Invariant measures on reductive real spherical homogeneous spaces

笹木 集夢 (Atsumu SASAKI)

東海大学理学部数学科

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Email: atsumu@tokai.ac.jp (domain changed)

Introduction

Calculate the following double integral:

$$\iint_D xy\,dxdy \quad \text{where } D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}.$$

Now, we consider the diffeomorphism

$$\begin{array}{ccccc} \Phi : (0,2\pi) & \times & (0,1) & \simeq & D - \{(x,0) : 0 \leq x \leq 1\} \subset D. \\ \cup & & & & \cup \\ (\theta, r) & & & \mapsto & (x,y) = (r\cos\theta, r\sin\theta) \end{array}$$

(open)

$$J_\Phi = \frac{\partial(x,y)}{\partial(\theta,r)} = r : \text{Jacobian of } \Phi.$$

Then the double integral can be calculated by

$$\iint_D xy\,dxdy = \int_0^{2\pi}\int_0^1 r^2\cos\theta\sin\theta \cdot J_\Phi\,drd\theta.$$

Introduction

Point.

- Any non-zero element of D is expressed by polar coordinates.

$$D - \{0\} = \Phi([0, 2\pi), (0, 1)) : \text{polar decomposition}.$$

- Measure $dxdy$ on D is characterized

  by measures $d\theta$ on $(0, 2\pi)$ and dr on $(0, 1)$.

Observation.

- Poincaré disk

$$D^\circ := \{\zeta \in \mathbb{C} : |\zeta| < 1\} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$$

  is an open set in D. Then,

$$\int_D xy\, dxdy = \int_{D^\circ} xy\, dxdy.$$

- $D^\circ \simeq SU(1,1)/S(U(1) \times U(1)) \simeq SL(2, \mathbb{R})/SO(2) =: G/K$.

Introduction

Poincaré disk $D^\circ := \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ is an open set in D.

- $\mathcal{H}_+ := \{z \in \mathbb{C} : \mathrm{Im}\, z > 0\}$: complex upper-half plane.
- $G = SL(2,\mathbb{R}) \supset K = SO(2)$: maximal compact subgroup.
- $G \curvearrowright \mathcal{H}_+$ by Möbius transform $\Rightarrow \mathcal{H}_+ \simeq G/K$.
- Cayley transform

$$\varphi : \mathcal{H}_+ \simeq D^\circ,\ z \mapsto \frac{z-\sqrt{-1}}{z+\sqrt{-1}} : \text{biholomorphic.}$$

Let $G = SL(2,\mathbb{R}) \curvearrowright D^\circ$ by

$$g \cdot \zeta = \varphi(g \cdot \varphi^{-1}(\zeta)).$$

Then, $K \curvearrowright D^\circ$ is written as

$$k_\theta \cdot \zeta = e^{-2\sqrt{-1}\theta}\zeta : \text{rotation for } k_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in K$$

Introduction

Poincaré disk $D^\circ := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$.

$$\begin{array}{ccccccc} \Phi : K \times (0,1) & \xrightarrow{\sim} & D^\circ - \{0\} & \subset & D^\circ \simeq & G/K \\ \cup & & \cup & \text{open} & & \cup \\ (k, r) & \mapsto & k \cdot r & \mapsto & & (k \exp(r I_{1,1}))K \end{array}$$

- dg_K : invariant measure on G/K (unique up to scalars).
- dk : (normalized) Haar measure on K.
- dr : Lebesgue measure on $\mathfrak{a}_+ := (0, 1) \subset \mathbb{R}$.
- $(J_\Phi)_{(k,r)}$ is the Jacobian of Φ at $(k, r) \in K \times \mathfrak{a}_+$.

$$\int_{G/K} f(gK)\, dg_K = \int_K \int_{\mathfrak{a}_+} f(k \exp(r I_{1,1})K)(J_\Phi)_{(k,r)}\, dr dk.$$

Invariant measure on reductive symmetric spaces

Let G/H be a reductive symmetric space.

