Subriemannian geodesic flows of the seven-dimensional sphere

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§1. Introduction

Riemannian geodesics and Hamiltonian flow

Riemannian geodesics

Q: smooth manifold, $\langle \cdot, \cdot \rangle$: Riemannian metric.
A curve $c : [\alpha, \beta] \to Q$ is called a geodesic if it solves the variational problem with fixed end points:

$$\delta \left(\int_\alpha^\beta \sqrt{\langle \dot{c}(t), \dot{c}(t) \rangle_{c(t)}} \mathrm{d}t \right) = 0$$
$$\iff \ddot{c}_i + \sum_{j,k} \Gamma^i_{jk} \dot{c}_j \dot{c}_k = 0.$$

The geodesics induce a vector field on the tangent bundle TQ, called the *geodesic spray*. Identifying the tangent and the cotangent bundles through the Riemannian metric as

$$TQ \supset T_xQ \ni v_x \mapsto \langle v_x, \cdot \rangle_x \in T^*_xQ \subset T^*Q,$$

we have a vector field on T^*Q which is a Hamiltonian vector field w.r.t. the canonical symplectic structure.

Hamiltonian systems on symplectic manifold

(M, ω): symplectic manifold, i.e. $\dim M = 2n$;
ω: non-degenerate closed two-form, $\underbrace{\omega \wedge \cdots \wedge \omega}_{n} \neq 0$, $\mathrm{d}\omega = 0$.

Hamiltonian vector field Ξ_H for the Hamiltonian $H \in \mathcal{C}^\infty(M)$ w.r.t. ω is given by $\iota_{\Xi_H}\omega = -\mathrm{d}H$.
Poisson bracket $\{\cdot, \cdot\}$ is defined through
$\{F, G\} = -\Xi_F(G) = -\omega(\Xi_F, \Xi_G)$, where $F, G \in \mathcal{C}^\infty(M)$.

Properties

- $\{F, G\} = -\{G, F\}$,
- Leibniz rule; $\{FG, H\} = \{F, H\} G + \{G, H\} F$,
- Jacobi identity;
  $\{\{F, G\}, H\} + \{\{H, F\}, G\} + \{\{G, H\}, F\} = 0$,

for arbitrary functions $F, G, H \in \mathcal{C}^\infty(M)$.

Hamiltonian vector field on cotangent bundle

On the cotangent bundle $M = T^*Q$ of a smooth manifold Q, we have the canonical symplectic structure.

The canonical (Liouville) one-form θ on T^*Q is given, at $\alpha_y \in T_yQ \subset TQ$, by $\theta(\alpha_y) \cdot X_{\alpha_y} = \alpha_y \cdot (\mathrm{d}\pi)_{\alpha_y} X_{\alpha_y}$, where $X_{\alpha_y} \in T_y(TQ)$ and $\pi : TQ \to Q$: canonical projection.

The canonical symplectic structure is given by $\omega = \mathrm{d}\theta$.

The Riemannian metric $\langle \cdot, \cdot \rangle$ defines a fibre metric $\langle \cdot, \cdot \rangle'$ on T^*Q and set $H(\alpha_y) := \frac{1}{2}\langle \alpha_y, \alpha_y \rangle'_y$, $\alpha_y \in T^*_yQ \subset T^*Q$.

The Hamiltonian vector field Ξ_H on T^*Q for this Hamiltonian H corresponds to the geodesic spray on TQ.
The corresponding flow is called the *geodesic flow*.

In local coordinates

In local coordinates $(q_1, \ldots, q_n)$ on an open set $U \subset Q$, consider the local trivialization $T^*Q \supset \pi^{-1}(U) \ni \sum_{i=1}^{n} p_i \, (\mathrm{d}q_i)_q \mapsto (q_1, \ldots, q_n; p_1, \ldots, p_n) \in U \times \mathbb{R}^n$. Then, the canonical symplectic structure is given as $\omega = \sum_{i=1}^{n} \mathrm{d}p_1 \wedge \mathrm{d}q_i$.

