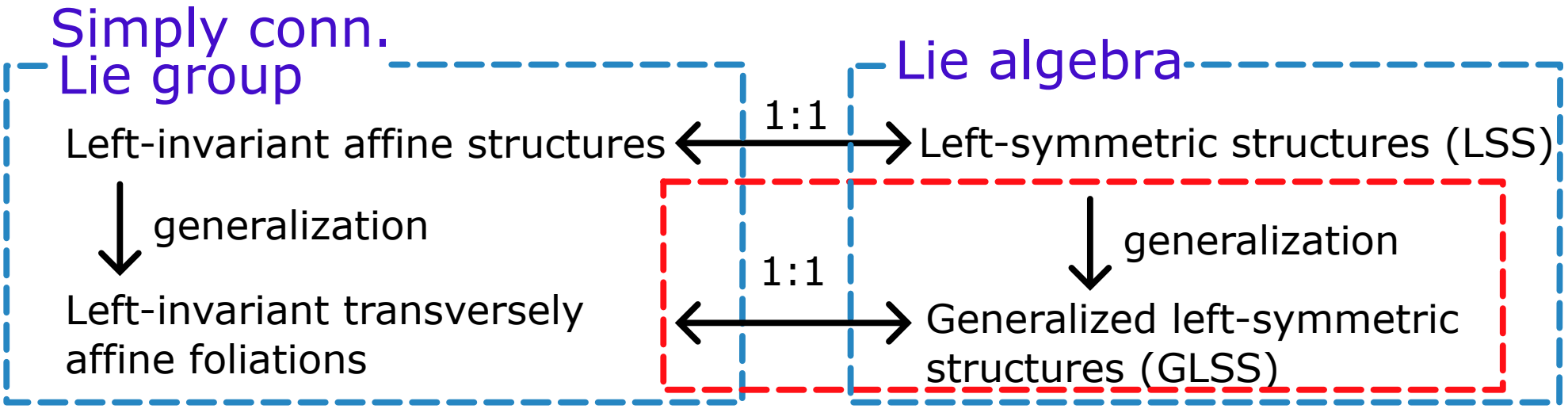
左不変アファイン葉層構造と左対称代数構造の一般化

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部分多様体幾何とリー群作用 2025
東京理科大学 神楽坂キャンパス 森戸記念館

Motivation



- Construction of GLSS
- (Non)existence of GLSS for low-dim. Lie algebras

N. Kato, *Left-invariant transversely affine foliations and a generalization of left-symmetric structures*, Beitr. Algebra Geom. (2025)

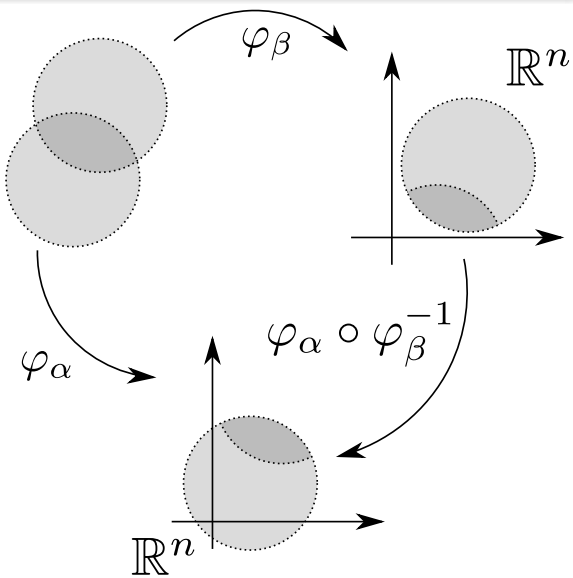
Affine structures

- M: n-dimensional smooth manifold

Definition

An atlas $\mathcal{U} = \{(U_\alpha, \varphi_\alpha)\}$ of M is an affine structure of M
$\overset{\text{def}}{\iff} \varphi_\alpha \circ \varphi_\beta^{-1} \in \mathrm{Aff}(\mathbb{R}^n) = \mathrm{GL}(\mathbb{R}^n) \ltimes \mathbb{R}^n$.