- σ : involution of G satisfying $(G^\sigma)_0 \subset H \subset G^\sigma$.
- θ : Cartan involution of G with $\sigma\theta = \theta\sigma \Rightarrow H$ is θ-stable.
- $K := G^\theta$ is a maximal compact subgroup of G.
- $\mathfrak{a} \subset \mathfrak{g}^{-\theta,-\sigma}$: maximal abelian subspace.
- $\Sigma \equiv \Sigma(\mathfrak{g}, \mathfrak{a})$: restricted root system $\supset \Sigma^+$: positive system.
- $\mathfrak{a}_+ \subset \mathfrak{a}$: positive Weyl chamber related to Σ^+ and $\overline{\mathfrak{a}}_+$: closure.

Cartan decomposition for symmetric space G/H

(1) Multiplication map $\Phi : K \times \exp \overline{\mathfrak{a}}_+ \times H \to G$ is surjective.

(2) For $k \in K,\ X \in \overline{\mathfrak{a}}_+$ and $h \in H$

$$\Phi^{-1}(\{k(\exp X)h\}) = \{(km^{-1}, X, mh) : m \in Z_{K\cap H}(X)\}.$$

In particular, $K\backslash G/H \simeq \overline{\mathfrak{a}}_+$.

(cf. Flensted-Jensen ('78), Rossmann ('79))

Invariant measure on reductive symmetric spaces

Let G/H be a reductive symmetric space.

- $\mathfrak{g}(\mathfrak{a}, \lambda)$: restricted root space with restricted root $\lambda \in \Sigma$.
- $m_{\pm}(\lambda) := \dim(\mathfrak{g}(\mathfrak{a}, \lambda) \cap \mathfrak{g}^{\pm\sigma\theta})$ $(\lambda \in \Sigma)$.
- dg_H : invariant measure on G/H (unique up to scalars).
- dk : Haar measure on K and dY : Lebesgue measure on $\mathfrak{a}$.

Invariant measure on symmetric G/H

We define a function $\delta : \mathfrak{a} \to \mathbb{R}$ by

$$\delta(Y) := \prod_{\lambda \in \Sigma^+} \left| (\sinh \lambda(Y))^{m_+(\lambda)} (\cosh \lambda(Y))^{m_-(\lambda)} \right| \quad (Y \in \mathfrak{a}).$$

Then, we get an integral formula related to $G = K(\exp \overline{\mathfrak{a}}_+)H$:

$$\int_{G/H} f(gH)\, dg_H = \int_K \int_{\mathfrak{a}_+} f(k(\exp Y)H)\delta(Y)\, dY\, dk.$$

Invariant measure on reductive symmetric spaces

Let G/H be a reductive symmetric space.

Key ingredient (due to [Schlichtkrull '84]):

(1) Cartan decomposition for G/H implies

$$\Phi : K \times \exp \mathfrak{a}_+ \to G/H,\ (k, a) \mapsto kaH$$

is a diffeomorphism onto an open subset of G/H.

(2) Linear map $A_{\exp Y} : \mathfrak{k} \times \mathfrak{a} \to \mathfrak{g}/\mathfrak{h}$ is given by

$$A_{\exp Y}(X_0, Y_0) = \mathrm{Ad}(\exp(-Y))X_0 + Y_0 + \mathfrak{h}.$$

$\Rightarrow$ Jacobian $(J_\Phi)_{(k, \exp Y)}$ equals $|\det A_{\exp Y}|$

(3) We take $X_\lambda \in \mathfrak{g}(\mathfrak{a}, \lambda) \cap \mathfrak{g}^{\varepsilon\sigma\theta}$ for $\lambda \in \Sigma^+$ and $\varepsilon \in \{\pm 1\}$.

We define $X_\lambda^\pm := X_\lambda \pm \theta(X_\lambda) \in \mathfrak{g}^{\pm\theta} \cap \mathfrak{g}^{\varepsilon\sigma}$.

