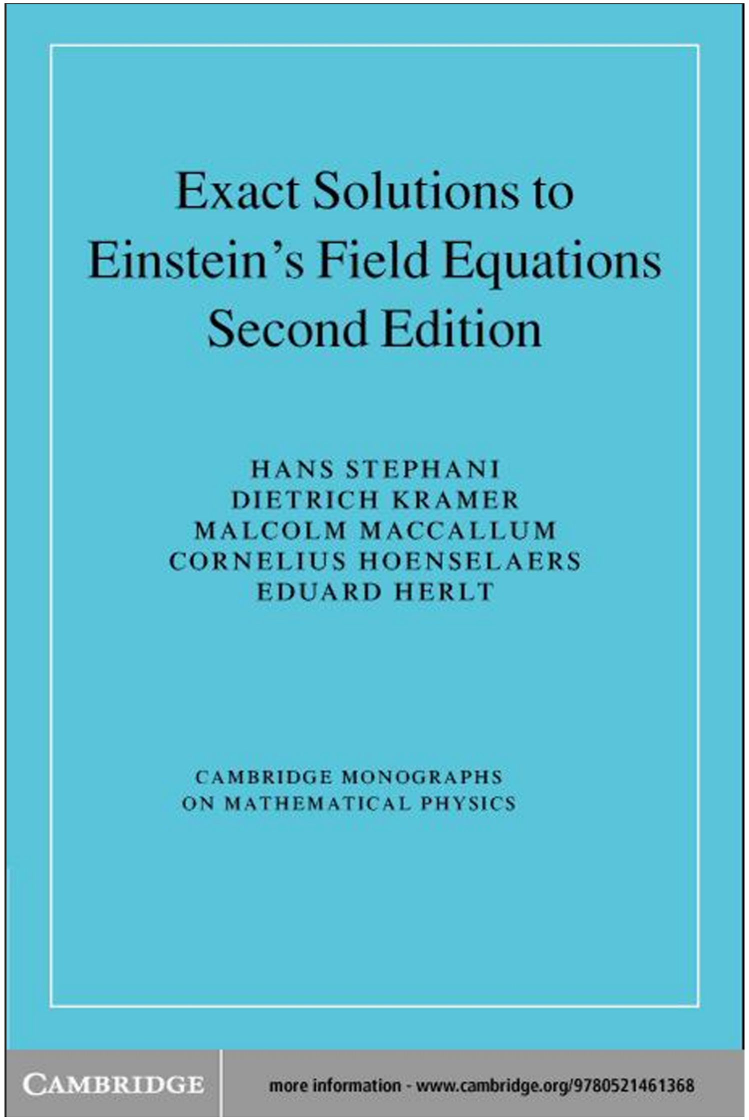
Table1: Cosmological Principle

For example,
The Robertson–Walker spacetime
(including Minkowski, deSitter and anti-deSitter spacetimes)
is spatially homogeneous and spatially isotropic.

The Taub–NUT spacetime
is spatially homogeneous but not spatially isotropic.

Can one classify Ricci-flat Lorentzian homogeneous spaces?

$\rightsquigarrow$ Symmetric case is completed (*Cahen–Wallach spaces*).
$\rightsquigarrow$ Almost abelian case is in today's talk.



H. Stephani et al., *Exact solutions of Einstein's field equations*, second edition, Cambridge Monographs on Mathematical Physics, Cambridge Univ. Press, Cambridge, 2003.

G : a 1-connected n-dim. Lie group.
$\mathfrak{g}$: the Lie algebra of G.
g_G : a left-invariant Lorentzian metric on G.

g_G : left-inv. Lor. met. on $G \overset{1:1}{\longleftrightarrow} \langle\ ,\ \rangle$: Lorentzian bi-lin. form on $\mathfrak{g}$

Then (G, g_G) is a homogeneous Lorentzian manifold.

Remark

(G, g_G) is not necessarily (geodesically-)complete.

G : compact $\Longrightarrow$ (G, g_G) : complete (Marsden, '73).
g_G : bi-invariant $\Longrightarrow$ (G, g_G) : complete.

When (G, g_G) is flat,
G : unimodular $\Longleftrightarrow$ (G, g_G) : complete (Aubert–Medina, '03).