- $(M, \mathcal{U})$ is called an affine manifold.
- M has an affine structure
  $\Leftrightarrow$ M has a flat and torsion-free connection



Developing maps and holonomy homomorphisms

- M: affine manifold
  $\widetilde{M}$: the universal cover of M
  - $\rightsquigarrow$ $\exists D\colon \widetilde{M} \to \mathbb{R}^n$: a local diffeomorphism called a developing map
    $\exists h\colon \pi_1(M) \to \mathrm{Aff}(\mathbb{R}^n)$: a homomorphism called a holonomy homomorphism
    s.t. $D(\alpha\widetilde{x}) = h(\alpha)D(\widetilde{x}), \quad \forall \widetilde{x} \in \widetilde{M}, \forall \alpha \in \pi_1(M).$
- $\{\text{Affine structure of } M\} \cong \{(D,h)\}/\sim$
  $(D,h) \sim (gD, ghg^{-1}), g \in \mathrm{Aff}(\mathbb{R}^n).$

Definition

An affine structure $[D,h]$ is complete $\overset{\text{def}}{\iff}$ D is a diffeomorphism.

Example

- G: n-dim. simply connected Lie group
- $\rho\colon G \to \mathrm{Aff}(\mathbb{R}^n)$: n-dim. affine action with an open orbit $\rho(G)x_0 \subset \mathbb{R}^n$
  - $\rightsquigarrow$ G has an affine structure defined by $D = \mathrm{ev}_{x_0} \circ \rho$.
- $\Gamma \subset G$: lattice, $M := \Gamma\backslash G$
  - $\rightsquigarrow$ M has an affine structure defined by $D = \mathrm{ev}_{x_0} \circ \rho, h = \rho|_\Gamma$.
- This affine structure is complete iff the action ρ is simply transitive

For example:

- $G = \mathbb{R}^2, \rho = \mathrm{id}$.
- $G = \mathbb{R}^2, \rho\colon \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \left(\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} x + \frac{1}{2}y^2 \\ y \end{pmatrix} \right)$.
- $G = \mathbb{R}^2, \rho\colon \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \left(\begin{pmatrix} e^x & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ y \end{pmatrix} \right)$.

Left-invariant affine structures

- G: n-dim. simply connected Lie group

Definition

An affine structure $[D, h] = [D]$ of G is left-invariant
$\overset{\text{def}}{\iff} \forall g \in G,\ \exists A_g \in \mathrm{Aff}(\mathbb{R}^n)$ such that $D \circ L_g = A_g \circ D$

- $\rho\colon G \to \mathrm{Aff}(\mathbb{R}^n)$: n-dim. affine action with an open orbit
  $\rightsquigarrow$ $D = \mathrm{ev}_{x_0} \circ \rho$ is a left-inv. affine structure on G.
- Conversely, $[D]$: left-inv. affine structure of G
  $\rightsquigarrow$ $\rho\colon G \to \mathrm{Aff}(\mathbb{R}^n)$ defiend by $\rho(g) = A_g$ is a hom. with $\mathrm{ev}_{x_0} \circ \rho = D$ $(x_0 := D(e))$
- G has a left-invariant affine structure
  $\Leftrightarrow$ G has an n-dim. affine action with an open orbit.
- G has a complete left-invariant affine structure
  $\Leftrightarrow$ G has a simply transitive affine action.

Existence of left-invariant affine structures

Theorem (Chu 1974)

G: semi-simple
$\Rightarrow G$ has no left-invariant affine structures.

Theorem (Scheuneman 1971, 1974)

G: 2 or 3-step nilpotent Lie group
$\Rightarrow G$ admits a complete left-invariant affine structure.

Conjecture (Milnor 1977)

Does every solvable Lie group admit a complete left-invariant affine structure?

Theorem (Yamaguchi 1979)

A, H: abelian, $G = H \ltimes A$: simply conn.
Then G has a complete left-inv. affine structure (simply transitive affine action of G) defined by

$$\Lambda(X) = \begin{cases} 0 & (X \in \mathfrak{a}) \\ \mathrm{ad}(X) & (X \in \mathfrak{h}) \end{cases},$$

$\phi(X) = (\Lambda(X), X) \in \mathfrak{aff}(\mathfrak{g}) \cong \mathfrak{aff}(\mathbb{R}^n), d\rho = \phi.$

Theorem (Benoist 1995, Burde-Grunewald 1995)

$\exists$ nilpotent Lie group G which has no left-invariant affine structures.

