# Subriemannian geodesic flows of the seven-dimensional sphere

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Abstract

This paper is a summary of the paper [BaLTa]. The main theorems state the complete integrability of geodesic flows of the seven-dimensional sphere associated to the trivializable subriemannian structures. The proof is carried out on the basis of Thimm's method proposed in early 1980's, which amounts the search for the constants of motion to the construction of a chain of non-degenerate Lie subalgebras in $\mathfrak{so}(8)$.
**Key words** subriemannian manifold, Lie groups, Thimm's method
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## 1 Introduction

This paper is a summary of the paper [BaLTa] by the author together with Wolfram Bauer and Abdellah Laaroussi.

In Riemannian geometry, the geodesics play very important roles in investigating the geometry of the Riemannian manifolds. From the view point of Hamiltonian systems, the geodesics are formulated as a Hamiltonian system, called the geodesic flow, on the (co)tangent bundle of the Riemannian manifold. In this context, it is natural to study the completely integrability of the flows and various methods have been proposed in the previous research. See e.g. [F, P] for the details on this topic.

The subriemannian geometry is an extension of the Riemannian geometry, consisting of a smooth manifold equipped with a metric defined only along an assigned tangential distribution, which is normally assumed to be bracket-generating. The geodesics are defined also as a solution to a variational problem with a non-holonomic constraint. Although there in general exists abnormal geodesics in the variational problem, the Hamiltonian framework can still be applied to the normal geodesics. Therefore, it is still natural to question the complete integrability of the geodesic flows of subriemannian geodesic flows.

Previously, there have been studies on integrable subriemannian geodesic flows on homogeneous spaces. For the Heisenberg groups, the paper [KOR] shows the integrability of geodesic flow for a left-invariant Riemannian metric and, as a direct consequence, the complete integrability is found also for the subriemannian geodesic flows. In the case of the pseudo $H$-type Lie groups, the complete integrability of the (sub-)Riemannian geodesic flow is showed in [BaTa]. The paper [O] studies several lower-dimensional nilpotent Lie groups of steps two and three, in view of completely integrable geodesic flows. On the other hand, there is also an example of subriemannian geodesic flow which is not completely integrable in [MoShSt].

In the present work, trivializable subriemannian structures on the seven-dimensional sphere are considered and the complete integrability of the geodesic flows is showed. Among arbitrary spheres,

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the seven-dimensional sphere is the highest-dimensional one whose tangent bundle is trivializable. In fact, there is a tangential distribution with the bracket-generating condition of rank $4, 5, 6$ on $S^7$. The trivializability of the tangential distributions permits explicit constructions of the constants of motion for the subriemannian geodesic flows. However, the key idea of the construction lies in the method by A. Thimm [Th] who proved the complete integrability for the Riemannian geodesic flows of several homogeneous spaces. By this method, the search for constants of motion amounts to the construction of so-called non-degenerate subalgebras in the Lie algebra $\mathfrak{so}(8)$ of the rotation group $SO(8)$ which transitively acts on $S^7$.

The present paper has the following structure:
In Section 2, the Hamiltonian framework for the normal geodesics of a subriemannian manifold is briefly reviewed. Section 3 is a review on the method by Thimm. It also addresses the complete integrability of the Riemannian geodesic flows of arbitrary spheres. Section 4 concerns the construction of trivializable subriemannian geodesic flows of rank $4, 5, 6$ and the complete integrability is showed by constructing a chain of non-degenerate subalgebras explicitly for each of the subriemannian structures.

# 2 Hamiltonian framework for the subriemannian geodesic flow

In this section, we briefly review the Hamiltonian framework of the geodesic flow on a subriemannian manifold. For the details, see e.g. [Mo].