Then, the Hamilton's equation is written as

$$\begin{cases} \dot{q}_i &= \dfrac{\partial H}{\partial p_i}, \\ \dot{p}_i &= -\dfrac{\partial H}{\partial q_i}, \end{cases}$$

$i = 1, \cdots, n$.

Liouville complete integrability

A Hamiltonian system (M, ω, H) on a $2n$-dim. sympl mfd is *(Liouville) completely integrable* if there exist n functionally independent $f_1, \ldots, f_{n-1}, f_n \equiv H \in \mathcal{C}^\infty(M)$ s.t. $\{f_i, f_j\} = 0$ for any $i, j = 1, \ldots, n$. Set $F = (f_1, \ldots, f_n) : M \to \mathbb{R}^n$.

Theorem (Liouville-Arnol'd)

Take $x \in M$, for which $F(x)$ is a regular value of F and $F^{-1}(F(x))$ is connected and compact. Then, $\exists U \subset \mathbb{R}^n$: neighbourhood of $F(x)$, $\exists \Psi : F^{-1}(U) \to U \times \mathbb{T}^n$: diffeo, s.t. $\Psi^* \left(\sum_{i=1}^n \mathrm{d}I_i \wedge \mathrm{d}\theta_i \right) = \omega|_{F^{-1}(U)}$,
where $(I_1, \ldots, I_n; \theta_1, \ldots, \theta_n)$: coordinates on $U \times \mathbb{T}^n$, $\theta_i + 2\pi = \theta_i$.
Further, the flow of the Hamiltonian vector field Ξ_H on $U \times \mathbb{T}^n \cong F^{-1}(U)$ is linear w.r.t. $(\theta_1, \ldots, \theta_n)$.

Subriemannian geodesic flow

Subriemannian manifolds

Q: smooth manifold. $\mathcal{D} \subset TQ$: tangential distribution (sub-bundle) of rank r.

Given a fibre metric $\langle \cdot, \cdot \rangle$ of $\mathcal{D}$, we call $(Q, \mathcal{D}, \langle \cdot, \cdot \rangle)$ a subriemannian manifold. We assume the bracket-generating condition:

$$\exists p \in \mathbb{N}_0 \text{ s.t. } \mathcal{D}^p = TQ,\ \mathcal{D}^{p-1} \neq TQ,$$

setting $\mathcal{D}^0 = \mathcal{D}$, $\mathcal{D}^{i+1} = \mathcal{D}^i + \left[\mathcal{D}^i, \mathcal{D}\right]$. Here, we identify the vector bundles as the sheaves of germs of sections.

Subriemannian geodesics 1

$(Q, \mathcal{D}, \langle \cdot, \cdot \rangle)$: a subriemannian manifold.
A curve $c : [\alpha, \beta] \to Q$ is a (subriemannian) geodesic if it solves the variational problem with fixed end points:

$$\delta \left(\int_\alpha^\beta \sqrt{\langle \dot{c}(t), \dot{c}(t) \rangle_{c(t)}} \mathrm{d}t \right) = 0, \quad \text{s.t. } \dot{c}(t) \in \mathcal{D}_{c(t)}, \ \forall t \in [\alpha, \beta].$$

To consider the Hamiltonian formalism for subriemannian geodesics, take the bundle homomorphism $\beta : T^*Q \to \mathcal{D}$ defined through

$$\alpha_y(v) = \langle \beta(\alpha_y), v \rangle, \quad \alpha_y \in T_y^*Q, v \in \mathcal{D}_y, y \in Q.$$

Then, we have the fibre metric (*cometric*) $\langle \cdot, \cdot \rangle' : T^*Q \times T^*Q \to \mathbb{R}$ defined through

$$\langle \alpha_y, \alpha_y' \rangle_y' = \langle \beta(\alpha_y), \beta(\alpha_y') \rangle, \quad \alpha_y, \alpha_y' \in T_yQ.$$