$\Rightarrow A_{\exp Y}(X_\lambda^+, 0) = \cosh \lambda(Y) X_\lambda^+ - \sinh \lambda(Y) X_\lambda^- + \mathfrak{h}.$ $\square$

Object of this talk is

Reductive real spherical homogeneous spaces

Background:

T. Kobayashi proposed the study of real spherical varieties.

— He recognized that real spherical varieties control the multiplicities of representations.

$\rightsquigarrow$ Finite multiplicity theorem (Kobayashi–T. Oshima '13).

Conjecture (Kobayashi ('95)).

G/H : reductive real spherical homogeneous space

$\Rightarrow$ There exists $\mathfrak{b} \subset \mathfrak{g}^{-\theta}$: abelian subspace s.t.

$$G = K(\exp \mathfrak{b})H.$$

Once one can find an analogue of a Cartan decomposition,

$\Rightarrow$ We can expect an integral formula (cf. Kobayashi ('97)).

Aim of this talk

G/H : reductive non-symmetric and real spherical.

In this talk, we explain:

Aim 1. How to find an abelian subspace $\mathfrak{b}$ in $\mathfrak{g}^{-\theta}$ explicitly s.t.

Cartan decomposition for G/H

$$G = K(\exp \mathfrak{b})H.$$

Aim 2. Integral formula related to Cartan decomposition for G/H; How to find a function $\delta : \mathfrak{b} \to \mathbb{R}$ explicitly s.t.

Integral formula

$$\int_{G/H} f(gH)\, dg_H = \int_K \int_{\mathfrak{b}_+} f(k(\exp Y)H)\delta(Y)\, dY\, dk.$$

for some suitable open subset $\mathfrak{b}_+$ in $\mathfrak{b}$.

Reductive real spherical homogeneous space

- G : non-cpt. reductive real algebraic group.
- H : real reductive algebraic subgroup of G.

Definition (Kobayashi ('95)).

G/H is real spherical $\overset{\text{def}}{\Leftrightarrow}$ $\underset{\text{min. parabolic}}{\exists P}$ $\curvearrowright G/H$ has an open orbit.

- Reductive symmetric spaces are real spherical.
- Classification:
  - Krämer ('79), Brion ('87) — complex spherical.
  - Kobayashi–Matsuki ('14)
    — $(G \times H)/\Delta(H)$ with (G, H) : symmetric paris.
  - Knop–Krötz–Pecher–Schlichtkrull ('19) ([KKPS '19] for short)
    — all real spherical G/H when G is real semisimple.

Reductive real spherical homogeneous space

Classification of non-symmetric reductive real spherical G/H [KKPS]

$\rightsquigarrow$ We divide non-symmetric reductive real spherical G/H into:

(1) We say that G/H is of rank-one type

$\overset{\text{def}}{\Leftrightarrow} (\mathfrak{g}, \mathfrak{h}) \simeq (\mathfrak{so}(7,\mathbb{C}), \mathfrak{g}_2(\mathbb{C})), (\mathfrak{g}_2(\mathbb{C}), \mathfrak{sl}(3,\mathbb{C})),$
$(\mathfrak{so}(4,3), \mathfrak{g}_{2(2)}), (\mathfrak{g}_{2(2)}, \mathfrak{sl}(3,\mathbb{R})), (\mathfrak{g}_{2(2)}, \mathfrak{su}(2,1)).$

(2) G/H is neither symmetric nor of rank-one type.

Fact. G/H is neither symmetric nor of rank-one type.

$\Rightarrow H \subsetneq {}^{\exists}L \subsetneq G$: such that G/L is a symmetric space.

Example. $G/H = SO^*(10)/Spin_0(2,5)$ is

- non-symmetric and reductive real spherical.
- $L = SO^*(2) \times SO^*(8)$ satisfies Fact.

Cartan decomposition for real spherical space

G/H : reductive real spherical homogeneous space.