Let $Q$ be a smooth manifold. We take a tangential distribution (sub-bundle of the tangent bundle) $\mathcal{D} \subset TQ$ of rank $r$. Given a fibre metric $\langle \cdot, \cdot \rangle$ of $\mathcal{D}$, we call the triple $(Q, \mathcal{D}, \langle \cdot, \cdot \rangle)$ a subriemannian manifold. We assume the bracket-generating (or Hörmander) condition:

$$\exists p \in \mathbb{N}_0 \,\text{s.t.}\, \mathcal{D}^p = TQ,\ \mathcal{D}^{p-1} \neq TQ,$$

setting $\mathcal{D}^0 = \mathcal{D}$, $\mathcal{D}^{i+1} = \mathcal{D}^i + \left[\mathcal{D}^i, \mathcal{D}\right]$. Here, we identify the vector bundles as the sheaves of germs of sections.

A curve $c : [\alpha, \beta] \to Q$ is called a (subriemannian) geodesic if it is a solution to the variational problem

$$\delta \left( \int_\alpha^\beta \sqrt{\langle \dot{c}(t), \dot{c}(t) \rangle_{c(t)}} \mathrm{d}t \right) = 0, \quad \text{subject to}\, \dot{c}(t) \in \mathcal{D}_{c(t)},\ \forall t \in [\alpha, \beta].$$

with fixed end points.

To consider the Hamiltonian formalism for subriemannian geodesics, we introduce the cometric associated to the subriemannian structure. We take the bundle homomorphism $\beta : T^*Q \to \mathcal{D}$ defined through

$$\alpha_y(v) = \langle \beta(\alpha_y), v \rangle, \quad \alpha_y \in T_y^*Q, v \in \mathcal{D}_y, y \in Q.$$

Then, we have the fibre metric $\langle \cdot, \cdot \rangle' : T^*Q \times T^*Q \to \mathbb{R}$ of $T^*Q$ defined through

$$\langle \alpha_y, \alpha'_y \rangle'_y = \langle \beta(\alpha_y), \beta(\alpha'_y) \rangle, \quad \alpha_y, \alpha'_y \in T_yQ.$$

This fibre metric is called the *cometric* associated to the subriemannian structure.

Now, we consider the function $H$ on $T^*Q$ defined through

$$H(\alpha_y) := \frac{1}{2} \langle \alpha_y, \alpha_y \rangle'_y, \qquad \alpha_y \in T_y^*Q \subset T^*Q.$$

On the cotangent bundle $T^*Q$, we consider the canonical symplectic structure $\omega$. It is known that a solution to Hamilton's equation for the Hamiltonian system $(T^*Q, \omega, H)$ induces a subriemannian geodesic of $(Q, \mathcal{D}, \langle \cdot, \cdot \rangle)$, usually called a *normal geodesic*.

**Remark 2.1.** There are subriemannian geodesic which does not solve Hamilton's equation. Such geodesics are called abnormal geodesics. See e.g. [Mo] for the details on abnormal geodesics. Below, we focus on normal geodesics and hence we only think about the Hamiltonian framework for the subriemannian geodesic flows.

# 3 Riemannian geodesic flows on homogeneous spaces

In this section, we address Riemannian geodesic flows of homogeneous spaces. As the most fundamental example, we start with the geodesic flow of a sphere with respect to the standard Riemannian metric. It is completely integrable in the sense of Liouville, as is widely known. We then review the method of Thimm by means of which several Riemannian geodesic flows are proved to be completely integrable in [Th]. This method also plays a central role in our study on subriemannian geodesic flow of $S^7$.

## 3.1 Riemannian geodesic flows on spheres

On the $n$-dimensional sphere $S^n \subset \mathbb{R}^{n+1}$, the geodesics with respect to the induced Riemannian metric are known to be (a part of) a great circle.

In the Hamiltonian formalism, the corresponding geodesic flow on $T^*S^n$ is also known to be completely integrable.

