平行移動写像と弱鏡映部分多様体のアファイン微分幾何学
Affine differential geometry of parallel transport maps and weakly reflective submanifolds

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部分多様体幾何とリー群作用 **2025**

東京理科大学 神楽坂キャンパス 森戸記念館第 **1** フォーラム

2025 年 **12** 月 **9** 日

Overview: The parallel transport map $\Phi_{G/K}$

Hilbert sp

$V_{\mathfrak{g}} := L^2([0,1], \mathfrak{g})$

$\Phi_{G/K}\downarrow$ **Riem sbmer**

G/K

compact Riem sym sp

(**R. S. Palais, C.-L. Terng; C.-L. Terng, G. Thorbergsson; 1980s–1990s**)

pseudo-Hilbert sp

$V_{\mathfrak{g}} := L^2([0,1], \mathfrak{g})$

$\Phi_{G/K}\downarrow$ **pseudo-Riem sbmer**

G/K

Riem sym sp of non-compact type

(**N. Koike 2000s–**)

This talk: We consider generalization of these theories in the framework of **affine differential geometry**.

Contents

1. Affine immersions and affine submersions
2. Parallel transport maps over Lie groups
3. Parallel transport maps over homogeneous spaces
4. Focal points and Fredholm properties
5. Regularized mean curvatures
6. Weakly reflective submanifolds

Sec. 1 - Affine immersions and affine submersions (1/3)

Definition (cf. Nomizu-Sasaki 1994)

M, N: smooth manifolds, $\varphi : M \to N$: immersion

$\mathcal{W}$: transversal bundle, i.e. subbundle of the pullback φ^*TN s.t.

$$\varphi^*TN = d\varphi(TM) \oplus \mathcal{W}.$$

∇^M, ∇^N: (torsion-free) affine connections on M, N respectively.

$\varphi : (M, \nabla^M) \to (N, \nabla^N)$ is an affine immersion with transversal bdl $\mathcal{W}$

$$\overset{\text{def}}{\iff} \quad \left(\nabla^{\varphi^*TN}_X d\varphi(Y)\right)^\top = d\varphi\left(\nabla^M_X Y\right) \quad \text{for } X, Y \in \mathfrak{x}(M).$$

Then affine fundamental form α, shape op A, transversal conn $\nabla^{\mathcal{W}}$ are defined by:

$$\nabla^{\varphi^*TN}_X d\varphi(Y) = d\varphi(\nabla^M_X Y) \oplus \alpha(X, Y), \qquad X, Y \in \mathfrak{x}(M).$$

$$\nabla^{\varphi^*TN}_X Z = -d\varphi(A_Z(X)) \oplus \nabla^{\mathcal{W}}_X Z, \qquad X \in \mathfrak{x}(M),\ Z \in \Gamma(\mathcal{W}).$$

Note $\varphi : M \to (N, \nabla^N)$: immersion with transversal bdl $\mathcal{W}$.

$\Rightarrow \exists^1 \nabla^M$ s.t. $\varphi : (M, \nabla^M) \to (N, \nabla^N)$: affine immersion with t-bdl $\mathcal{W}$.

Sec. 1 - Affine immersions and affine submersions (2/3)

Definition (Abe-Hasegawa 2001)

$\hat{N}$, N: smooth manifolds, $\pi : \hat{N} \to N$: submersion.

$\mathcal{V}$: the vertical distribution of π, i.e. $\mathcal{V}_u := \mathrm{Ker}(d\pi)_u = T_u(F_u)$

where F_u: fiber through $u \in \hat{N}$

$\mathcal{H}$: a horizontal distribution of π, i.e. a subbundle of $T\hat{N}$ satisfying

$$T\hat{N} = \mathcal{V} \oplus \mathcal{H}$$

$\hat{\nabla}$, ∇: (torsion-free) affine connections on $\hat{N}$, N respectively.