Conjecture (Kobayashi ('95)).

G/H : reductive real spherical homogeneous space

$\Rightarrow$ There exists $\mathfrak{b} \subset \mathfrak{g}^{-\theta}$: abelian subspace s.t.

$$G = K(\exp \mathfrak{b})H.$$

- S. ('10, '15, etc): reductive complex spherical case

  — case-by-case argument according to the classification and find $\mathfrak{b}$ explicitly in connection with visible actions.

  Example. $SO(2n+1,\mathbb{C})/GL(n,\mathbb{C})$, $E_6(\mathbb{C})/Spin(10,\mathbb{C})$.
- Tanaka ('22) proves the existence of $\mathfrak{b}$ for any real spherical

  — by the induction on the dimension of G.

Rank-one type non-symmetric real spherical

(1) G/H is of rank-one type

$$\overset{\text{def}}{\Leftrightarrow} (\mathfrak{g},\mathfrak{h}) \simeq (\mathfrak{so}(7,\mathbb{C}),\mathfrak{g}_2(\mathbb{C})),(\mathfrak{g}_2(\mathbb{C}),\mathfrak{sl}(3,\mathbb{C})),$$
$$(\mathfrak{so}(4,3),\mathfrak{g}_{2(2)}),(\mathfrak{g}_{2(2)},\mathfrak{sl}(3,\mathbb{R})),(\mathfrak{g}_{2(2)},\mathfrak{su}(2,1)).$$

Theorem A-1 (Progr. Math. '25).

There exists a unique $\mathfrak{b} \subset \mathfrak{p} \cap \mathfrak{h}^\perp$: 1-dim abelian subspace such that

$$K\backslash G/H \simeq \overline{\mathfrak{b}}_+.$$

Here, for $\mathfrak{b} = \mathbb{R}E$ we write $\mathfrak{b}_+ = \mathbb{R}_+E$ and $\overline{\mathfrak{b}}_+ = \mathbb{R}_{\geq 0}E$.

Idea of Proof.

- $K \cap H \curvearrowright \underset{\text{unit sphere}}{S(\mathfrak{p} \cap \mathfrak{h}^\perp)}$ is transitive $\rightsquigarrow (K \cap H)\backslash(\mathfrak{p} \cap \mathfrak{h}^\perp) \simeq \mathbb{R}_{\geq 0}$.
- $K \times (\mathfrak{p} \cap \mathfrak{h}^\perp) \times (\mathfrak{p} \cap \mathfrak{h}) \underset{\text{diffeo.}}{\simeq} G,\ (k,Y,X) \mapsto k \exp Y \exp X.$ □

(cf. Mostow, Loos, Kobayashi ('89))

General non-symmetric real spherical G/H

(2) G/H is neither symmetric nor of rank-one type.

$$H \subsetneq {}^{\exists}L \subsetneq G \text{ : such that } G/L \text{ is a symmetric space.}$$

Theorem A-2.

Let a non-symmetric and reductive real spherical G/H.

We can construct $\mathfrak{b} \subset \mathfrak{g}^{-\theta}$ explicitly such that

$$G = K(\exp \mathfrak{b})H.$$

In particular,

$$\mathfrak{b} \to K\backslash G/H,\ Y \mapsto K(\exp Y)H \text{ is surjective.}$$

Cartan decomposition for $SO^*(10)/Spin_0(5,2)$

$$G/H = SO^*(10)/Spin_0(5,2)$$
$$\Rightarrow K = U(5) \text{ and } H \subsetneq L := SO^*(8) \times SO^*(2) \subsetneq G.$$

(Step 1) $G/L = SO^*(10)/(SO^*(8) \times SO^*(2))$: symmetric space

$$\Rightarrow G = K(\exp \overline{\mathfrak{a}}_+)L \text{ with } \dim \mathfrak{a} = 1.$$ ▸ symmetric

(Step 2) $M = SO^*(6) \times U(1)$ and $M_H = M \cap H = SU(2,1)$.

$$\text{Key lemma (2)} \Rightarrow \frac{SO^*(6) \times U(1)}{SU(2,1)} \simeq \frac{SO^*(8) \times SO^*(2)}{Spin_0(5,2)}.$$

$M/M_H = \dfrac{SO^*(6) \times U(1)}{SU(2,1)}$ is real spherical but NOT symmetric.