Identifying $T^*S^n$ with $TS^n \subset \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ through the Riemannian metric, the Poisson commuting constants of motion can be given as

$$f_j(x,y) = \sum_{k,\ell \geq n+2-j} (x_k y_\ell - x_\ell y_k)^2, \qquad j = 2, \ldots, n+1,$$

where $(x,y) = (x_1, \ldots, x_{n+1}; y_1, \ldots, y_{n+1}) \in TS^n \subset \mathbb{R}^n \times \mathbb{R}^n$ with $x \in S^n$, $x^{\mathrm{T}} y = 0$. Note that $x_k y_\ell - x_\ell y_k$ is nothing but the angular momentum in physical terminology.

## 3.2 Thimm's method

In the beginning of 1980's, Thimm [Th] introduced a method to obtain constants of motion for the geodesic flows on certain class of homogeneous spaces $Q = G/H$. We assume that $G$ is equipped with a bi-invariant pseudo-Riemannian metric which induces a $G$-invariant Riemannian metric on $Q$. Equivalently, we have an inner product $B$ on the Lie algebra $\mathfrak{g}$ of $G$ which is invariant with respect to the adjoint action. We assume that the restriction of $B$ to the Lie algebra $\mathfrak{h}$ of $H$ is non-degenerate and hence the orthogonal complement $\mathfrak{m} \subset \mathfrak{g}$ to $\mathfrak{h}$ is invariant with respect to the adjoint action by $H$. The restriction $B|_{\mathfrak{m}\times\mathfrak{m}}$ is non-degenerate and induces a (pseudo-)Riemannian metric on $Q = G/H$. We identify the tangent and the cotangent bundles of $Q$ with respect to this metric. Then, the canonical symplectic structure on $T^*Q$ induces a symplectic structure on $TQ$. In this situation, the momentum mapping $P : TQ \to \mathfrak{g}$ is given as

$$P(g\xi) = Ad_g \xi,$$

where $g\xi \in T_{gH}Q$, $g \in G$, $\xi \in \mathfrak{m}$. Note that the mapping $P$ is a Poisson mapping with respect to the Lie-Poisson bracket on $\mathfrak{g}$ and the above Poisson structure on $TQ$.

The key idea of Thimm's method lies in the following observation. We call a Lie subalgebra $\mathfrak{k} \subset \mathfrak{g}$ *non-degenerate* if the restriction $B|_{\mathfrak{k}\times\mathfrak{k}} : \mathfrak{k} \times \mathfrak{k} \to \mathbb{R}$ is non-degenerate. Now, we assume to have two *non-degenerate* subalgebras $\mathfrak{g}_0, \mathfrak{g}_1 \subset \mathfrak{g}$ such that $[\mathfrak{g}_0, \mathfrak{g}_1] \subset \mathfrak{g}_1$. For a pair of Ad-invariant

functions $h_i \in \mathcal{C}^\infty(\mathfrak{g}_i)$ on $\mathfrak{g}_i$, $i = 0, 1$, the Poisson bracket between the pull-backs $\pi_i^* h_i$ of $h_i$ through the orthogonal projection $\pi_i : \mathfrak{g} \to \mathfrak{g}_i$, $i = 0, 1$, vanishes:

$$\{h_0 \circ \pi_0, h_1 \circ \pi_1\} = 0.$$

Here, the Lie-Poisson bracket on $\mathfrak{g}$ is defined through

$$\{f, g\}(\xi) = B(\xi, [\nabla f(\xi), \nabla g(\xi)]), \qquad f, g \in \mathfrak{C}^\infty(\mathfrak{g}),$$

where $\nabla f(\xi)$ denotes the gradient of $f$ at $\xi \in \mathfrak{g}$ with respect to $B$.

Note that the condition $[\mathfrak{g}_0, \mathfrak{g}_1] \subset \mathfrak{g}_1$ holds e.g. if $\mathfrak{g}_0 \subset \mathfrak{g}_1$. If there is a sequence of non-degenerate subalgebras

$$\mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \cdots \subset \mathfrak{g}_k \subset \mathfrak{g},$$

we can obtain many constants of motion, by virtue of the Ad-invariant functions on $\mathfrak{g}_i$, $i = 1, 2, \ldots, k$.