Foliations

- M^n: n-dim. manifold
- $L_\lambda \subset M$: conn. imm. codimension q submanifold (called a leaf)

Definition

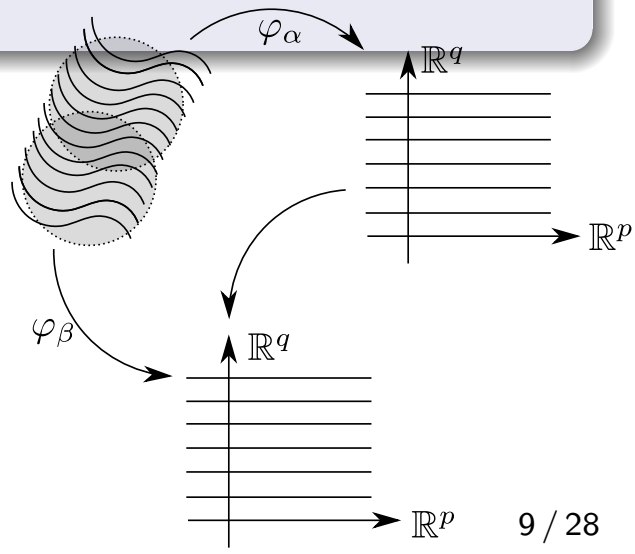
$\mathcal{F} := \{L_\lambda\}$ is a codimension q foliation of M

$\overset{\text{def}}{\iff}$

(i) $M = \bigsqcup_{\lambda\in\Lambda} L_\lambda$,

(ii) $\exists \mathcal{U} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha\in A}$: atlas of M such that $\mathcal{F}|_{U_\alpha} = \varphi_\alpha^{-1}(\{\mathbb{R}^p \times \{y\} \mid y \in \mathbb{R}^q\})$.

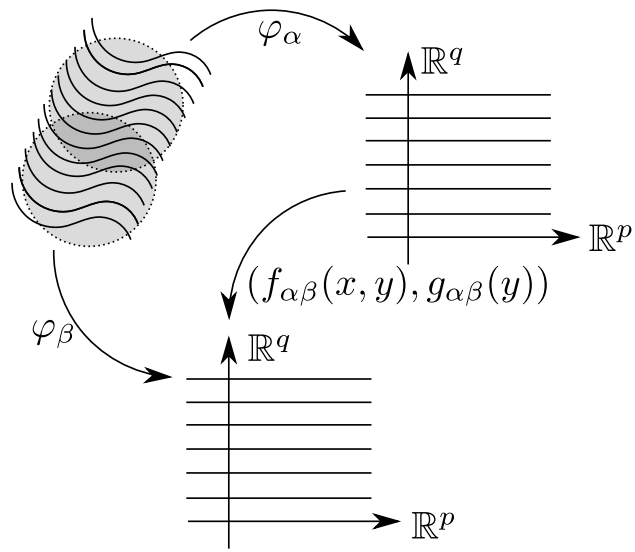
- An atlas satisfying (ii) is called a foliated atlas of $\mathcal{F}$.



Transversely affine foliations

- $\mathcal{F}$: a codim. q foliation of M^n
- $\mathcal{U} := \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$: a foliated atlas of $\mathcal{F}$

$\rightsquigarrow$ $\varphi_\alpha \circ \varphi_\beta^{-1}(x, y) = (f_{\alpha\beta}(x, y), g_{\alpha\beta}(y))$



Definition

$\mathcal{U}$ is a transversely affine structure of $\mathcal{F}$
$\overset{\text{def}}{\iff} g_{\alpha\beta} \in \mathrm{Aff}(\mathbb{R}^q) = \mathrm{GL}(\mathbb{R}^q) \ltimes \mathbb{R}^q$.

- $(\mathcal{F}, \mathcal{U})$ is called a transversely affine foliation (TAF).
- $\mathcal{F}_M := \{x \mid x \in M\}$: the codim. n foliation

  $\rightsquigarrow$ {Affine structures of M} = {Transv. affine structures of $\mathcal{F}_M$}
  = {Codim. n TAF of M}

Developing maps and holonomy homomorphisms

- $\mathcal{F}$: a TAF of M

  $\widetilde{M}$: the universal cover of M

  $\rightsquigarrow$ $\exists D\colon \widetilde{M} \to \mathbb{R}^q$: a submersion called a developing map

  $\exists h\colon \pi_1(M) \to \mathrm{Aff}(\mathbb{R}^q)$: a homomorphism called a holonomy homomorphism

  s.t. $D(\alpha\widetilde{x}) = h(\alpha)D(\widetilde{x}), \quad \forall \widetilde{x} \in \widetilde{M}, \forall \alpha \in \pi_1(M).$
- $\{\text{Codim. } q \text{ TAF of } M\} \cong \{(D,h)\}/\sim$

  $(D,h) \sim (gD, ghg^{-1}), g \in \mathrm{Aff}(\mathbb{R}^q).$

※ $\mathcal{F}$ is defined by $\pi_1(M)\backslash\{\text{conn. comp. of } D^{-1}(y) \mid y \in \mathbb{R}^q\}$.

