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曲面の離散微分幾何の拡がり

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ワークショップ「離散幾何学〜理論から物質を探求する〜」

内容

1. **研究の概要**
2. Previous works
3. Discrete ZMC surfaces in $\mathbb{R}^{2,1}$
4. ~~Future works~~ Erased, because they are in progress.

参考文献

[BP] A.I. Bobenko and U. Pinkall, *Discrete isothermic surfaces*, J. Reine Angew. Math., **475** (1996), 187-208.

[Y1] Y-, *Discrete maximal surfaces with singularities in Minkowski space*, DGA **43** (2015), 130-154.

[Y2] Y-, *Discrete timelike minimal surfaces and discrete wave equations*, preprint.

1. 研究の概要
※本セクションのみ日本語で記載

そもそも離散微分幾何とは？

離散微分幾何 $\overset{???}{=}$ 連続的対象 (曲線，曲面，etc...) と同様の性質を持つ離散幾何の構築

マニフェスト（指導原理）

Discretize the whole theory, not just the equations.

(Bobenko, Suris)

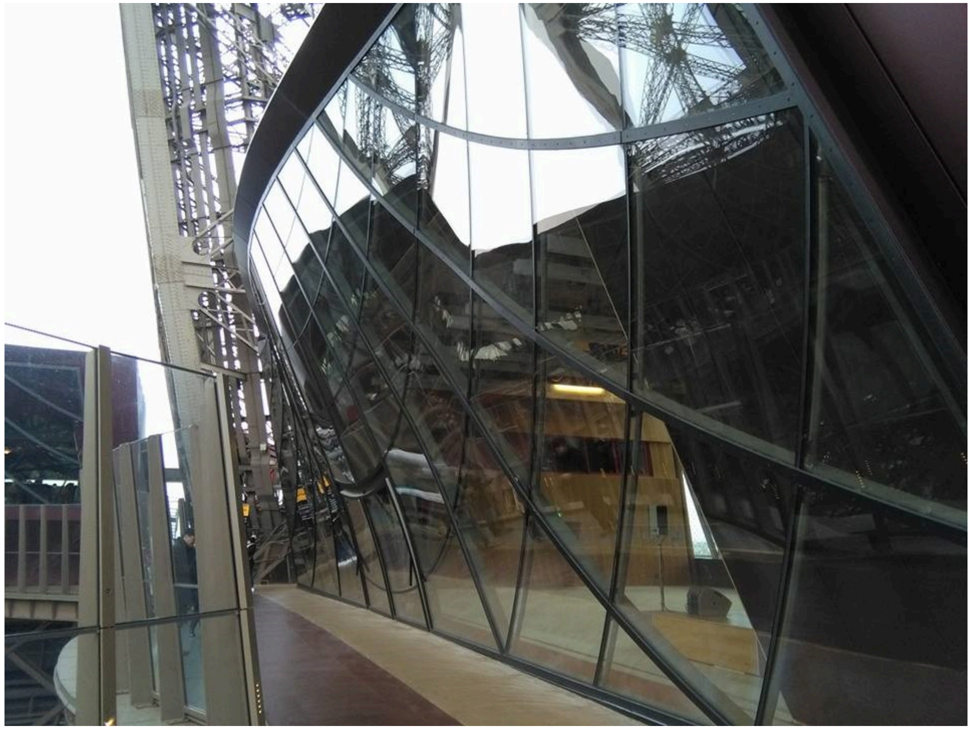
例　極小曲面

性質

1. 与えられた境界を持つ曲面のうち，表面積最小 (変分原理)
2. 平均曲面一定 0 (微分幾何)
3. 正則関数を用いた構成法: Weierstrass の表現公式 (複素関数)
4. 特別な座標系や変換の存在性 (可積分系)