Using the above idea, Thimm [Th] showed the complete integrability of geodesic flows of the several homogeneous spaces, such as the real and the complex Grassmannian manifolds.

In the case of sphere $S^n = SO(n+1)/SO(n)$, we naturally have the sequence of subalgebras:

$$\mathfrak{so}(2) \subset \mathfrak{so}(3) \subset \cdots \subset \mathfrak{so}(n-1) \subset \mathfrak{so}(n) \, (\subset \mathfrak{so}(n+1)),$$

taking the skew-symmetric matrices in different sizes. The functions $f_j$ in the previous subsection are nothing but the pull-backs of the quadratic Ad-invariant functions on the subalgebras $\mathfrak{so}(j)$ through the orthogonal projections.

# 4 Subriemannian geodesic flows on spheres

We here consider the subriemannian geodesic flows of the spheres $S^3$ and $S^7$. The details in the case of $S^7$ can be found in [BaLTa].

## 4.1 Subriemannian geodesic flow of $S^3$

Recall the three-dimensional sphere is diffeomorphic to the special unitary group: $S^3 \cong SU(2)$. By the left-trivialization, the tangent bundle is trivialized as $TS^3 \cong SU(2) \times \mathfrak{su}(2)$.

We take an orthonormal generators $e_1, e_2, e_3 \in \mathfrak{su}(2)$ for which

$$[e_1, e_2] = e_3, \quad [e_2, e_3] = e_1, \quad [e_3, e_1] = e_2.$$

The subspaces $\mathcal{D}_q := \mathtt{d}L_q \mathrm{span}_{\mathbb{R}}\{e_1, e_2\}$ in the tangent space $T_q SU(2)$, at $q \in SU(2) \cong S^3$, give rise to a left-invariant subriemannian structure on $S^3 \cong SU(2)$. The subriemannian metric is the induced metric from the bi-invariant metric on $SU(2)$.

Identifying $T^*S^3 \cong TS^3 \cong SU(2) \times \mathfrak{su}(2)$, the Hamiltonian for the subriemannian geodesic flow is given as

$$H(q, p) = \frac{1}{2}\left(p_1^2 + p_2^2\right), \quad p = p_1 e_1 + p_2 e_2 + p_3 e_3 \in \mathfrak{su}(2).$$

As $p_1$, $H$, and $p_1^2 + p_2^2 + p_3^2$ Poisson commute with respect to the Lie-Poisson bracket on $\mathfrak{su}(2)$, the subriemannian geodesic flow is completely integrable.

## 4.2 Subriemannian geodesic flows on $S^7$

It is known that $S^n$ is equipped with a trivializable bracket-generating subriemannian structure induced by a Clifford module structure only if $n = 1, 3, 7, 15, 23, 31, 63$. See [BaF] and [BaFI] for the details.

We here focus on the case of rank $r$ trivializable subriemannian structure on $S^7$. Recall that $S^7$ is the highest-dimensional sphere whose tangent bundle is trivializable. The subriemannian metric is the induced metric from the one of $S^7$. The rank $r$ can only be either $4, 5, 6, 7$, to obtain a bracket-generating subriemannian structure, as is pointed out in [BaF], [BaFI]. We consider the cases $r = 4, 5, 6$.

**Trivializable subriemannian structures on $S^7$.** For a $(n+1) \times (n+1)$ matrix $A = (a_{ij}) \in \mathbb{R}^{(n+1)\times(n+1)}$, we consider the vector field

$$X(A)_x := \sum_{i,j=1}^{n} a_{ij} x_j \frac{\partial}{\partial x_i}, \quad x \in \mathbb{R}^{n+1},$$

on $\mathbb{R}^{n+1}$, which can be restricted to the sphere $S^n$ if the matrix $A$ is skew-symmetric.