Definition

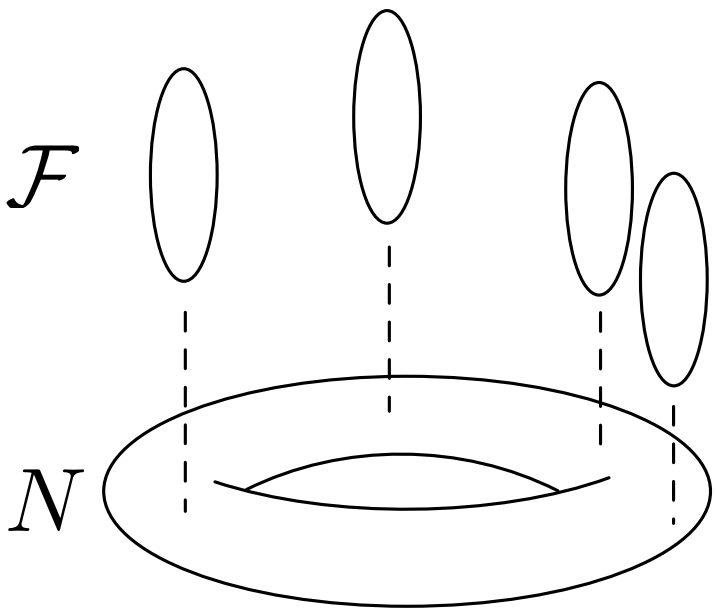
A TAF $\mathcal{F} = [D,h]$ of M is complete

$\overset{\text{def}}{\iff}$ $D\colon \widetilde{M} \to \mathbb{R}^q$ is a fiber bundle.

Example (Bundle foliation)

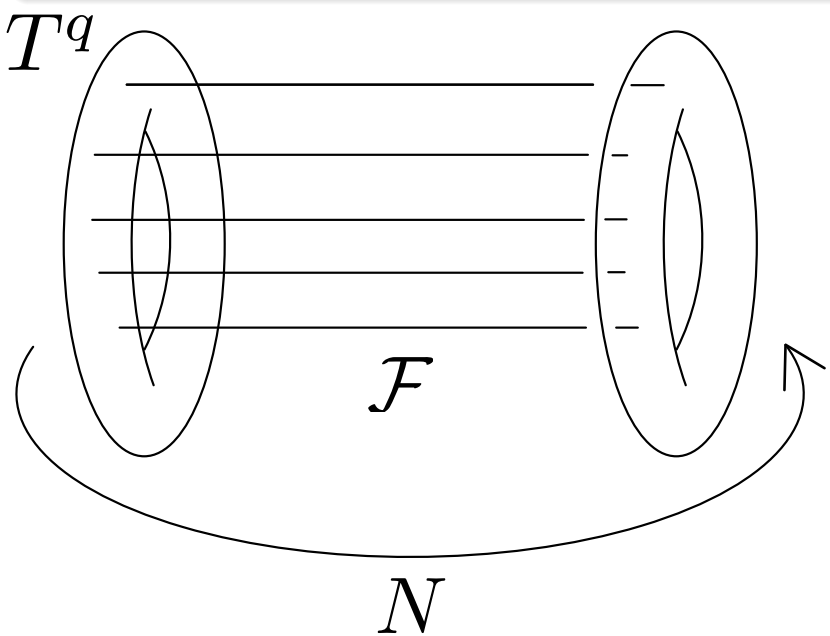
- $f\colon M^n \to N^q$: fiber bundle
- $\mathcal{F} := \{f^{-1}(y) \mid y \in N\}$: bundle foliation
- N has an affine structure $[D_0, h_0]$
  $\Rightarrow \mathcal{F}$ is a TAF.

  D, h are defined by $D = D_0 \circ \widetilde{f}, h = h_0 \circ f_*$.
- Conversely, $\mathcal{F}$ is a TAF $\Rightarrow N$ has an affine structure.



Example (Suspension foliation)

- $\rho\colon \pi_1(N) \to \mathrm{GL}(q,\mathbb{Z}) \subset \mathrm{Diff}(T^q)$: hom.
- $M := N \times_\rho T^q = \pi_1(N) \ltimes_\rho \mathbb{Z}^q \backslash \widetilde{N} \times \mathbb{R}^q$
- $p\colon \widetilde{N} \times \mathbb{R}^q \to N \times_\rho T^q$: the quotient map
- $\mathcal{F} := \{p(\widetilde{N} \times \{y\}) \mid y \in \mathbb{R}^q\}$: suspension foliation
- $\mathcal{F}$ is a TAF.
  D, h are defined by $D = \mathrm{pr}_2\colon \widetilde{M} = \widetilde{N} \times \mathbb{R}^q \to \mathbb{R}^q$,
  $h = (\rho, \mathrm{id})\colon \pi_1(M) = \pi_1(N) \ltimes_\rho \mathbb{Z}^q \to \mathrm{GL}(q,\mathbb{Z}) \ltimes \mathbb{Z}^q \subset \mathrm{Aff}(\mathbb{R}^q)$.