Subriemannian geodesics 2

Take the function H on T^*Q defined through

$$H(\alpha_y) := \frac{1}{2}\langle \alpha_y, \alpha_y \rangle'_y, \qquad \alpha_y \in T^*_y Q \subset T^*Q.$$

A solution to Hamilton's equation for the Hamiltonian system (T^*Q, ω, H) induces a subriemannian geodesic of $(Q, \mathcal{D}, \langle \cdot, \cdot \rangle)$, usually called a *normal geodesic*.

N.B. There are subriemannian geodesic which does not solve Hamilton's equation, which is called *abnormal geodesic*. Below, we focus on normal geodesics.

See R. Montgomery's book [Mo] for the details on subriemannian geometry.

§2. Riemannian geodesic flows on homogeneous spaces

Riemannian geodesic flows on spheres

On spheres $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ equipped with the induced metric, the geodesics are known to be (a part of) a great circle.

In the Hamiltonian formalism, the corresponding geodesic flow on $T^*\mathbb{S}^n$ is also known to be completely integrable.
Identifying $T^*\mathbb{S}^n$ with $T\mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$, the Poisson commuting constants of motion can be given as

$$f_j(x,y) = \sum_{k,\ell \geq n+2-j} (x_k y_\ell - x_\ell y_k)^2, \qquad j = 2, \ldots, n+1,$$

where $(x,y) = (x_1, \ldots, x_{n+1}; y_1, \ldots, y_{n+1}) \in \mathbb{R}^n \times \mathbb{R}^n$ with $x \in \mathbb{S}^n$, $x^{\mathrm{T}} y = 0$.

Method by Thimm 1

A. Thimm [Th] introduced a method to obtain first integrals for the geodesic flows on homogeneous spaces G/K, where G is equipped with a bi-invariant metric.

Key idea:

Suppose the Lie algebra $\mathfrak{g}$ of G has bi-invariant metric B. Given two *non-degenerate* subalgebras $\mathfrak{g}_0, \mathfrak{g}_1 \subset \mathfrak{g}$ s.t. $[\mathfrak{g}_0, \mathfrak{g}_1] \subset \mathfrak{g}_1$,

$$\{h_0 \circ \pi_0, h_1 \circ \pi_1\} = 0,$$

for the Lie-Poisson bracket, where $h_i \in \mathcal{C}^\infty(\mathfrak{g}_i)$: bi-invariant functions, and $\pi_i : \mathfrak{g} \to \mathfrak{g}_i$: orthogonal projection, $i = 0, 1$. The condition $[\mathfrak{g}_0, \mathfrak{g}_1] \subset \mathfrak{g}_1$ holds e.g. if $\mathfrak{g}_0 \subset \mathfrak{g}_1$.

Optimally, if there is a sequence of non-deg. subalgebras

$$\mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \cdots \subset \mathfrak{g}_k \subset \mathfrak{g},$$

one may obtain many first integrals.

Method by Thimm 2

Using the idea, Thimm [Th] showed the complete integrability of geodesic flows of the real and the complex Grassmannian manifolds.
In the case of sphere $\mathbb{S}^n = SO(n+1)/SO(n)$, one can use the sequence of subalgebras:
$\mathfrak{so}(2) \subset \mathfrak{so}(3) \subset \cdots \subset \mathfrak{so}(n-1) \subset \mathfrak{so}(n) \, (\subset \mathfrak{so}(n+1))$.

Remark:
Thimm's method only fits bi-invariant Riemannian metrics.
Mishchenko and Fomenko [MiF] introduced a class of left-invariant metrics on semi-simple Lie groups whose geodesic flows are completely integrable.
Using their method, Brailov [Br] constructed a class of Riemannian metrics on compact homogeneous spaces, which allow completely integrable geodesic flows.