$\pi : (\hat{N}, \hat{\nabla}) \to (N, \nabla)$ is an **affine submer with horizontal distribution $\mathcal{H}$**

$\underset{\text{def}}{\iff} (\hat{\nabla}_{\hat{X}}\hat{Y})^{\mathcal{H}} = (\nabla_X Y)\hat{}$ for $X, Y \in \mathfrak{x}(N)$. ($\hat{X}$: the horizontal lift of X)

Then **fundamental tensors** (of O'Neill) are defined by

$$\mathcal{T}_Z W := (\hat{\nabla}_{Z^{\mathcal{V}}} W^{\mathcal{V}})^{\mathcal{H}} + (\hat{\nabla}_{Z^{\mathcal{V}}} W^{\mathcal{H}})^{\mathcal{V}},$$

$$\mathcal{A}_Z W := (\hat{\nabla}_{Z^{\mathcal{H}}} W^{\mathcal{H}})^{\mathcal{V}} + (\hat{\nabla}_{Z^{\mathcal{H}}} W^{\mathcal{V}})^{\mathcal{H}}, \qquad Z, W \in \mathfrak{x}(\hat{N})$$

Lemma (Abe-Hasegawa 2001)

If $\mathcal{A}|_{\mathcal{H}\times\mathcal{H}}$ is alternating, then "γ: geodesic $\Leftrightarrow$ h-lift $\hat{\gamma}$: geodesic".

Sec. 1 - Affine immersions and affine submersions (3/3)

Remark

$\hat{M} := \pi^{-1}(M)$ $\subset$ $(\hat{N}, \nabla^{\hat{N}})$ $\hat{\mathcal{W}}$: horizontal lift of $\mathcal{W}$

submfd with $\hat{\mathcal{W}}$

$\downarrow$ $\pi \downarrow$ affine submer with $\mathcal{H}$

M $\subset$ (N, ∇^{N})

submfd with $\mathcal{W}$

In particular, the transversal bundle of each fiber $F = \pi^{-1}(pt)$ is defined as the restriction of $\mathcal{H}$ to F.

Sec. 2 - Parallel transport maps over Lie groups (1/4)

Setting

G: (fin dim) connected Lie group. $\mathfrak{g}$: its Lie algebra.

The space of L^2-paths

Choose $\langle \cdot, \cdot \rangle$: inner product of $\mathfrak{g}$ $\rightsquigarrow$ $|\cdot|$: induced norm on $\mathfrak{g}$.

$$\langle u, v \rangle_{L^2} := \int_0^1 \langle u(t), v(t) \rangle dt, \quad \rightsquigarrow \quad \|u\|_{L^2} := \left(\int_0^1 |u(t)|^2 dt \right)^{\frac{1}{2}}$$

$$V_{\mathfrak{g}} := L^2([0,1], \mathfrak{g}) := \left\{ u : [0,1] \to \mathfrak{g} : \text{measurable} \;\middle|\; \|u\|_{L^2} < \infty \right\} \Big/ \underset{\text{a.e.}}{\sim}$$

$\Rightarrow (V_{\mathfrak{g}}, \langle \cdot, \cdot \rangle_{L^2})$ becomes a Hilbert space.

As a topological vector sp, $V_{\mathfrak{g}}$ does not depend on $\langle \cdot, \cdot \rangle$.

Thus, we can (and will) regard $V_{\mathfrak{g}}$ as a **Hilbertable space**.
(i.e. a topological vector sp s.t. a complete inner product exists.)

Sec. 2 - Parallel transport maps over Lie groups (2/4)

The Hilbert Lie group of Sobolev H^1-paths

A continuous map $g : [0,1] \to G$ is of Sobolev H^1

$\iff$ g is of Sobolev H^1 in each local chart of G

$\overset{\text{def}}{\Longrightarrow}$ the derivative $g'(t) \in T_{g(t)}G$ is defined for almost every $t \in [0,1]$ and $\int_0^1 \rho(g'(t), g'(t))\, dt < \infty$ for any Riemannian metric ρ on G.

Moreover, the set

$$\mathcal{G} := H^1([0,1], G) = \{g : [0,1] \to G \text{: Sobolev } H^1\text{-map}\}$$

becomes a Hilbert mfd with smooth group str $(g \cdot h)(t) := g(t)h(t)$.

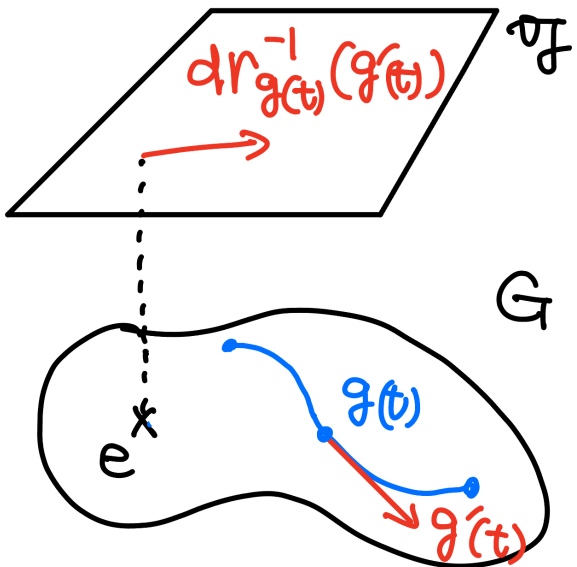
Lie group action (essentially Palais-Terng)