We next consider a Cartan decomposition for $M/M_H \rightsquigarrow$ (Step $1'$).

Cartan decomposition for $SO^*(10)/Spin_0(5,2)$

$$(G', H') := (M, M_H) = (SO^*(6) \times U(1), SU(2,1)).$$
$$\Rightarrow M_K = K' = U(3) \times U(1) \text{ and } H' \subsetneq L' := U(2,1) \times U(1) \subsetneq G'.$$

(Step $1'$) $G'/L' \simeq SO^*(6)/U(2,1)$: symmetric space
$$\Rightarrow G' = K'(\exp \overline{\mathfrak{a}}'_+)L' \text{ with } \dim \mathfrak{a}' = 1.$$

(Step $2'$) $M' = U(2) \times U(1)$ and $M'_{H'} = SU(2)$.
$$(U(2) \times U(1))/SU(2) \simeq (U(2,1) \times U(1))/SU(2,1).$$
Hence, $M'_{K'} = M'$ and then $M' = M'_{K'}M'_{H'}$.

(Step $3'$) $M'_{K'} \backslash M'/M'_{H'} \simeq M'_{K'} \backslash L'/H' \Rightarrow L' = M'_{K'}H'$.

(Step $4'$) Using Herringbone stitch method (Kobayashi ('02))
$$G' = K'(\exp \overline{\mathfrak{a}}'_+)M'_{K'}H' = K'M'_{K'}(\exp \overline{\mathfrak{a}}'_+)H' = K'(\exp \overline{\mathfrak{a}}_+)H'.$$

Cartan decomposition for $SO^*(10)/Spin_0(5,2)$

(Step $3''$) $K'\backslash G'/H' = M_K\backslash M/M_H \simeq M_K\backslash L/H$

(Step $4'$) $\Rightarrow L = M_K(\exp \mathfrak{a}')H$

(Step $4''$) Using Herringbone stitch method (Kobayashi ('02))

$$\begin{aligned} G &= K(\exp \overline{\mathfrak{a}}_+)L = K(\exp \overline{\mathfrak{a}}_+)M_K(\exp \overline{\mathfrak{a}}'_+)H \\ &= KM_K(\exp \overline{\mathfrak{a}}_+)(\exp \overline{\mathfrak{a}}'_+)H \\ &= K(\exp(\overline{\mathfrak{a}}_+ + \overline{\mathfrak{a}}'_+))H. \end{aligned}$$

We take $\overline{\mathfrak{b}}_+ := \overline{\mathfrak{a}}_+ + \overline{\mathfrak{a}}'_+ \subset \mathfrak{b} = \mathfrak{a} + \mathfrak{a}'$: 2-dim. abelian

Then, we get $G = K(\exp \overline{\mathfrak{b}}_+)H$.

Herringbone stitch method for G/H

$$G = K(\exp \overline{\mathfrak{a}_0^+})L = K(\exp \overline{\mathfrak{a}_0^+})K_M(\exp \overline{(\mathfrak{a}_0')^+})H$$
$$= K K_M(\exp \overline{\mathfrak{a}_0^+})(\exp \overline{(\mathfrak{a}_0')^+})H = K \exp(\overline{\mathfrak{a}_0^+} + \overline{(\mathfrak{a}_0')^+})H.$$