For a homogeneous Riemannian manifold (M, g_M),
it is Ricci-flat $\Longleftrightarrow$ flat (Alekseevskii–Kimel'fel'd '75).

G : *almost abelian* $:\Longleftrightarrow \exists \mathfrak{a} \subset \mathfrak{g}$: codim. 1 abelian ideal.

By definition, we have
G : almost abelian Lie group/iso. $\stackrel{1:1}{\longleftrightarrow}$ A : real square matrix/equiv.
Note that A is an $(n-1) \times (n-1)$ real square matrix.

Here A is equivalent to B if A is similar to B up to scaling.
We call the equivalence class of A the *associated matrix* of G.

Remark

- G is isomorphic to $\mathbb{R}^n$ w/ a suitable Lie group structure.
- almost abelian $\Longrightarrow$ two-step solvable.
- $n = 2 \Longrightarrow$ almost abelian.
- $n = 3 \Longrightarrow$ almost abelian, or semisimple.

When G is almost abelian,
G : nilpotent $\Longleftrightarrow$ A : nilpotent $\qquad$ G : unimodular $\Longleftrightarrow \operatorname{tr} A = 0$

Theorem 1 [Otero-Casal '23]

G : a 4-dim. Lie group.
$\mathfrak{g}$: the Lie algebra of G.
g_G : a left-invariant Lorentzian metric on G.

If (G, g_G) is Ricci-flat but neither locally symmetric nor a plane wave, then it is locally homothetic to the Lie group determined by

$$[X_1, X_4] = -2X_1,\ [X_2, X_4] = X_2 + \sqrt{3}X_3,\ [X_3, X_4] = -\sqrt{3}X_2 + X_3,$$

where $\{X_1, X_2, X_3, X_4\}$: a generator of $\mathfrak{g}$.

This solution is called the *Petrov solution* (Petrov, '62).

Moreover, the Lie group is almost abelian, and is the only vacuum solution of Einstein equations admitting a (almost) simply-transitive four-dimensional maximal group of isometry.

Lemma 1

Let G be an $(n+1)$-dimensional almost abelian Lie group, g_G a left-invariant Lorentzian metric on G.

Then there exists a generator $\{X_1, \dots, X_n, X_{n+1}\}$ of $\mathfrak{g}$ such that

$$\mathrm{span}\{X_1, \dots, X_n\} = \mathfrak{a}, \quad \mathfrak{g} = \mathfrak{a} \rtimes \mathbb{R}X_{n+1}$$

and the bilinear form $\langle\, , \,\rangle$ on $\mathfrak{g}$ can be expressed by one of the following

$$\text{(a)} \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & -1 \end{bmatrix}, \quad \text{(b)} \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}, \quad \text{(c)} \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & 0 & 1 \\ & & & 1 & 0 \end{bmatrix},$$

where $\mathfrak{a}$ is an abelian ideal of $\mathfrak{g}$.

$M_n\mathbb{R}$ denotes the set of $n \times n$ real square matrices.

2. Classification

For the three metrics (a), (b) and (c) in Lemma,
we obtain the following by directly solving Ricci-flat conditions.

Theorem 2 [S.–Tsuyuki] (1/3)

In case (a), denoting the associated matrix by $[A]$,
an almost abelian Lie group (G, g_G) is Ricci-flat iff,
setting

$$A = S + T,$$

where S, T denote the symmetric, anti-symmetric matrices,

it holds that

$$S = O.$$

Moreover, (G, g_G) is actually flat.

Theorem 2 [S.–Tsuyuki] (2/3)

In case (b), denoting the associated matrix by $[A]$, an almost abelian Lie group (G, g_G) is Ricci-flat iff, setting

$$J = \operatorname{diag}(-1, 1, \ldots, 1) \in M_n\mathbb{R}$$

and

$$A = S_L + T_L,$$

where S_L, T_L satisfy

$${}^tS_L = JS_LJ, \quad {}^tT_L = -JT_LJ,$$

it holds that

$$\operatorname{tr} S_L = \operatorname{tr} S_L^2 = 0, \quad [S_L, T_L] = O.$$

Theorem 2 [S.–Tsuyuki] (3/3)