離散化の方向性その１

実社会への応用

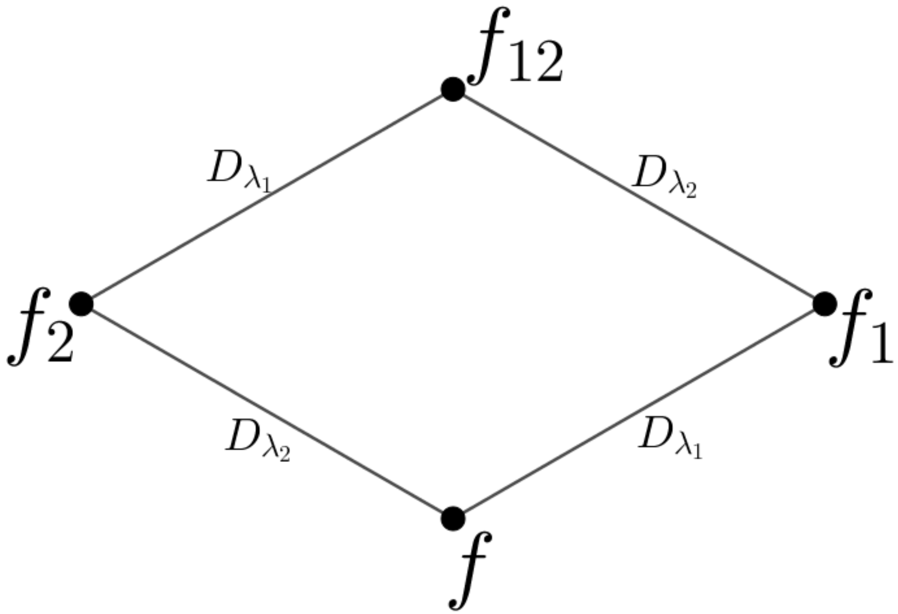


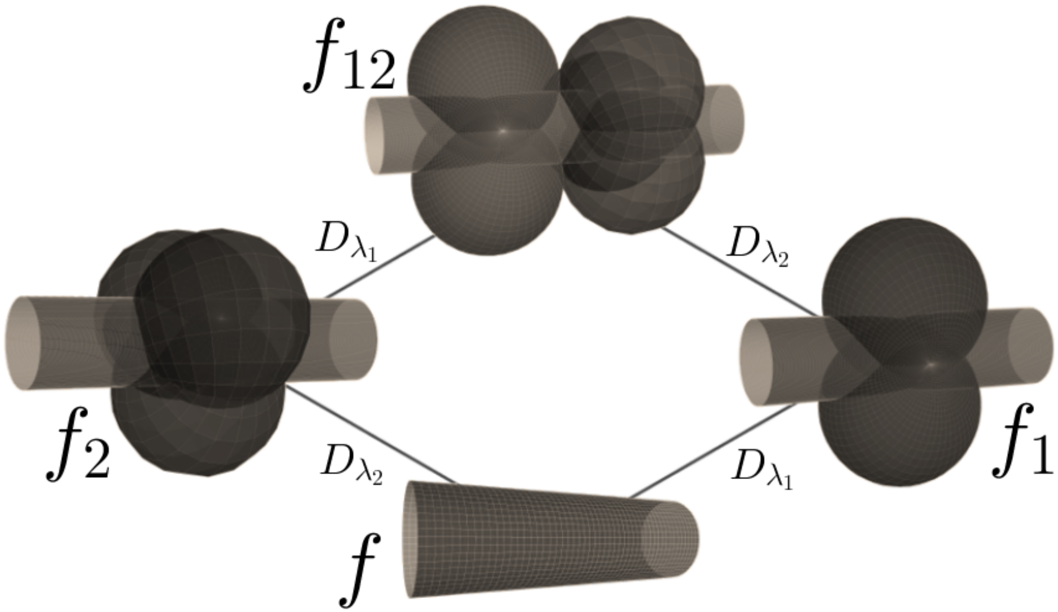
図：エッフェル塔（パリ，フランス）

例　離散変分原理 ⟶ （連続的な）微分幾何に基づいて，離散的な土台のもとで理論を再整備・再構築する必要性

離散化の方向性その2

（連続，離散，半離散）可積分系の背後に潜む幾何学の構築



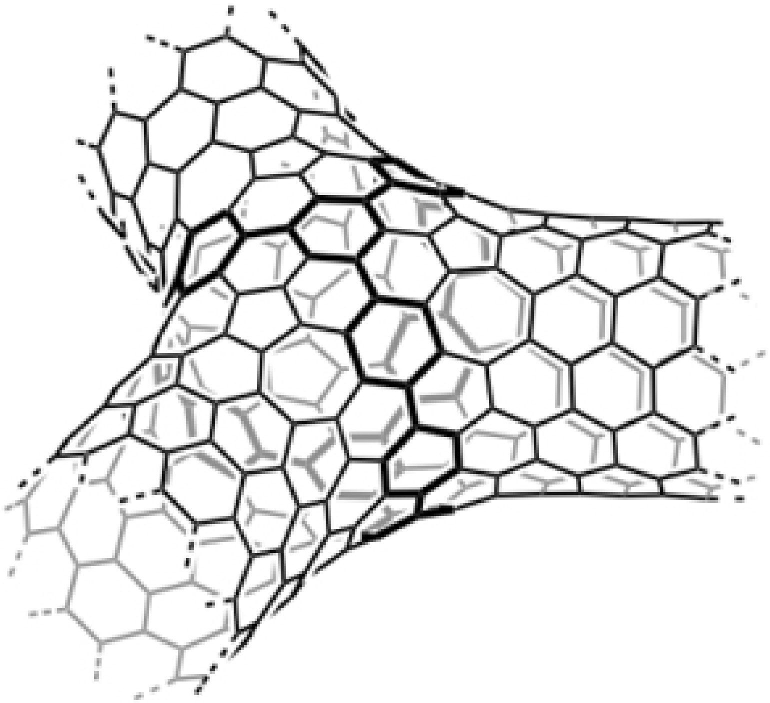


左図：双等温曲面（後述）に対するダルブー変換のBianchi Permutability
※（連続的）微分幾何学に現れる離散対称性（可積分性）