We take a set of $(n+1) \times (n+1)$ skew-symmetric matrices $A_k$, $k = 1, \dots, r$, satisfying the Clifford relations:

$$A_k A_\ell + A_\ell A_k = -2\delta_{k\ell}, \quad k, \ell = 1, \dots, r.$$

The maximal number $r$ of the associated vector fields $X_i := X(A_i)$, $i = 1, \dots, r$, which are orthonormal everywhere on $S^n$, coincides with the maximal dimension of a trivial sub-bundle of $TS^n$ (cf. [A]).

To construct subriemannian structures on $S^7$, we take $8 \times 8$ skew-symmetric matrices $A_k$, $k = 1, \dots, 7$, which induce linear endomorphisms of $\mathbb{R}^8 \cong \mathbb{H}^2$, with the relations $A_k A_\ell + A_\ell A_k = -2\delta_{k\ell}$, $k, \ell = 1, \dots, 7$. Such matrices can be taken for example as

$$A_1 := \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix}, \quad A_2 := \begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix} \quad A_3 := \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix},$$

where

$$C := \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \; D := \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \; E := \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

and

$$A_4 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \; A_5 := \begin{pmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{pmatrix}, \; A_6 := \begin{pmatrix} \mathbf{j} & 0 \\ 0 & -\mathbf{j} \end{pmatrix}, \; A_7 := \begin{pmatrix} \mathbf{k} & 0 \\ 0 & -\mathbf{k} \end{pmatrix}.$$

Here, $\mathbf{i}, \mathbf{j}, \mathbf{k}$ stands for the standard generators of the quaternions $\mathbb{H} = \mathrm{span}_{\mathbb{R}} \{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$.

The tangent bundle $TS^7$ to the seven-dimensional sphere is trivialized through the mapping

$$TS^7 \supset T_x S^7 \ni \sum_{k=1}^{7} \alpha_k X_k \mapsto (x, \alpha) \in S^7 \times \mathbb{R}^7,$$

where $\alpha = (\alpha_1, \dots, \alpha_7)$. Then, the tangential distribution

$$\mathcal{D}^r := \mathrm{span}\, \{X_k = X(A_k) \mid k = 1, \dots, r\}$$

is showed to be bracket-generating and hence we have a bracket-generating trivializable subriemannian structure on $S^7$ for $r = 4, 5, 6$, with the induced metric from the standard Riemannian metric of $S^7$.

**Geodesic flow for the rank 5 Subriemannian structure.** We think of the following non-degenerate Lie subalgebras $\mathfrak{g}_\ell$ for $\ell = 1, \dots, 6$ in $\mathfrak{g}_7 := \mathfrak{so}(8)$:

$$\begin{aligned}
\mathfrak{g}_1 &:= \mathrm{Lie}\{A_6A_7\} \cong \mathbb{R},\\
\mathfrak{g}_2 &:= \mathrm{Lie}\{A_1A_2, A_6A_7\} \cong \mathbb{R}^2\\
\mathfrak{g}_3 &:= \mathrm{Lie}\{A_1A_2, A_1A_3, A_2A_3, A_6A_7\} \cong \mathfrak{so}(3) \oplus \mathbb{R},\\
\mathfrak{g}_4 &:= \mathrm{Lie}\{A_iA_j, A_6A_7 \,:\, i,j = 1, \dots, 4,\ i < j\} \cong \mathfrak{so}(4) \oplus \mathbb{R},\\
\mathfrak{g}_5 &:= \mathrm{Lie}\{A_iA_j, A_6A_7 \,:\, i,j = 1, \dots, 5,\ i < j\} \cong \mathfrak{so}(5) \oplus \mathbb{R},\\
\mathfrak{g}_6 &:= \mathrm{Lie}\{A_iA_j, A_6, A_7, A_6A_7 \,:\, i,j = 1, \dots, 5,\ i < j\} \cong \mathfrak{so}(5) \oplus \mathfrak{so}(3).
\end{aligned}$$

They form a chain

$$\mathfrak{g}_1 \hookrightarrow \mathfrak{g}_2 \hookrightarrow \mathfrak{g}_3 \hookrightarrow \mathfrak{g}_4 \hookrightarrow \mathfrak{g}_5 \hookrightarrow \mathfrak{g}_6 \hookrightarrow \mathfrak{g}_7 = \mathfrak{so}(8)$$

of non-degenerate subalgebras in $\mathfrak{so}(8)$.