Define a transitive, affine action $\mathcal{G} \curvearrowright V_{\mathfrak{g}}$ by

$$g * u := \mathrm{Ad}(g)u - dr_g^{-1}(g') \quad \in V_{\mathfrak{g}}$$

Here $(\mathrm{Ad}(g)u)(t) := \mathrm{Ad}^G(g(t))u(t)$,

r_g: the right translation by $g \in \mathcal{G}$



Sec. 2 - Parallel transport maps over Lie groups (3/4)

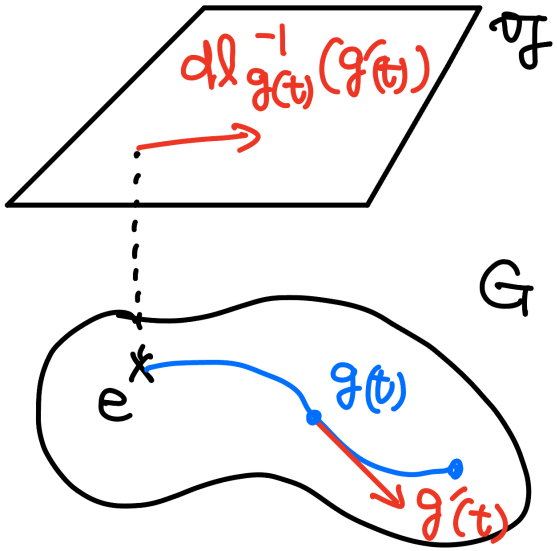
Definition (essentially Palais and Terng)

The **parallel transport map**

$$\Phi : V_{\mathfrak{g}} \to G, \quad u \mapsto \Phi(u)$$

is a submersion defined by $\Phi(u) = g(1)$ where $g \in \mathcal{G}$ is the unique solution to ODE

$$dl_g^{-1}(g') = u, \quad g(0) = e.$$



Proposition (essentially due to Palais and Terng 1988)

The diagram commutes:

$$\begin{array}{ccccc} g & \in & \mathcal{G} & \curvearrowright & V_{\mathfrak{g}} \\ \downarrow & & \downarrow & & \Phi \downarrow \\ (g(0), g(1)) & \in & G \times G & \curvearrowright & G \end{array} \qquad \begin{array}{l} g * u := \mathrm{Ad}(g)u - dr_g^{-1}(g') \\ \\ (b, c) \cdot a := bac^{-1} \end{array}$$

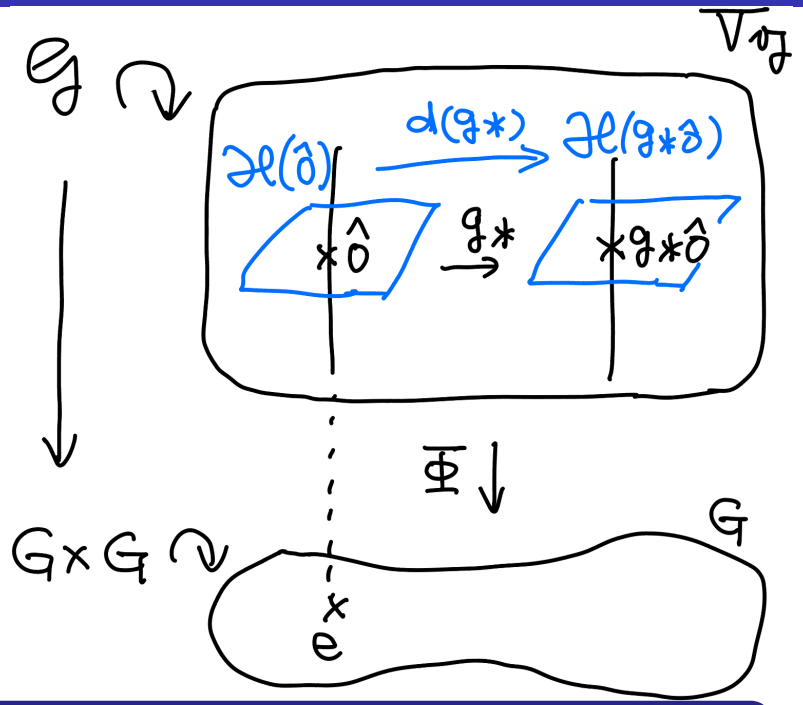
Remark

The unique existence for ODE (L^2-coefficients, H^1-solution) is not trivial, especially when G is non-compact.