§3. Subriemannian geodesic flows on $\mathbb{S}^7$
(Details can be found in [BaLTa2].)

Integrable subriemannian geodesic flows

There are studies on integrable subriemannian geodesic flows on homogeneous spaces as in the following examples.

- For Heisenberg groups of any dimension, Kocsard, Ovando, Reggiani [KOR] have shown the integrability of (sub-)Riemannian geodesic flow.
- For the pseudo H-type Lie groups, the complete integrability of the (sub-)Riemannian geodesic flow is showed in [BaTa1].
- A more recent work by Ovando [O].

Remark:
There is an example of subriemannian geodesic flow which is not completely integrable. ([MoShSt])

Subriemannian geodesic flow of $\mathbb{S}^3$

Recall $\mathbb{S}^3 \cong SU(2)$ and hence $T\mathbb{S}^3 \cong SU(2) \times \mathfrak{su}(2)$.
We take an orthonormal generators $e_1, e_2, e_3 \in \mathfrak{su}(2)$ for which

$$[e_1, e_2] = e_3, \quad [e_2, e_3] = e_1, \quad [e_3, e_1] = e_2.$$

Then, $\mathcal{D}_q := \mathrm{d}L_q \mathrm{span}_{\mathbb{R}} \{e_1, e_2\}$, $q \in SU(2) \cong \mathbb{S}^3$, gives rise to a (left-invariant) subriemannian structure on $\mathbb{S}^3 \cong SU(2)$.
Identifying $T^*\mathbb{S}^3 \cong T\mathbb{S}^3 \cong SU(2) \times \mathfrak{su}(2)$, we can write the Hamiltonian as

$$H(q,p) = \frac{1}{2}\left(p_1^2 + p_2^2\right), \quad p = p_1 e_1 + p_2 e_2 + p_3 e_3 \in \mathfrak{su}(2).$$

As p_1, H, and $p_1^2 + p_2^2 + p_3^2$ Poisson commute, the subriemannian geodesic flow is completely integrable.

Trivializable subriemannian structures on $\mathbb{S}^7$ 1

It is known that $\mathbb{S}^n$ is equipped with a trivializable bracket-generating subriemannian structure induced by a Clifford module structure only if $n = 1, 3, 7, 15, 23, 31, 63$. See [BaF] and [BaFI] for the details.

From now on, we focus on the case of rank r trivializable subriemannian structure on $\mathbb{S}^7$. (N.B. $\mathbb{S}^7$ is the highest dimensional parallelizable sphere.) The subriemannian metric is the induced metric from the one of $\mathbb{S}^7$. The rank r can only be either $4, 5, 6, 7$, to obtain a bracket-generating subriemannian structure. ([BaF], [BaFI].)
We consider the cases $r = 4, 5, 6$.

Trivializable subriemannian structures on $\mathbb{S}^7$ 2

Given a $(n+1)\times(n+1)$ matrix $A=(a_{ij})\in\mathbb{R}^{(n+1)\times(n+1)}$, we associate a vector field

$$X(A)_x := \sum_{i,j=1}^{n} a_{ij}x_j\frac{\partial}{\partial x_i}, \quad x\in\mathbb{R}^{n+1},$$

on $\mathbb{R}^{n+1}$. The vector field can be restricted to the sphere $\mathbb{S}^n$ if A is skew-symmetric.
Take a set of $(n+1)\times(n+1)$ skew-symmetric matrices A_k, $k=1,\dots,r$, satisfying the Clifford relations:

$$A_kA_\ell + A_\ell A_k = -2\delta_{k\ell}, \quad k,\ell = 1,\dots,r.$$

The maximal number r of the associated vector fields $X_i := X(A_i)$, $i=1,\dots,r$, which are orthonormal everywhere on $\mathbb{S}^n$, coincides with the maximal dimension of a trivial sub-bundle of $T\mathbb{S}^n$. ([A].)