Example

- G: n-dim. simply connected Lie group
- $\rho\colon G \to \mathrm{Aff}(\mathbb{R}^q)$: q-dim. affine action with an open orbit $\rho(G)x_0 \subset \mathbb{R}^q$
  - $\rightsquigarrow$ G has a TAF $\mathcal{F}$ defined by $D = \mathrm{ev}_{x_0} \circ \rho \quad (\mathcal{F} = \{gG_{x_0}\}_{g\in G})$
- $\Gamma \subset G$: lattice, $M := \Gamma\backslash G$
  - $\rightsquigarrow$ M has a TAF defined by $D = \mathrm{ev}_{x_0} \circ \rho, h = \rho|_\Gamma$
- This TAF is complete iff the action ρ is transitive.

For example:

- $G = \mathbb{R}^n, \rho\colon \mathbb{R}^n \to \mathbb{R}^q \subset \mathrm{Aff}(\mathbb{R}^q)$: a projection
- $G = \left\{ \left(\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right) \right\} \subset \mathrm{Aff}(\mathbb{R}^2)$, $\rho = \iota\colon G \hookrightarrow \mathrm{Aff}(\mathbb{R}^2)$
- Horocycle flow:
  $G = \widetilde{\mathrm{SL}(2,\mathbb{R})}, \rho = \iota \circ p\colon G \to \mathrm{SL}(2,\mathbb{R}) \hookrightarrow \mathrm{Aff}(\mathbb{R}^2), x_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Left-invariant TAFs

Definition

G: n-dim. simply connected Lie group
A TAF $[D, h] = [D]$ of G is left-invariant
$\overset{\text{def}}{\iff} \forall g \in G, \exists A_g \in \mathrm{Aff}(\mathbb{R}^q)$ such that $D \circ L_g = A_g \circ D$.

- $\rho\colon G \to \mathrm{Aff}(\mathbb{R}^q)$: q-dim. affine action with an open orbit
  $\rightsquigarrow$ $D = \mathrm{ev}_{x_0} \circ \rho$ is a left-inv. TAF of G
- Conversely, $[D]$: left-inv. TAF of G
  $\rightsquigarrow$ $\rho\colon G \to \mathrm{Aff}(\mathbb{R}^q)$ defined by $\rho(g) = A_g$ is a hom. with $\mathrm{ev}_{x_0} \circ \rho = D \quad (x_0 := D(e))$
- G has a codim. q left-inv. TAF
  $\Leftrightarrow$ G has a q-dim. affine action with an open orbit
- G has a codim. q left-inv. complete TAF
  $\Leftrightarrow$ G has a q-dim. transitive affine action

Left-symmetric structures

- $\mathfrak{g}$: Lie algebra

Definition

A bilinear map $\cdot : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is a left-symmetric structure (LSS)
$\overset{\text{def}}{\iff}$

(1) $X \cdot (Y \cdot Z) - (X \cdot Y) \cdot Z = Y \cdot (X \cdot Z) - (Y \cdot X) \cdot Z$,

(2) $[X, Y] = X \cdot Y - Y \cdot X$.

- $(X, Y, Z) := X \cdot (Y \cdot Z) - (X \cdot Y) \cdot Z$: associator
  Then (1) $\Leftrightarrow (X, Y, Z) = (Y, X, Z)$.
- Under (2),
  (1)$\Leftrightarrow$ (1)': $\ell : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is a homomorphism of Lie algebras.
  $\ell_X : Y \mapsto X \cdot Y$ is the left translation.

Theorem (Matsushima 1968, Helmstetter 1971)

Left-invariant affine structures of $G \overset{1:1}{\longleftrightarrow}$ LSSs on $\mathfrak{g}$.

Theorem (Medina 1981)

A left-invariant affine structure is complete
$\Leftrightarrow$ the corresponding LSS satisfies that
$\forall Y \in \mathfrak{g}, r_Y + \mathrm{id} \colon \mathfrak{g} \to \mathfrak{g}$ is bijective.