Sec. 2 - Parallel transport maps over Lie groups (4/4)

Theorem (M. 2025)

G: connected Lie grp, ∇^G: cano conn
$(\nabla^G_X Y = \frac{1}{2}[X,Y],\ X,Y \in \mathfrak{g})$
D: flat conn on the Hilbertable sp $V_{\mathfrak{g}}$.
$\Rightarrow \Phi : (V_{\mathfrak{g}}, D) \to (G, \nabla^G)$: affine submer
with $\mathcal{G}$-inv horizontal dist $\mathcal{H}$ defined by $\mathcal{H}(\hat{0}) := \hat{\mathfrak{g}}$ (: const paths in $\mathfrak{g}$)
Moreover, $\mathcal{A}|_{\mathcal{H}\times\mathcal{H}}$ is alternating.



Remark

The above corollary extends the following known results:

- G: compact, ρ: bi-inv Riem met on G, $\langle\cdot,\cdot\rangle^\rho$: Ad-inv inner prod on $\mathfrak{g}$

$\Rightarrow \Phi : (V_{\mathfrak{g}}, \langle\cdot,\cdot\rangle^\rho_{L^2}) \to (G,\rho)$: Riem submer [Terng-Thorbergsson 1995]

- G: semisimple, ρ: bi-inv pseudo-Riem metric on G, $\langle\cdot,\cdot\rangle^\rho$: Ad-inv indefinite inner prod on $\mathfrak{g}$

$\Rightarrow \Phi : (V_{\mathfrak{g}}, \langle\cdot,\cdot\rangle^\rho_{L^2}) \to (G,\rho)$: pseudo-Riem submer [Koike 2004]

Here, the horizontal dist induced by metric coincides with $\mathcal{H}$ above.

Sec. 3 - Parallel transport maps over homog sp (1/3)

Definition

G: connected Lie group, K: closed subgroup of G with Lie alg $\mathfrak{k}$

The parallel transport map over the homog sp G/K is defined by

$$\Phi_{G/K} := \pi \circ \Phi : V_{\mathfrak{g}} \to G \to G/K, \quad \text{(where } \pi : G \to G/K \text{: proj)}$$

Remark

$\Phi_{G/\{e\}} = \Phi$. Moreover $\Phi_{(G\times G)/\Delta G}$ and Φ are naturally identified.

Proposition

Set $\mathcal{G}_K := \{g \in \mathcal{G} \mid g(1) \in K\}$. Then the diagram commutes:

$$\begin{array}{ccccc} g & \in & \mathcal{G}_K & \curvearrowright & V_{\mathfrak{g}} \\ \downarrow & & \downarrow & & \Phi_{G/K} \downarrow \\ g(0) & \in & G & \curvearrowright & G/K \end{array} \qquad \begin{array}{l} g \cdot u := \mathrm{Ad}(g)u - dr_g^{-1}(g') \\ \\ b \cdot aK := (ba)K \end{array}$$

Sec. 3 - Parallel transport maps over homog sp (2/3)

Recall (Nomizu 1954, Kobayashi-Nomizu)

The homogeneous space G/K is reductive

$\Leftrightarrow$ $\mathfrak{p}$: $\mathrm{Ad}(K)$-inv subspace of $\mathfrak{g}$ s.t. $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$

$\overset{\text{def}}{\Rightarrow}$ $\exists^1$ $\nabla^{G/K}$: G-invariant affine connection on G/K s.t.

(1) Every geodesic through eK is expressed as $(\exp tX)K$ for $X \in \mathfrak{p}$,

(2) $\nabla^{G/K}$ is torsion-free

Here $\nabla^{G/K}$ is called the natural torsion-free connection on G/K.