Trivializable subriemannian structures on $\mathbb{S}^7$ 3

Take 8×8 skew-symmetric matrices A_k, $k = 1, \dots, 7$, acting on $\mathbb{R}^8 \cong \mathbb{H}^2$, s.t. $A_k A_\ell + A_\ell A_k = -2\delta_{k\ell}, \quad k, \ell = 1, \dots, 7.$
Such matrices are e.g. given as

$$A_1 := \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix}, \quad A_2 := \begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix} \quad A_3 := \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix},$$

where

$$C := \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \; D := \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \; E := \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

and

$$A_4 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \; A_5 := \begin{pmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{pmatrix}, \; A_6 := \begin{pmatrix} \mathbf{j} & 0 \\ 0 & -\mathbf{j} \end{pmatrix}, \; A_7 := \begin{pmatrix} \mathbf{k} & 0 \\ 0 & -\mathbf{k} \end{pmatrix}.$$

Trivializable subriemannian structures on $\mathbb{S}^7$ 4

The tangent bundle is trivialized as

$$T\mathbb{S}^7 \supset T_x\mathbb{S}^7 \ni \sum_{k=1}^{7} \alpha_k X_k \mapsto (x, \alpha) \in \mathbb{S}^7 \times \mathbb{R}^7,$$

where $\alpha = (\alpha_1, \ldots, \alpha_7)$.
The tangential distribution

$$\mathcal{D}^r := \operatorname{span}\{X_k = X(A_k) \mid k = 1, \ldots, r\}$$

equipped with the induced metric is a bracket-generating trivializable subriemannian structure on $\mathbb{S}^7$ for $r = 4, 5, 6$.

Non-deg. subalgebras for the rank 5 SR structure

Consider the non-degenerate Lie subalgebras $\mathfrak{g}_\ell$ for $\ell = 1, \dots, 6$ in $\mathfrak{g}_7 := \mathfrak{so}(8)$:

$$
\begin{aligned}
\mathfrak{g}_1 &:= \mathrm{Lie}\{A_6 A_7\} \cong \mathbb{R}, \\
\mathfrak{g}_2 &:= \mathrm{Lie}\{A_1 A_2, A_6 A_7\} \cong \mathbb{R}^2 \\
\mathfrak{g}_3 &:= \mathrm{Lie}\{A_1 A_2, A_1 A_3, A_2 A_3, A_6 A_7\} \cong \mathfrak{so}(3) \oplus \mathbb{R}, \\
\mathfrak{g}_4 &:= \mathrm{Lie}\{A_i A_j, A_6 A_7 \ : \ i, j = 1, \dots, 4, \ i < j\} \cong \mathfrak{so}(4) \oplus \mathbb{R}, \\
\mathfrak{g}_5 &:= \mathrm{Lie}\{A_i A_j, A_6 A_7 \ : \ i, j = 1, \dots, 5, \ i < j\} \cong \mathfrak{so}(5) \oplus \mathbb{R}, \\
\mathfrak{g}_6 &:= \mathrm{Lie}\{A_i A_j, A_6, A_7, A_6 A_7 \ : \ i, j = 1, \dots, 5, \ i < j\} \\
&\qquad\qquad\qquad\qquad \cong \mathfrak{so}(5) \oplus \mathfrak{so}(3).
\end{aligned}
$$

We have the chain of non-deg. subalgebras

$$\mathfrak{g}_1 \hookrightarrow \mathfrak{g}_2 \hookrightarrow \mathfrak{g}_3 \hookrightarrow \mathfrak{g}_4 \hookrightarrow \mathfrak{g}_5 \hookrightarrow \mathfrak{g}_6 \hookrightarrow \mathfrak{g}_7 = \mathfrak{so}(8).$$