Example

$\mathfrak{g} = \mathbb{R}^2 = \mathbb{R}\langle X, Y \rangle$

(1) $X \cdot X = X \cdot Y = Y \cdot X = Y \cdot Y = 0$: complete LSS

(2) $X \cdot X = Y$: complete LSS

(3) $X \cdot X = X$: non-complete LSS

(1) corresponds to $\rho = \mathrm{id}$

(2) corresponds to $\rho \colon \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \left(\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} x + \frac{1}{2} y^2 \\ y \end{pmatrix} \right)$

(3) corresponds to $\rho \colon \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \left(\begin{pmatrix} e^x & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ y \end{pmatrix} \right)$

Theorem (Yamaguchi 1979)

A, H: abelian, $G = H \ltimes A$: simply conn.
Then G has a complete left-inv. affine structure (simply transitive affine action of G) defined by

$$\Lambda(X) = \begin{cases} 0 & (X \in \mathfrak{a}) \\ \mathrm{ad}(X) & (X \in \mathfrak{h}) \end{cases},$$

$\phi(X) = (\Lambda(X), X) \in \mathfrak{aff}(\mathfrak{g}) \cong \mathfrak{aff}(\mathbb{R}^n), d\rho = \phi.$

Example (Consequence of Yamaguchi's Theorem)

$\mathfrak{a}, \mathfrak{h}$: abelian, $\mathfrak{g} = \mathfrak{h} \ltimes \mathfrak{a}$.
$(H_1 + A_1) \cdot (H_2 + A_2) := [H_1, H_2 + A_2] = [H_1, A_2]$
is a complete LSS of $\mathfrak{g}$ $(H_i \in \mathfrak{h}, A_i \in \mathfrak{a})$.

Generalized left-symmetric structures

- $\mathfrak{g}$: a Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$: a subalgebra

Definition

A bilinear map $\bullet\colon \mathfrak{g} \times \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}/\mathfrak{h}$ is a generalized left-symmetric structure (GLSS) of $\mathfrak{g}$ w.r.t. $\mathfrak{h}$

$\overset{\text{def}}{\iff}$

(1) $[X,Y] \bullet \overline{Z} = X \bullet (Y \bullet \overline{Z}) - Y \bullet (X \bullet \overline{Z})$

(2) $\overline{[X,Y]} = X \bullet \overline{Y} - Y \bullet \overline{X}$

- (1) $\Leftrightarrow \ell\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}/\mathfrak{h})$ is a homomorphism of Lie algebras
- GLSS of $\mathfrak{g}$ w.r.t. $\{0\}$ = LSS

Left-invariant TAF and GLSS

Theorem (K. 2025)

Codim. q left-invariant TAFs of $G \overset{1:1}{\longleftrightarrow}$ codim. q GLSSs of $\mathfrak{g}$.
More precisely, $\mathrm{LTAF}'_q(G) \cong \mathrm{GLSS}_q(\mathfrak{g})$.

- $\mathrm{LTAF}'_q(G) := \{[D]\}$: the set of codim. q left-invariant TAF of G
- $\mathrm{GLSS}_q(\mathfrak{g})$
  $:= \{(\bullet, \mathfrak{h}) \mid \mathfrak{h} \subset \mathfrak{g}$: codim. q subalgebra, $\bullet$: GLSS of $\mathfrak{g}$ w.r.t. $\mathfrak{h}\}$

Outline of Proof

- $\mathrm{LTAF}_q(G)$
  $:= \{(\rho, x_0) \in \mathrm{Hom}(G, \mathrm{Aff}(\mathbb{R}^q)) \times \mathbb{R}^q \mid \mathrm{ev}_{x_0} \circ \rho$ is a sbm.$\}/\mathrm{Aff}(\mathbb{R}^q)$
- $\mathrm{LTAF}_q(\mathfrak{g})$
  $:= \{(\phi, x_0) \in \mathrm{Hom}(\mathfrak{g}, \mathfrak{aff}(\mathbb{R}^q)) \times \mathbb{R}^q \mid \mathrm{ev}_{x_0} \circ \phi$ is a surj.$\}/\mathrm{Aff}(\mathbb{R}^q)$
- Prove that $\mathrm{LTAF}'_q(G) \cong \mathrm{LTAF}_q(G) \cong \mathrm{LTAF}_q(\mathfrak{g}) \cong \mathrm{GLSS}_q(\mathfrak{g})$