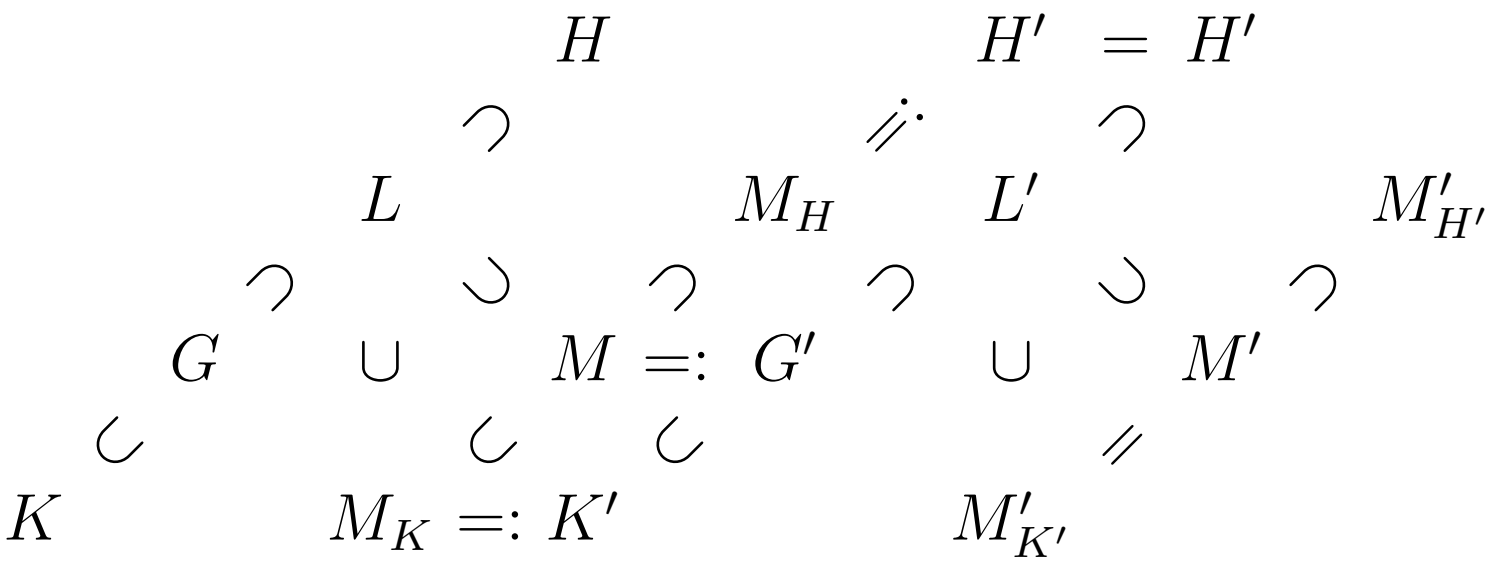


Figure: Herringbone stitch method for $SO^*(10)/Spin(2,5)$

Invariant measure on reductive real spherical space

G/H : reductive real spherical.

Theorem A. We can find $\mathfrak{b} \subset \mathfrak{g}^{-\theta}$: abelian subspace explicitly s.t.

$$G = K(\exp \mathfrak{b})H.$$

- dg_H : invariant measure (which exists unique up to constant).
- dk : (normalized) Haar measure on the compact group K.

  dY : Lebesgue measure on the abelian $\mathfrak{b}$.

Theorem B.

One can find a function $\delta : \mathfrak{b} \to \mathbb{R}$ explicitly such that

$$\int_{G/H} f(gH)\, dg_H = \int_K \int_{\mathfrak{b}} f(k(\exp Y)H)\delta(Y)\, dY\, dk$$

for $f \in C_c(G/H)$ (up to constant).

Invariant measure on rank-one type

(1) G/H is of rank-one type $\overset{\text{def}}{\Leftrightarrow}$ $(\mathfrak{g},\mathfrak{h})$

$\simeq$ (1) $(\mathfrak{so}(7,\mathbb{C}),\mathfrak{g}_2(\mathbb{C}))$, (2) $(\mathfrak{g}_2(\mathbb{C}),\mathfrak{sl}(3,\mathbb{C}))$,

(3) $(\mathfrak{so}(4,3),\mathfrak{g}_{2(2)})$, (4) $(\mathfrak{g}_{2(2)},\mathfrak{sl}(3,\mathbb{R}))$, (5) $(\mathfrak{g}_{2(2)},\mathfrak{su}(2,1))$.