右図：ダルブー変換によって得られた曲面

離散化の方向性その3

連続的ではなく，離散的な土台のもとで研究すべき微分幾何的対象物の研究（cf. 梶ヶ谷さん，内藤さん）



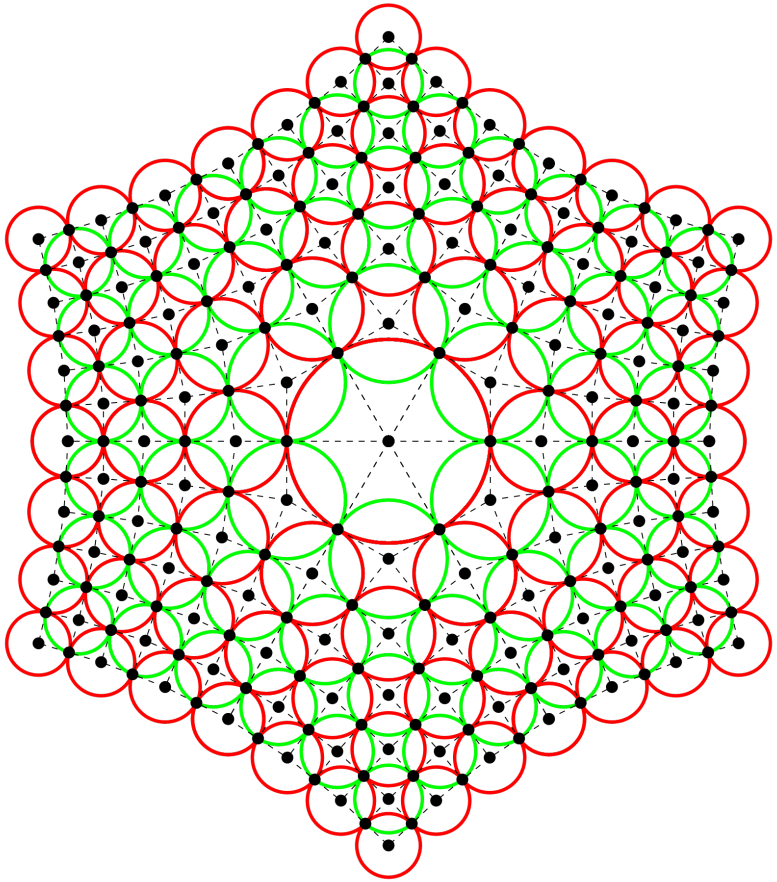
図：分岐したカーボンナノチューブ（図はK. Matsui, et al. Chemical Science 4.1 (2013): 84-88より）

離散化の方向性その4

離散幾何的対象に現れる連続的構造の解明（専門外）

サークルパッキング，より
一般にサークルパターンの
幾何（Thurston, etc...）
(幾何トポロジー，双曲幾何，
etc...)