$\mathrm{LTAF}_q(\mathfrak{g}) \to \mathrm{GLSS}_q(\mathfrak{g})$

- $[\phi, x_0] \in \mathrm{LTAF}_q(\mathfrak{g})$, $f := \mathrm{ev}_{x_0} \circ \phi \colon \mathfrak{g} \to \mathbb{R}^q$
- $\mathfrak{h} := \mathrm{Ker}(f) \subset \mathfrak{g}$: subalgebra
  - $\rightsquigarrow$ $\overline{f} \colon \mathfrak{g}/\mathfrak{h} \xrightarrow{\sim} \mathbb{R}^q$
- Define $\bullet$ by $X \bullet \overline{Y} = \overline{f}^{-1}(\phi_L(X)\overline{f}(\overline{Y}))$
  - $\rightsquigarrow$ $(\bullet, \mathfrak{h}) \in \mathrm{GLSS}_q(\mathfrak{g})$ □

$\mathrm{GLSS}_q(\mathfrak{g}) \to \mathrm{LTAF}_q(\mathfrak{g})$

- $(\bullet, \mathfrak{h}) \in \mathrm{GLSS}_q(\mathfrak{g})$
- Fix $\overline{f} \colon \mathfrak{g}/\mathfrak{h} \xrightarrow{\sim} \mathbb{R}^q$: linear isom, $f := \overline{f} \circ p \colon \mathfrak{g} \to \mathbb{R}^q$
- Define $\phi_L \colon \mathfrak{g} \to \mathfrak{gl}(\mathbb{R}^q)$ by $\phi_L(X) = \overline{f} \circ \ell_X \circ \overline{f}^{-1}$
  - $\rightsquigarrow$ $\phi = (\phi_L, f) \colon \mathfrak{g} \to \mathfrak{aff}(\mathbb{R}^q)$ is a hom. and $[\phi, \boldsymbol{o}] \in \mathrm{LTAF}_q(\mathfrak{g})$

Completeness of TAF and GLSS

- Recall:
  - **Thm(Medina 1981)**: A left-invariant affine structure is complete
    $\Leftrightarrow$ the corresponding LSS satisfies that
    $\forall Y \in \mathfrak{g}, r_Y + \mathrm{id} \colon \mathfrak{g} \to \mathfrak{g}$ is bijective.
  - **Def**: A TAF $[D, h]$ of M is complete
    $\overset{\text{def}}{\Longleftrightarrow} D \colon \widetilde{M} \to \mathbb{R}^q$ is a fiber bundle projection.

Theorem (K. 2025)

A left-invariant TAF of G is complete
$\Leftrightarrow$ the corresponding GLSS satisfies that
$\forall \overline{Y} \in \mathfrak{g}/\mathfrak{h}, r_{\overline{Y}} + p \colon \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ is surjective.

Construction of GLSS

- Suppose that $\mathfrak{h}$ is an ideal of $\mathfrak{g}$.

Example 1

$\cdot$: LSS on $\mathfrak{g}/\mathfrak{h}$
Then $X \bullet \overline{Y} := \overline{X} \cdot \overline{Y}$ is a GLSS of $\mathfrak{g}$ w.r.t. $\mathfrak{h}$

Proposition

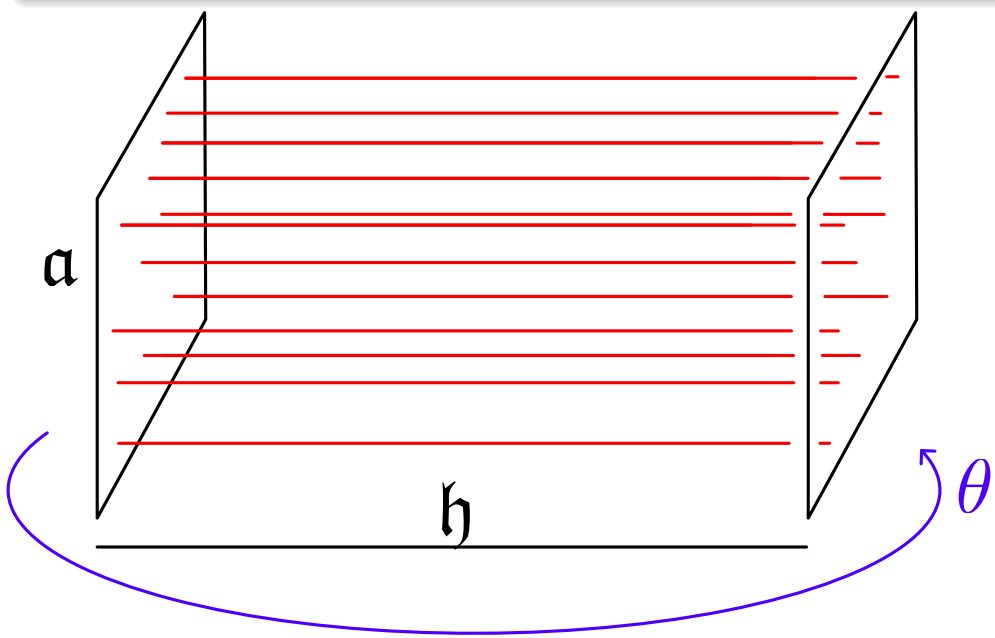
Conversely, for any GLSS $\bullet$ of $\mathfrak{g}$ w.r.t. $\mathfrak{h}$, $\exists$ LSS $\cdot$ on $\mathfrak{g}/\mathfrak{h}$ s.t. $X \bullet \overline{Y} = \overline{X} \cdot \overline{Y}$