Theorem (M.: preprint)

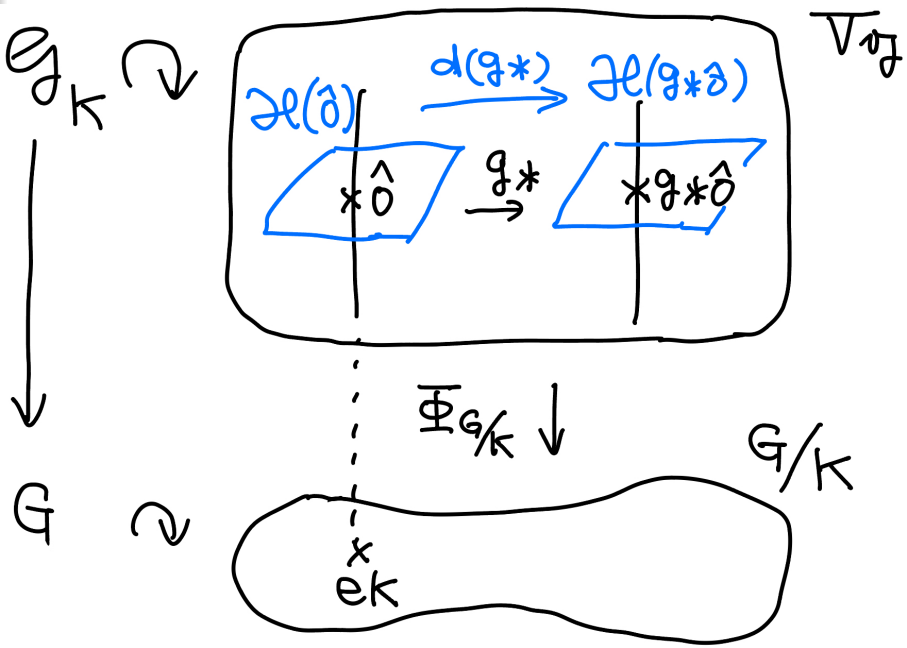
G/K: reductive homog sp

with decomp $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$.

$\nabla^{G/K}$: natural torsion-free conn.

$\Longrightarrow$ $\Phi_{G/K} : (V_{\mathfrak{g}}, D) \to (G/K, \nabla^{G/K})$ is an affine submersion with $\mathcal{G}_K$-inv h-dist $\mathcal{H}$ defined by $\mathcal{H}(\hat{0}) := \hat{\mathfrak{p}}$ (: const paths in $\mathfrak{p}$)

Moreover, $\mathcal{A}|_{\mathcal{H}\times\mathcal{H}}$ is alternating.



Sec. 3 - Parallel transport maps over homog sp (3/3)

Corollary (M. 2025)

G/K: symmetric space, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$: canonical decomp
$\nabla^{G/K}$: canonical connection (i.e. $(G/K, \nabla^{G/K})$: affine symmetric sp)
$\Longrightarrow \Phi_{G/K} : (V_{\mathfrak{g}}, D) \to (G/K, \nabla^{G/K})$ is an affine submersion
with $\mathcal{G}_K$-inv h-dist $\mathcal{H}$ defined by $\mathcal{H}(\hat{0}) := \hat{\mathfrak{p}}$ (: const paths in $\mathfrak{p}$)
Moreover, $\mathcal{A}|_{\mathcal{H}\times\mathcal{H}}$ is alternating.

Remark

The above corollary extends the following two results:

- $(G/K, \rho^{G/K})$: Riem sym sp of compact type,
  $\rho = \rho^G$: bi-inv Riem metric on G which induces $\rho^{G/K}$

$\Rightarrow \Phi_{G/K} : (V_{\mathfrak{g}}, \langle \cdot, \cdot \rangle^{\rho}_{L^2}) \to (G/K, \rho^{G/K})$: Riem sbmer [Terng-Thorbergsson]

- $(G/K, \rho^{G/K})$: Riem sym sp of non-compact type
  $\rho = \rho^G$: bi-inv pseudo-Riem metric on G which induces ρ

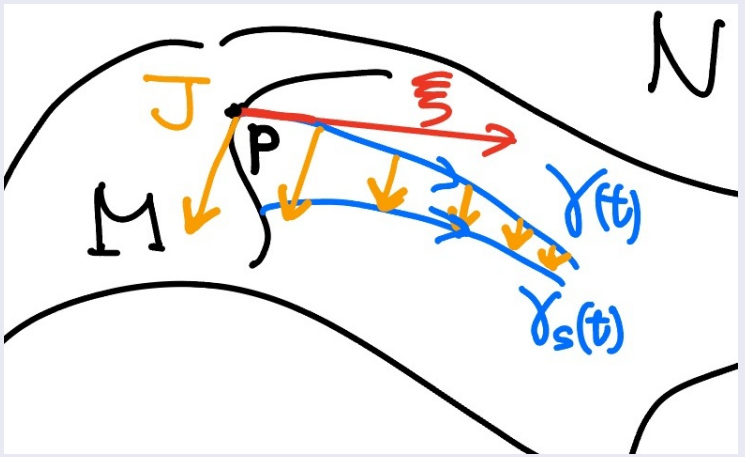
$\Rightarrow \Phi_{G/K} : (V_{\mathfrak{g}}, \langle \cdot, \cdot \rangle^{\rho}_{L^2}) \to (G/K, \rho^{G/K})$: pseudo-Riem submer. [Koike]

Here, the horizontal dist induced by metric coincides with $\mathcal{H}$ above.