Next, we take the Ad-invariant functions $f_i(X_i) := \dfrac{1}{2}B_{\mathfrak{g}_i}(X_i, X_i)$ on $\mathfrak{g}_i$ and the orthogonal projections $\pi_i : \mathfrak{so}(8) \to \mathfrak{g}_i$, $i = 1, \dots, 7$. The functions $f_i \circ \pi_i$, $i = 1, \dots, 7$, on $\mathfrak{so}(8)$ are functionally independent and Poisson commute. Pulling back these functions through the momentum mapping $P : TS^7 \cong T^*S^7 \to \mathfrak{so}(8)^* \cong \mathfrak{so}(8)$ defined though $P(q,\xi)(\eta) = \langle \eta q, \xi\rangle$, $q \in S^7, \xi \in T_qS^7 \subset \mathfrak{so}(8), \eta \in \mathfrak{so}(8)$, we have functionally independent and Poisson commuting functions $f_i \circ \pi_i \circ P$, $i = 1, \dots, 7$, on $TS^7$.

On the other hand, the Hamiltonian $H_{5,sub}$ on $T^*S^7$ associated to the rank 5 subriemannian geodesic flow can be written as

$$H_{5,sub} = \frac{1}{4}f_7 \circ P - f_6 \circ \pi_6 \circ P + f_5 \circ \pi_5 \circ P,$$

which of course Poisson commutes with $f_i \circ \pi_i \circ P$, $i = 1, \dots, 7$. Thus, we have the following theorem.

**Theorem 4.1.** *The subriemannian geodesic flow on $T^*S^7$ associated to the trivializable rank 5 subriemannian structure is completely integrable. A set of functionally independent and Poisson commuting constants of motion is given by $f_i \circ \pi_i \circ P$, $i = 1, \dots, 7$.*

**Geodesic flow for the rank 4 Subriemannian structure.** As to the rank 4 trivializable subriemannian structure, we consider the following non-degenerate subalgebras

$$\begin{aligned}
\mathfrak{h}_\ell &= \mathrm{span}_{\mathbb{R}}\{A_iA_j \mid 1 \le i < j \le \ell + 1\} \cong \mathfrak{so}(\ell + 1), \quad \text{if } 1 \le \ell \le 3;\\
\mathfrak{h}_4 &= \mathrm{span}_{\mathbb{R}}\{A_iA_j \mid 1 \le i < j \le 4 \,\text{or}\, (i,j) = (5,6)\} \cong \mathfrak{so}(4) \oplus \mathbb{R},\\
\mathfrak{h}_5 &= \mathrm{span}_{\mathbb{R}}\{A_iA_j \mid 1 \le i < j \le 4 \,\text{or}\, 5 \le i < j \le 7\} \cong \mathfrak{so}(4) \oplus \mathfrak{so}(3),\\
\mathfrak{h}_6 &= \mathrm{span}_{\mathbb{R}}\Big(\{A_iA_j \mid 1 \le i < j \le 5 \,\text{or}\, 5 \le i < j \le 7\} \cup \{A_5, A_6, A_7\}\Big) \cong \mathfrak{so}(4) \oplus \mathfrak{so}(4),
\end{aligned}$$

which form the chain of non-degenerate subalgebras:

$$\mathfrak{h}_1 \subset \mathfrak{h}_2 \subset \mathfrak{h}_3 \subset \mathfrak{h}_4 \subset \mathfrak{h}_5 \subset \mathfrak{h}_6 \subset \mathfrak{h}_7 := \mathfrak{so}(8).$$