（その他）近年，離散微分幾何
の研究は微分幾何の研究にも
応用されている．



曲面の離散微分幾何の現状（主観）

- 様々な離散曲面論（曲面の離散化）が展開されている
- 離散化の目的が異なるため，統一的な離散曲面論は「ない」

$\longrightarrow$ 新たな離散曲面論の構築や，これまでの離散曲面論を包括する理論の創出が今後の課題

※前者については多くの研究があるが，後者については難しく，現状ではほとんどない．純粋数学的には，後者が面白い．

2. Previous works

Symbols

$\mathbb{R}^3$: Euclidean 3-space with standard Euclidean metric
$\mathbb{R}^{2,1}$: Minkowski $(2+1)$-space with the Lorentz metric

$$\langle x, y\rangle = x_1 y_1 + x_2 y_2 - x_0 y_0$$

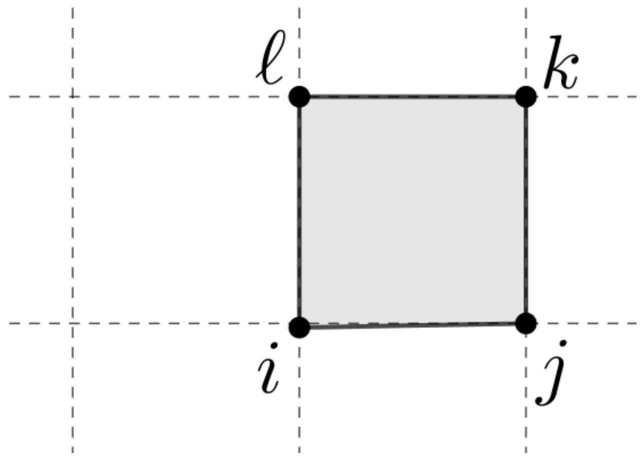
for $x = (x_1, x_2, x_0),\ y = (y_1, y_2, y_0) \in \mathbb{R}^{2,1}$
$\mathbb{C}$: complex plane
$\mathbb{C}'$: split-complex plane, that is,

$$\mathbb{C}' := \{a + j'b | a, b, \in \mathbb{R},\ (j')^2 = +1, j' \ni \mathbb{R}\}$$

$\mathbb{H}^2 := \{x \in \mathbb{R}^{2,1} |\ \langle x, x\rangle = -1\}$: hyperbolic 2-space
$\mathbb{S}^{1,1} := \{x \in \mathbb{R}^{2,1} |\ \langle x, x\rangle = +1\}$: de Sitter 2-space

Abbreviation

$^{\forall}(m,n) \in \mathbb{Z}^2,$
$f(m,n) = f_{m,n} = f_i,$
$f_{m+1,n} = f_j,$
$f_{m+1.n+1} = f_k,\ f_{m,n+1} = f_\ell$



Setting in the smooth case

Def 1. Let $x : \mathbb{R}^2 \ni (u, v) \mapsto x(u, v) \in \mathbb{R}^3$ be an isothermic surface. Then a surface x^* satisfying