In case (c), denoting the associated matrix by $[A]$, an almost abelian Lie group (G, g_G) is Ricci-flat iff, setting

$$A = \begin{bmatrix} A' & \boldsymbol{b} \\ {}^t\boldsymbol{c} & d \end{bmatrix} \quad (A' \in M_{n-1}\mathbb{R},\ \boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}^{n-1},\ d = 0, 1),$$

and

$$A' = S' + T',$$

where S', T' denote the symmetric, anti-symmetric matrices,

it holds that

$$\boldsymbol{b} = \boldsymbol{0}, \quad S' = O \text{ (when } d = 0), \quad \operatorname{tr} S' = \operatorname{tr} S'^2 \text{ (when } d = 1).$$

Remark

The result is exactly the same as Riemannian situation in this case (a).

The Petrov solution is included in this case (b).

Calvaruso and Zaeim showed in 4-dim. case.
However, a slight deficiency exists in this case (c).

G. Calvaruso and A. Zaeim, *Four-dimensional Lorentzian Lie groups*, Differential Geom. Appl. **31** (2013), no. 4, 496–509.

Recall

If (G^4, g_G) is Ricci-flat but neither locally symmetric nor a plane wave, then it is locally homothetic to the Petrov solution.

$\rightsquigarrow$ Find a higher-dimensional version of the Petrov solution in case (b).

3. Generalization

Lemma 2 [S.–Tsuyuki]

For an almost abelian Lie group (G, g_G) w/ left-inv. Lor. met.,
it is Ricci-flat and locally symmetric iff it is flat.

Lemma 3 [S.–Tsuyuki]

$(G = \mathbb{R}^{n+1}, g_G)$: Ricci-flat almost abelian Lie group satisfying case (b).
$A = S_L + T_L$: associated matrix.
Then the metric g_G satisfies

$$g_G = {}^t d\boldsymbol{x} \begin{bmatrix} e^{-{}^t A x_{n+1}} & \mathbf{0} \\ {}^t\mathbf{0} & 1 \end{bmatrix} J \begin{bmatrix} e^{-A x_{n+1}} & \mathbf{0} \\ {}^t\mathbf{0} & 1 \end{bmatrix} d\boldsymbol{x}$$

$$= {}^t d\boldsymbol{x} J \begin{bmatrix} e^{-2S_L x_{n+1}} & \mathbf{0} \\ {}^t\mathbf{0} & 1 \end{bmatrix} d\boldsymbol{x},$$

where $\boldsymbol{x} = {}^t(x_1, \ldots, x_n, x_{n+1}) \in \mathbb{R}^{n+1}$.

(M, g_M) : n-dim. spacetime.

(M, g_M) : a *pp-wave* $:\Longleftrightarrow \exists V \in \Gamma(TM)$: parallel null vector field s.t. it is transversally flat, i.e.

$$R(X, Y) = 0 \quad (\forall X, Y \in V^{\perp}).$$

(M, g_M) : a *plane wave* $:\Longleftrightarrow (M, g_M)$: a pp-wave and it satisfies

$$\nabla_X R = 0 \quad (\forall X \in V^{\perp}).$$

Here ∇, R denote the Levi–Civita connection, curvature tensor of (M, g_M), resp.

Remark

Pp-waves have special holonomy, i.e.
the (restricted) Lorentzian holonomy group is non-irreducible, indecomposable and abelian.

Prop [Singer–Steinberg '94]

Let $A \in M_n\mathbb{R}$ satisfy

$$ {}^tA = JAJ. $$

Then A is conjugate by $P \in \mathrm{O}(1, n-1)$ to a diagonal matrix, or a direct sum of a diagonal matrix and one of the following:

$$ \text{(i)} \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}, \quad \text{(ii)} \begin{bmatrix} -1+r & -1 \\ 1 & 1+r \end{bmatrix}, \quad \text{(iii)} \begin{bmatrix} -\gamma & -\gamma & -1 \\ \gamma & \gamma & 1 \\ 1 & 1 & 0 \end{bmatrix}. $$

Here $\alpha \neq 0$, $\beta, r, \gamma \in \mathbb{R}$ and

$$ \mathrm{O}(1, n-1) := \{P \in \mathrm{GL}_n\mathbb{R} \mid {}^tPJP = J\}. $$

Remark

Lorentzian ver. of the normal forms for real normal matrices.