Sec. 4 - Focal points and Fredholm properties (1/3)

Definitions (well-known in the Riemannian case)

$(M, \nabla^M) \hookrightarrow (N, \nabla^N)$: affine immersion with transversal bdl $\mathcal{W}$.
$p \in M,\ \xi \in \mathcal{W}_p,\ \gamma(t) := \mathrm{Exp}_p^N\, t\xi.$



- J is a M-Jacobi field along γ
  $\overset{\text{def}}{\iff}$ J is a variational vector field of a geodesic variation γ_s of γ s.t. $\gamma_s(0) \in M$ and $\gamma_s'(0) \in \mathcal{W}_{\gamma_s(0)}$
  $\iff$ J is a Jacobi field along γ s.t. $J(0) \in T_pM$ and $J'(0)^\top = -A^M_{\gamma'(0)} J(0)$.
  $\mathfrak{J}^M_\gamma := \{J : M\text{-Jacobi field along } \gamma\} \cong T_pM \oplus \mathcal{W}_p \cong T_\xi \mathcal{W}$
- ξ is a focal direction of M
  $\overset{\text{def}}{\iff} 0 \neq \exists J \in \{J \in \mathfrak{J}^M_\gamma \mid J(1) = 0\}$
  $\iff (d\,\mathrm{Exp}^N|_{\mathcal{W}})_\xi : T_\xi \mathcal{W} \to T_{\mathrm{Exp}_p^N \xi} N$ is not injective

  Then $\gamma(1) = \mathrm{Exp}_p^N \xi$ is called a focal point of M.
  $\dim\{J \in \mathfrak{J}^M_\gamma \mid J(1) = 0\} = \dim(\mathrm{Ker}(d\,\mathrm{Exp}^N|_{\mathcal{W}})_\xi)$: its multiplicity.

Sec. 4 - Focal points and Fredholm properties (2/3)

Prop (M. 2025, see also Heintze-Liu-Olmos)

V: Hilbertable space, D: flat connection on V.
$\pi : (V, D) \to (N, \nabla^N)$: affine submer with h-dist $\mathcal{H}$ s.t. $\mathcal{A}|_{\mathcal{H}\times\mathcal{H}}$ is alternating.
M submanifold of N with transversal bundle $\mathcal{W}$
$\hat{M} := \pi^{-1}(M)$: submfd with lifted t-bundle $\hat{\mathcal{W}}$
Then the following conditions are equivalent:

(1) $\hat{\xi} \in \hat{\mathcal{W}}$: focal direction of $\hat{M}$ with multip m.

(2) $d\pi(\hat{\xi}) = \xi \in \mathcal{W}$: focal direction of M with multip m.

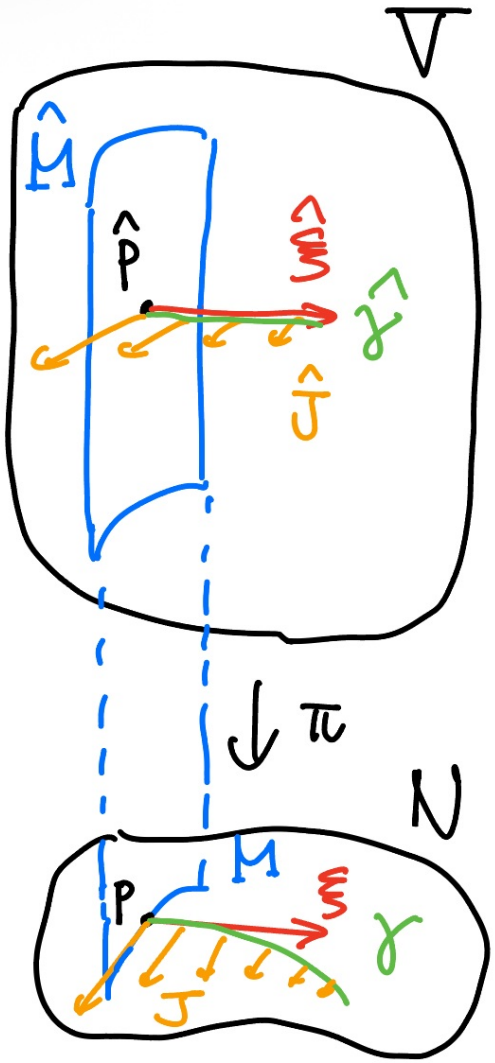
Corollary

The following are equivalent for $\lambda \in \mathbb{R}\backslash\{0\}$:

(1) λ is an eigenvalue of $A^{\hat{M}}_{\hat{\xi}}$ with multip m.