We consider the Ad-invariant functions $g_i(Y_i) := \dfrac{1}{2}B_{\mathfrak{h}_i}(Y_i, Y_i)$, $\quad Y_i \in \mathfrak{h}_i$. Then, the functions $g_i \circ \pi_{\mathfrak{h}_i} \circ P$, $i = 1, \dots, 7$, are functionally independent and Poisson commuting with each other. The Hamiltonian for the subriemannian geodesic flow is written as $H_{4,sub} = \dfrac{1}{4}g_7 \circ P - g_6 \circ \pi_{\mathfrak{h}_6} \circ P + g_5 \circ \pi_{\mathfrak{h}_5} \circ P$.

Similarly to the previous case, we have the following theorem:

**Theorem 4.2.** *The subriemannian geodesic flow on $T^*S^7$ associated to the trivializable rank 4 subriemannian structure is completely integrable. A set of functionally independent and Poisson commuting constants of motion is given by $g_i \circ \pi_i \circ P$, $i = 1, \ldots, 7$.*

**Geodesic flow for the rank 6 Subriemannian structure.** For the rank 6 trivializable SR structure, we take the following non-degenerate subalgebras

$$\mathfrak{k}_\ell = \mathrm{span}_{\mathbb{R}} \{A_i A_j \mid 1 \leq i < j \leq \ell + 1\} \cong \mathfrak{so}(\ell + 1), \quad \text{if } 1 \leq \ell \leq 5;$$
$$\mathfrak{k}_6 = \mathrm{span}_{\mathbb{R}} (\{A_i A_j \mid 1 \leq i < j \leq 6\} \cup \{A_7\}) \cong \mathfrak{so}(6) \oplus \mathbb{R},$$

which gives rise to the chain of non-degenerate subalgebras:

$$\mathfrak{k}_1 \subset \mathfrak{k}_2 \subset \mathfrak{k}_3 \subset \mathfrak{k}_4 \subset \mathfrak{k}_5 \subset \mathfrak{k}_6 \subset \mathfrak{k}_7 := \mathfrak{so}(8).$$

We consider the Ad-invariant functions $h_i(Z_i) := \frac{1}{2} B_{\mathfrak{k}_i}(Z_i, Z_i), \quad Z_i \in \mathfrak{h}_i$. The functions $h_i \circ \pi_{\mathfrak{k}_i} \circ P$, $i = 1, \ldots, 7$, are functionally independent and they Poisson commute with each other. We have the Hamiltonian for the subriemannian geodesic flow as $H_{6,sub} = \frac{1}{4} h_7 \circ P - h_6 \circ \pi_{\mathfrak{k}_6} \circ P + h_5 \circ \pi_{\mathfrak{k}_5} \circ P$.

**Theorem 4.3.** *The subriemannian geodesic flow on $T^*S^7$ associated to the trivializable rank 6 subriemannian structure is completely integrable. A set of functionally independent and Poisson commuting constants of motion is given by $k_i \circ \pi_i \circ P$, $i = 1, \ldots, 7$.*

**Remarks.** Note that the chains of non-degenerate Lie subalgebras used in the above constructions are different from those in Thimm's paper [Th].

Similar methods can be applied to the geodesic flows associated to the subriemannian structures induced from the Hopf bundle and quaternionic Hopf bundle and the corresponding geodesic flows are also completely integable. The details can be found in [BaLTa].

In the paper [BaLTa], it is also showed that the above subriemannian geodesic flows have non-periodic orbits. The chains of non-degenerate subalgebras is also apploed to have some lower bounds for the dimension of the subriemannian isometry groups. Further, the paper also studies commuting differential operators whose symbols are the constants of motion constructed in the previous theorems. See [BaLTa] for the details on these topics.

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