Integrability for rk 5 trivializable SR structure 1

Take the bi-invariant functions $f_i(X_i) := \frac{1}{2} B_{\mathfrak{g}_i}(X_i, X_i)$ on $\mathfrak{g}_i$ and the orthogonal projections $\pi_i : \mathfrak{so}(8) \to \mathfrak{g}_i$, $i = 1, \ldots, 7$. The functions $f_i \circ \pi_i$, $i = 1, \ldots, 7$, on $\mathfrak{so}(8)$ (and hence $f_i \circ \pi_i \circ P$, $i = 1, \ldots, 7$, on $T\mathbb{S}^7$) are functionally independent and Poisson commute.
Here, $P : T\mathbb{S}^7 \cong T^*\mathbb{S}^7 \to \mathfrak{so}(8)^* \cong \mathfrak{so}(8)$ is the momentum mapping defined though
$P(g, \xi)(\eta) = \langle \eta q, \xi \rangle$, $q \in \mathbb{S}^7, \xi \in T_q\mathbb{S}^7 \subset \mathfrak{so}(8), \eta \in \mathfrak{so}(8)$.

Integrability for rk 5 trivializable SR structure 2

The Hamiltonian $H_{5,sub}$ on $T^*\mathbb{S}^7$ w.r.t. the rank 5 SR geodesic flow can be written as

$$H_{5,sub} = \frac{1}{4} f_7 \circ P - f_6 \circ \pi_6 \circ P + f_5 \circ \pi_5 \circ P,$$

Theorem

The subriemannian geodesic flow on $T^*\mathbb{S}^7$ w.r.t. the trivializable rank 5 SR structure is completely integrable with the functionally independent and Poisson commuting first integrals $f_i \circ \pi_i \circ P$, $i = 1, \ldots, 7$.

Other subriemannian structures on $\mathbb{S}^7$ 1

Theorem

Geodesic flows for the following SR structures on $\mathbb{S}^7$ are completely integrable:

- *(Quaternionic) Hopf distributions,*
- *Trivializable SR structures of rk $4, 5, 6$.*

Other subriemannian structures on $\mathbb{S}^7$ 2

- Quaternionic Hopf distribution $\mathcal{D}_Q$ on $\mathbb{S}^7$:
  Same chain of subalgebras

$$\mathfrak{g}_1 \hookrightarrow \mathfrak{g}_2 \hookrightarrow \mathfrak{g}_3 \hookrightarrow \mathfrak{g}_4 \hookrightarrow \mathfrak{g}_5 \hookrightarrow \mathfrak{g}_6 \hookrightarrow \mathfrak{g}_7 = \mathfrak{so}(8).$$

  can still be used.
  N.B. $H_Q = \frac{1}{4} f_7 \circ P - f_6 \circ \pi_6 \circ P + f_4 \circ \pi_4 \circ P$.
- Hopf distribution on $\mathbb{S}^7$:
  Still the same chain can be applicable.
  N.B. $H_{Hopf} = \frac{1}{4} f_7 \circ P - f_1 \circ \pi_1 \circ P$.

Other subriemannian structures on $\mathbb{S}^7$ 3

- The rank 4 trivializable SR structure:
  Consider the non-deg. subalgebras

$$\mathfrak{h}_\ell = \operatorname{span}_{\mathbb{R}} \{A_i A_j \mid 1 \le i < j \le \ell + 1\}, \quad \text{if } 1 \le \ell \le 3;$$
$$\mathfrak{h}_4 = \operatorname{span}_{\mathbb{R}} \{A_i A_j \mid 1 \le i < j \le 4 \text{ or } (i,j) = (5,6)\},$$
$$\mathfrak{h}_5 = \operatorname{span}_{\mathbb{R}} \{A_i A_j \mid 1 \le i < j \le 4 \text{ or } 5 \le i < j \le 7\},$$
$$\mathfrak{h}_6 = \operatorname{span}_{\mathbb{R}} \Big(\{A_i A_j \mid 1 \le i < j \le 5 \text{ or } 5 \le i < j \le 7\} \cup \{A_5, A_6, A_7\} \Big).$$

  forming the chain

$$\mathfrak{h}_1 \subset \mathfrak{h}_2 \subset \mathfrak{h}_3 \subset \mathfrak{h}_4 \subset \mathfrak{h}_5 \subset \mathfrak{h}_6 \subset \mathfrak{h}_7 := \mathfrak{so}(8).$$