Recall

In case (b), since we may set $A = S_L$ from Lemma 3, the Ricci-flat condition is

$$\operatorname{tr} S_L = \operatorname{tr} S_L^2 = 0,$$

where S_L satisfy ${}^t S_L = J S_L J$.

When S_L is a diagonal matrix, we have $S_L = O \rightsquigarrow$ flat.

When S_L is in case (ii), that is,

$$S_L = \begin{bmatrix} -1+r & -1 \\ 1 & 1+r \end{bmatrix} \oplus \operatorname{diag}(\lambda_3, \ldots, \lambda_n),$$

we have $r = \lambda_3 = \cdots = \lambda_n = 0 \rightsquigarrow$ flat.

When S_L is in case (iii), that is,

$$S_L = \begin{bmatrix} -\gamma & -\gamma & -1 \\ \gamma & \gamma & 1 \\ 1 & 1 & 0 \end{bmatrix} \oplus \operatorname{diag}(\lambda_4, \ldots, \lambda_n),$$

we have $\lambda_4 = \cdots = \lambda_n = 0 \rightsquigarrow$ a non-flat plane wave.

Lemma 4 [S.–Tsuyuki]

(G, g_G) : Ricci-flat almost abelian Lie group satisfying case (c).
Then it is a plane wave (possibly flat).

Remark

If (M^n, g_M) be a non-flat homogeneous plane wave,
then $\dim \mathrm{Isom}(M, g_M) \geq n + 1$.

M Hanounah, L Mehidi and A Zeghib, *On homogeneous plane waves*, arXiv:2311.07459.

From Lemma 2 and Lemma 4, by considering the case (i) we obtain:

Theorem 3 [S.–Tsuyuki]

(G^{n+1}, g_G) : indecomposable, Ricci-flat almost abelian.

(G^{n+1}, g_G) is neither locally symmetric nor a plane wave iff it is homothetic to the almost abelian Lie group whose associated matrix $[A]$ given by

$$A = \begin{bmatrix} \alpha & -1 \\ 1 & \alpha \end{bmatrix} \oplus \operatorname{diag}(\lambda_3, \dots, \lambda_n) \in M_n\mathbb{R},$$

where

$$2\alpha + \sum_{i=3}^{n} \lambda_i = 0, \quad 2\alpha^2 + \sum_{i=3}^{n} \lambda_i^2 = 2, \quad \prod_{i=3}^{n} \lambda_i \neq 0.$$

Moreover, the matrix representation of the metric is the form of case (b) in Lemma 1.

Corollary [S.–Tsuyuki]

(G^{n+1}, g_G) : almost abelian defined by

$$A = \begin{bmatrix} \alpha & -1 \\ 1 & \alpha \end{bmatrix} \oplus \mathrm{diag}(\lambda_3, \dots, \lambda_n)$$

with

$$2\alpha + \sum_{i=3}^{n} \lambda_i = 0, \quad 2\alpha^2 + \sum_{i=3}^{n} \lambda_i^2 = 2, \quad \prod_{i=3}^{n} \lambda_i \neq 0.$$

and the metric is of the form of case (b) in Lemma 1.

Then it admits an almost simply-transitive isometry group
iff it satisfies $\lambda_3 > \ldots > \lambda_n$ up to conjugacy by $\mathrm{O}(1, n-1)$.

Key of proof

$$\mathrm{Isom}(G, g_G)_e \subset \mathrm{Sym}(G, g_G) \subset \mathrm{O}(1, n-1) \cap \mathrm{GL}(\mathfrak{g}).$$

4. Dissatisfaction

In case (b), the Ricci-flat condition is equivalent to
A is trace-free and satisfies that

$$ {}^t(AJ)(JA) - (AJ)^t(JA) = O, \quad \mathrm{tr}\, A^2 + \mathrm{tr}\, \left[(AJ)^t(JA)\right] = 0. $$

In addition, when we define the set as

$$ N := \{A \in M_n\mathbb{R} \mid {}^t(AJ)(JA) - (AJ)^t(JA) = O\}, $$

we have the Lie group action

$$ \mathrm{O}(1, n-1) \curvearrowright N; \quad A \mapsto P^{-1}AP. $$

Thus, what is the canonical forms of $\mathrm{O}(1, n-1) \curvearrowright N$?

Thank you for your attention!!