Theorem A-1 (Recall): $K\backslash G/H \simeq \overline{\mathfrak{b}}_+ = \mathbb{R}_{\geq 0}E$.

- $\varepsilon \in \mathfrak{b}^*$: dual of E.

- $0 \neq \lambda \in \mathfrak{b}^*$ is a restricted root of $(\mathfrak{g},\mathfrak{h})$

$\overset{\text{def}}{\Leftrightarrow} 0 \neq {}^{\exists}X \in \mathfrak{g}$ such that $[E,X] = \lambda(E)X$.

- Σ : set of all restricted roots of $(\mathfrak{g},\mathfrak{h})$.

Invariant measure on rank-one type

(1) G/H is of rank-one type $\overset{\text{def}}{\Leftrightarrow}$ $(\mathfrak{g},\mathfrak{h})$

$\simeq$ (1) $(\mathfrak{so}(7,\mathbb{C}),\mathfrak{g}_2(\mathbb{C}))$, (2) $(\mathfrak{g}_2(\mathbb{C}),\mathfrak{sl}(3,\mathbb{C}))$,

(3) $(\mathfrak{so}(4,3),\mathfrak{g}_{2(2)})$, (4) $(\mathfrak{g}_{2(2)},\mathfrak{sl}(3,\mathbb{R}))$, (5) $(\mathfrak{g}_{2(2)},\mathfrak{su}(2,1))$.

Theorem B-1 (Progr. Math. '25).

$(\mathfrak{g},\mathfrak{h})$	Σ	$\delta(Y)$
(1)	$\{\pm\varepsilon,\pm2\varepsilon\}$	$\lvert\sinh 3\varepsilon(Y)\rvert^6$
(2)	$\{\pm\varepsilon,\pm2\varepsilon,\pm3\varepsilon\}$	$\lvert\sinh 4\varepsilon(Y)\rvert^5$
(3)	$\{\pm\varepsilon,\pm2\varepsilon\}$	$\lvert\sinh 3\varepsilon(Y)\rvert^3$
(4)	$\{\pm\varepsilon,\pm2\varepsilon,\pm3\varepsilon\}$	$\lvert\sinh 4\varepsilon(Y)\rvert^2\lvert\cosh 2\varepsilon(Y)\rvert$
(5)	$\{\pm\varepsilon,\pm2\varepsilon,\pm3\varepsilon\}$	$\lvert\sinh 4\varepsilon(Y)\rvert^2\lvert\sinh 2\varepsilon(Y)\rvert$

Invariant measure on $SO^*(10)/Spin_0(5,2)$

(2) G/H is neither symmetric nor of rank-one type.

$$G/H = SO^*(10)/Spin_0(5,2) \Rightarrow \mathfrak{b} = \mathfrak{a} + \mathfrak{a}'$$

Therefore, we define $\delta : \mathfrak{b} \to \mathbb{R}$ by

$$\delta(Y_1, Y_2) := |(\sinh \varepsilon_1(Y_1))^6 (\cosh \varepsilon_1(Y_1))^6 \sinh 2\varepsilon_1(Y_1)| \cdot |(\cosh \varepsilon_2(Y_2))^4 \sinh 2\varepsilon_2(Y_2)| \quad (Y_1 \in \mathfrak{a}, Y_2 \in \mathfrak{a}').$$

Then, we get an integral formula

$$\int_{G/H} f(gH)\, dg_H = \int_K \int_{\mathfrak{b}_+} f(k(\exp(Y_1 + Y_2))H)\, \delta(Y_1, Y_2)\, dY_1 dY_2 dk.$$

where $\mathfrak{b}_+ = \mathfrak{a}_+ + \mathfrak{a}'_+$.