Other subriemannian structures on $\mathbb{S}^7$ 4

- The rank 4 trivializable SR structure (continued):
  Take the bi-invariant functions
  $g_i(Y_i) := \frac{1}{2} B_{\mathfrak{h}_i}(Y_i, Y_i), \quad Y_i \in \mathfrak{h}_i.$
  Then, $g_i \circ \pi_{\mathfrak{h}_i} \circ P$, $i = 1, \ldots, 7$, are functionally independent and Poisson commute.
  N.B. $H_{4,sub} = \frac{1}{4} g_7 \circ P - g_6 \circ \pi_{\mathfrak{h}_6} \circ P + g_5 \circ \pi_{\mathfrak{h}_5} \circ P.$

Other subriemannian structures on $\mathbb{S}^7$ 5

- The rank 6 trivializable SR structure:
  Consider the non-deg. subalgebras

$$\mathfrak{k}_\ell = \operatorname{span}_{\mathbb{R}} \{A_i A_j \mid 1 \leq i < j \leq \ell + 1\}, \quad \text{if } 1 \leq \ell \leq 5;$$
$$\mathfrak{k}_6 = \operatorname{span}_{\mathbb{R}} (\{A_i A_j \mid 1 \leq i < j \leq 6\} \cup \{A_7\}).$$

  forming the chain

$$\mathfrak{k}_1 \subset \mathfrak{k}_2 \subset \mathfrak{k}_3 \subset \mathfrak{k}_4 \subset \mathfrak{k}_5 \subset \mathfrak{k}_6 \subset \mathfrak{k}_7 := \mathfrak{so}(8).$$

Other subriemannian structures on $\mathbb{S}^7$ 6

- The rank 6 trivializable SR structure (continued):
  Take the bi-invariant functions
  $h_i(Z_i) := \frac{1}{2} B_{\mathfrak{k}_i}(Z_i, Z_i), \quad Z_i \in \mathfrak{h}_i.$
  Then, $h_i \circ \pi_{\mathfrak{k}_i} \circ P$, $i = 1, \ldots, 7$, are functionally independent and Poisson commute.
  N.B. $H_{6,sub} = \frac{1}{4} h_7 \circ P - h_6 \circ \pi_{\mathfrak{k}_6} \circ P + h_5 \circ \pi_{\mathfrak{k}_5} \circ P.$

Remarks

- The chain of non-deg. Lie aubalgebras used in this talk is different from those in Thimm's paper [Th].
- The subriemannian geodesic flows have non-periodic orbits.
- Some lower bounds are given for the dimension of the subriemannian isometry groups.
- There are commuting operators whose symbols are the first integrals constructed in the previous theorems.

Suriemannian isometry 1

Theorem

$\mathrm{Iso}_{\mathrm{sub}}\left(\mathbb{S}^7_{\mathrm{T},5}\right)$: *subriemannian isometry group for the rank* 5 *trivializable subriemannian structure. The Lie algebra of* $\mathrm{Iso}_{\mathrm{sub}}\left(\mathbb{S}^7_{\mathrm{T},5}\right) \cap O(8)$ *coincides with* $\mathfrak{g}_5$ *and hence* $\dim \mathrm{Iso}_{\mathrm{sub}}\left(\mathbb{S}^7_{\mathrm{T},5}\right) \geq \dim \mathfrak{g}_5 = 11$.