Example (Consequence of Yamaguchi's Theorem)

$\mathfrak{a}, \mathfrak{h}$: abelian, $\mathfrak{g} = \mathfrak{h} \ltimes \mathfrak{a}$.
$(H_1 + A_1) \cdot (H_2 + A_2) := [H_1, H_2 + A_2] = [H_1, A_2]$
is a complete LSS of $\mathfrak{g}$ $(H_i \in \mathfrak{h}, A_i \in \mathfrak{a})$.

Example 2

$\mathfrak{a}$: abelian, $\mathfrak{g} = \mathfrak{h} \ltimes_\theta \mathfrak{a}$ $(\theta \colon \mathfrak{h} \to \mathrm{Der}(\mathfrak{a}))$
$(H_1 + A_1) \bullet \overline{A_2} := \overline{[H_1, A_2]}$ is a complete GLSS of $\mathfrak{g}$ w.r.t. $\mathfrak{h}$.



- Horocycle flow:

$$G = \widetilde{\mathrm{SL}(2,\mathbb{R})}, \rho = \iota \circ p \colon G \to \mathrm{SL}(2,\mathbb{R}) \hookrightarrow \mathrm{Aff}(\mathbb{R}^2), x_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Example 3 (Horocycle flow)

$\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R}) = \mathbb{R}\langle H, X, Y\rangle$

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$\rightsquigarrow [H, X] = 2X, [H, Y] = -2Y, [X, Y] = H$

$\mathfrak{h} = \mathbb{R}\langle X\rangle, \mathfrak{g}/\mathfrak{h} = \mathbb{R}\langle \overline{H}, \overline{Y}\rangle$

$$H \bullet \overline{H} = \overline{H},\ H \bullet \overline{Y} = -\overline{Y},\ X \bullet \overline{H} = 0,\ X \bullet \overline{Y} = \overline{H},$$
$$Y \bullet \overline{H} = \overline{Y},\ Y \bullet \overline{Y} = 0$$

Then $\bullet$ is a GLSS of $\mathfrak{sl}(2,\mathbb{R})$ w.r.t. $\mathfrak{h}$.

Existence of GLSSs for low-dim. Lie algebras

- $\mathfrak{g}$ admits the codim. 0 GLSS ($n = \dim \mathfrak{g}, \mathfrak{h} = \mathfrak{g}$).
- $\mathfrak{g}$ admits a codim. n GLSS $\Leftrightarrow$ $\mathfrak{g}$ admits a LSS.
- $\mathbb{R}^n$ admits a codim. q GLSS for any q.

$\mathfrak{g}$	$q = \operatorname{codim} \mathfrak{h}$	GLSS
$\mathfrak{aff}(2) = \mathbb{R} \ltimes \mathbb{R}$	1	○ (Example 1, $\mathfrak{h} = \mathbb{R}$)
	2	○ (Yamaguchi '79)
$\mathbb{R} \ltimes_\theta \mathbb{R}^2$ $\theta = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & h \end{pmatrix}$	1	○ (Example 1, $\mathfrak{h} = \mathbb{R}^2$)
	2	○ (Example 2, $\mathfrak{h} = \mathbb{R}$)
	3	○ (Yamaguchi '79)

Existence of GLSSs for low-dim. Lie algebras

- Any semi-simple Lie algebra has no LSS (Chu'74).
- $\mathfrak{aff}(2)$ is solvable.
  - $\rightsquigarrow$ Any simple Lie algebra has no codim. 1 GLSS.
- For a compact manifold M, $\nexists$ submersion $f\colon M \to \mathbb{R}^q$.
  - $\rightsquigarrow$ Any compact simple Lie algebra has no GLSS.

$\mathfrak{g}$	$q = \operatorname{codim} \mathfrak{h}$	GLSS
$\mathfrak{so}(3)$	1	$\times$
	2	$\times$
	3	$\times$ (Chu'74)
$\mathfrak{sl}_2(\mathbb{R})$	1	$\times$
	2	$\bigcirc$ (horocycle flow)
	3	$\times$ (Chu'74)

Thank you for your attention!