(2) $\frac{1}{\lambda}\hat{\xi} \in \hat{\mathcal{W}}$: focal direction of $\hat{M}$ with multip m

(3) $\frac{1}{\lambda}\xi \in \mathcal{W}$: focal direction of M with multip m



Sec. 4 - Focal points and Fredholm properties (3/3)

Proposition (M. 2025)

$\pi : (V, D) \to (N, \nabla^N)$: affine submer with h-dist $\mathcal{H}$ s.t. $\mathcal{A}|_{\mathcal{H}\times\mathcal{H}}$ is alt.
M: submanifold of N with t-bundle $\mathcal{W}$. Suppose: N has finite dim.
$\Rightarrow \hat{M} := \pi^{-1}(M)$: Fredholm submanifold of Fredholm index 0.
(i.e. The map $\hat{\mathcal{W}} \to V, (\hat{p}, \hat{\xi}) \mapsto \hat{p} + \hat{\xi}$ is Fredholm and has index 0.)

Theorem (M.: preprint)

G/K: reductive homg sp. with decmop $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$
$\Phi_{G/K} : V_{\mathfrak{g}} \to G \to G/K$: the parallel transport map.
M: submanifold of G/K with transversal bundle $\mathcal{W}$.
$\Rightarrow$ the shape op $A^{\hat{M}}_{\hat{\xi}}$ of $\hat{M} = \Phi^{-1}_{G/K}(M)$ is a compact op ($\forall \hat{\xi} \in \hat{\mathcal{W}}$).

Remark

The above theorem generalizes the known results when
- G/K: Riemannian sym sp of compact type [Terng-Thorbergsson]
- G/K: Riemannian sym sp of non-compact type [Koike]

Sec. 5 - Regularized mean curvatures (1/2)

H: (separable) Hilbertable sp over $\mathbb{R}$, $A : H \to H$: compact operator.
$A^{\mathbb{C}} : H^{\mathbb{C}} \to H^{\mathbb{C}}$: complexification , Λ: the set of all eigenvalues of $A^{\mathbb{C}}$
Note that $\lambda \in \Lambda \Rightarrow \bar{\lambda} \in \Lambda$ (because $A^{\mathbb{C}}(v) = \lambda v \Leftrightarrow A^{\mathbb{C}}(\bar{v}) = \bar{\lambda}\bar{v}$)
$\Lambda_{>0} := \{\mathrm{Re}\,\lambda \mid \lambda \in \Lambda,\ \mathrm{Re}\,\lambda > 0\}$ and $\Lambda_{<0} := \{\mathrm{Re}\,\lambda \mid \lambda \in \Lambda,\ \mathrm{Re}\,\lambda < 0\}$.
Write $\Lambda_{>0} = \{\mu_k\}_{k=1}^{\infty}$ and $\Lambda_{<0} = \{\nu_k\}_{k=1}^{\infty}$ (counted with multiplicities) s.t.

$$\nu_1 \leq \nu_2 \leq \cdots < 0 < \cdots \leq \mu_2 \leq \mu_1$$

Definition (M.: preprint)

A compact operator $A : H \to H$ is called **regularizable** if the series

$$\sum_{k=1}^{\infty}(\mu_k + \nu_k)$$

converges. Then its value is called the **regularized trace** of A and written as $\mathrm{tr}_r\, A$. We say that A is **regularizable** if $\mathrm{tr}_r\, A$ exists.

Remark

When A: self-adjoint, the above definition coincides with that of Heintze-Liu-Olmos, (except that we do not assume $\mathrm{tr}\, A^2 < \infty$).

Sec. 5 - Regularized mean curvatures (2/2)

Definition (M.: preprint)

M: submanifold of a Hilbertable space V with transversal bundle $\mathcal{W}$.
Suppose: the shape op $A_\xi : T_pM \to T_pM$ is compact for any $\xi \in \mathcal{W}$.
M is **regularizable**
$\underset{\text{def}}{\iff}$ A_ξ is regularizable for any $\xi \in \mathcal{W}$.
$\implies \operatorname{tr}_r A_\xi$ is called the **regularized mean curvatures** in direction ξ.