Theorem

$\mathrm{Iso}_{\mathrm{sub}}\left(\mathbb{S}^7_{\mathrm{T},4}\right)$: *subriemannian isometry group for the rank* 4 *trivializable subriemannian structure. The Lie algebra of* $\mathrm{Iso}_{\mathrm{sub}}\left(\mathbb{S}^7_{\mathrm{T},4}\right) \cap O(8)$ *coincides with* $\mathfrak{h}_5$ *and hence* $\dim \mathrm{Iso}_{\mathrm{sub}}\left(\mathbb{S}^7_{\mathrm{T},4}\right) \geq \dim \mathfrak{h}_5 = 9$.

Suriemannian isometry 2

Theorem

$\mathrm{Iso}_{\mathrm{sub}}\left(\mathbb{S}^7_{\mathrm{H}}\right)$: *subriemannian isometry group for the rank* 6 *trivializable subriemannian structure associated to the Hopf fibration. The Lie algebra of* $\mathrm{Iso}_{\mathrm{sub}}\left(\mathbb{S}^7_{\mathrm{H}}\right) \cap O(8)$ *coincides with* $\mathfrak{k}_6$ *and hence* $\dim \mathrm{Iso}_{\mathrm{sub}}\left(\mathbb{S}^7_{\mathrm{H}}\right) \geq \dim \mathfrak{k}_6 = 16$.

Theorem

$\mathrm{Iso}_{\mathrm{sub}}\left(\mathbb{S}^7_{\mathrm{QH}}\right)$: *subriemannian isometry group for the subriemannian structure associated to the quaternionic Hopf fibration. The Lie algebra of* $\mathrm{Iso}_{\mathrm{sub}}\left(\mathbb{S}^7_{\mathrm{QH}}\right) \cap O(8)$ *coincides with* $\mathfrak{g}_6$ *and hence* $\dim \mathrm{Iso}_{\mathrm{sub}}\left(\mathbb{S}^7_{\mathrm{QH}}\right) \geq \dim \mathfrak{h}_6 = 13$.

Commuting differential operators

For the rank 5 trivializable subriemannian structure, we consider the following 2nd order differential operators

$$\mathcal{G}_\ell := \sum_{E \in \mathfrak{G}_\ell} X(E)^2, \quad \ell = 1, \dots, 7.$$

$\mathfrak{G}_\ell$: generators of $\mathfrak{g}_\ell$.

Proposition

The operators $\mathcal{G}_\ell$ for $\ell = 1, \dots, 7$ pairwise commute and commute with the sublaplacian $\Delta^{\text{sub}}_{\text{T},5} = -\sum_{j=1}^{5} X(A_j)^2$.

We find the same type of results for the rank 4 trivializable structure and the ones obtained by the (quaternionic) Hopf fibration.

Sublaplace heat kernel

For rank 5 trivializable SR structure, the heat kernel for the sublaplace operator can explicitly be obtained [BaLTa1]:

$$K^{\Delta_5^{\mathrm{sub}}}(t,\omega,\mathrm{np})$$
$$= \frac{e^{-t}}{2\pi t}\int_{\mathbb{R}} e^{-\frac{(z-\alpha)^2}{4t}} \int_0^{\infty} \frac{\sinh s \cdot \sin\left(\frac{\theta(\kappa,\alpha)s}{2t}\right)}{\sin\theta(\kappa,\alpha)} \cdot e^{-\frac{s^2-\theta(\kappa,\alpha)^2}{4t}} q_t(\cos r \cosh s)\mathrm{d}s\mathrm{d}\alpha.$$

$q_t := k_t^{\mathbb{S}^7} \circ \arccos$, $\theta(\kappa,\alpha) = \arccos(\cos\kappa \cdot \cos\alpha)$.
$(r, c, \kappa, \sigma, z)$: local coordinates of $\omega \in \mathbb{S}^7$ around the north pole np. ($c \in \mathbb{S}^3 \subset \mathbb{H}$.)
Idea:
Existence of commuting operators.

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