M is **minimal**
$\underset{\text{def}}{\iff}$ M is regularizable and $\operatorname{tr}_r A_\xi = 0$ for any $\xi \in \mathcal{W}$.

Question

G/K: reductive homg sp. with decmop $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$
$\Phi_{G/K} : V_{\mathfrak{g}} \to G \to G/K$: the parallel transport map.
M: submanifold of G/K with transversal bundle $\mathcal{W}$.
$\Rightarrow$ Is $\hat{M} = \Phi_{G/K}^{-1}(M)$ regularizable? Does $\operatorname{tr}_r A^{\hat{M}}{}_{\hat{\xi}} = \operatorname{tr} A^{M}{}_{\xi}$ hold?

Sec. 6 - Weakly reflective submanifolds (1/3)

Def (Ikawa-Sakai-Tasaki 2009)

M: submanifold of a Riemannian manifold (N, g).

M is **weakly reflective**

$\underset{\text{def}}{\Leftrightarrow}$ $\forall (p, \xi) \in T^{\perp}M$, $\exists \nu : N \to N$: isometry s.t.

$$\nu(p) = p, \quad \nu(M) = M, \quad d\nu(\xi) = -\xi$$

Prop (Ikawa-Sakai-Tasaki 2009)

weakly reflective $\Rightarrow$ austere $\Rightarrow$ minimal.

Here, **austere** $\underset{\text{def}}{\Leftrightarrow}$ For $\forall (p, \xi) \in T^{\perp}M$, the set of all eigenvalues with multiplicities of the shape operator A_{ξ} is invariant under $\times(-1)$.

Example (Podesta 1997, Ikawa-Sakai-Tasaki 2009)

Any singular orbit of a cohomogeneity one action is a weakly reflective submanifold.

Sec. 6 - Weakly reflective submanifolds (2/3)

Def (M.: 2025)

M: submanifold of affine manifold (N, ∇) with transversal bundle $\mathcal{W}$

M is **weakly reflective**

$\underset{\text{def}}{\Leftrightarrow}$ $\forall (p, \xi) \in \mathcal{W}$, $\exists \nu : N \to N$: affine transf. s.t.

$$\nu(p) = p, \quad \nu(M) = M, \quad d\nu(\xi) = -\xi, \quad d\nu(\mathcal{W}) = \mathcal{W}.$$

Prop (M. 2025)

(1) "weakly reflective $\Rightarrow$ austere " always holds.

(2) "austere $\Rightarrow$ minimal" holds in the following cases

- (a) (N, ∇): finite dim
- (b) $(N, \nabla) = (V, D)$ and the shape operators of M are compact.

Thm (M.: 2025)

G/K: reductive homogeneous space with decomp $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$,

$\Longrightarrow$ Each fiber of $\Phi_{G/K} : V_{\mathfrak{g}} \to G/K$ is a **weakly reflective** (and thus **minimal**) submfd of $(V_{\mathfrak{g}}, D)$ whose t-bdl is the restriction of $\mathcal{H}$.

Sec. 6 - Weakly reflective submanifolds (3/3)

Thm (M.: 2025)

(N, ∇): affine symmetric space. Fix $p \in N$

$$\begin{array}{ccc} \tilde{G}\text{: affine transf. group} & \supset & G\text{: identity component} \\ \cup & & \cup \\ \tilde{K}\text{: isotropy subgp at } p & \supset & K := \tilde{K} \cap G \end{array}$$

$\Phi_{G/K} : V_{\mathfrak{g}} \to G \to G/K = N$: parallel transport map
M: submanifold of (N, ∇) with transversal bundle $\mathcal{W}$.
$\Longrightarrow$ The following conditions are equivalent:

(1) M is weakly reflective

(2) $\Phi_{G/K}^{-1}(M)$ is weakly reflective s.t. every ν can be chosen from

$$\{g \in H^1([0,1], \tilde{G}) \mid g(1) \in \tilde{K}\}.$$

Remark

The above theorem generalizes my previous result
when N is an irreducible Riemannian symmetric space of cpt type.

Future problems

Problem 1

Affine isoparametric submanifolds in Hilbertable space?

$\rightsquigarrow$ **PTM**

Affine equifocal submanifolds in affine symmetric space?

Problem 2

Affine immersions into Euclidean space

$\rightsquigarrow$ **generalize**

Affine immersions into Hilbert space?

$\rightsquigarrow$ **PTM**

Affine immersion into affine symmetric space ??

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Thank you for your attention!