$$x^*_u = \frac{x_u}{\|x_u\|^2}, \quad x^*_v = -\frac{x_v}{\|x_v\|^2}$$

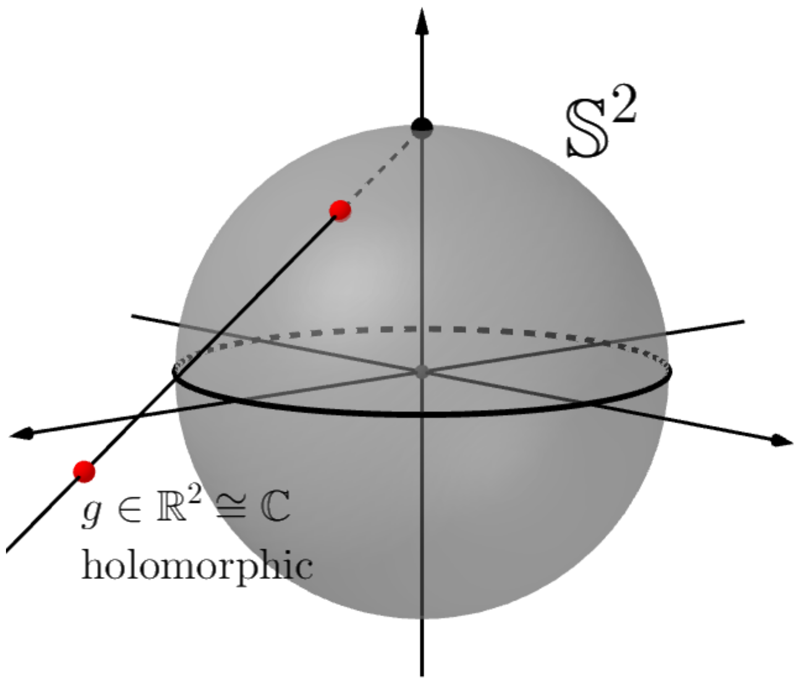
is called a Christoffel transform (C-transform) of x.

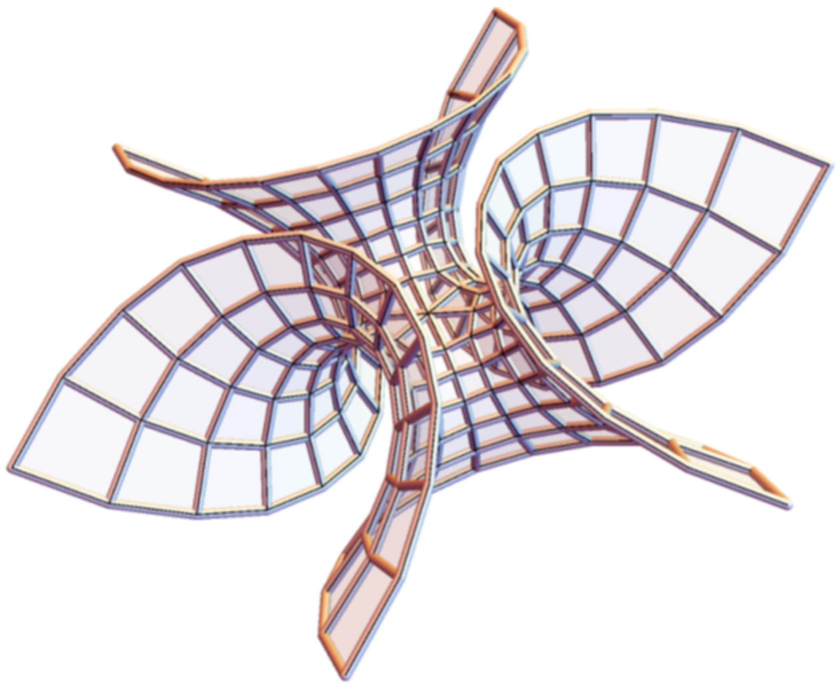
Important properties

1. The surface x^* is also isothermic.
2. x is isothermic $\Leftrightarrow \exists x^*$, and it is uniquely determined, up to homotherties and dilations.
3. x is isothermic minimal $\Leftrightarrow x^* \in \mathbb{S}^2$

Application of C-transforms

1. Choose a holomorphic function $g \in \mathbb{C} \cong \mathbb{R}^2$.
2. Take the inverse image of the stereographic projection of g to $\mathbb{S}^2 \subset \mathbb{R}^3$.
3. Take the C-transform of item 2, and we have a Weierstrass representation for isothermic minimal surfaces.

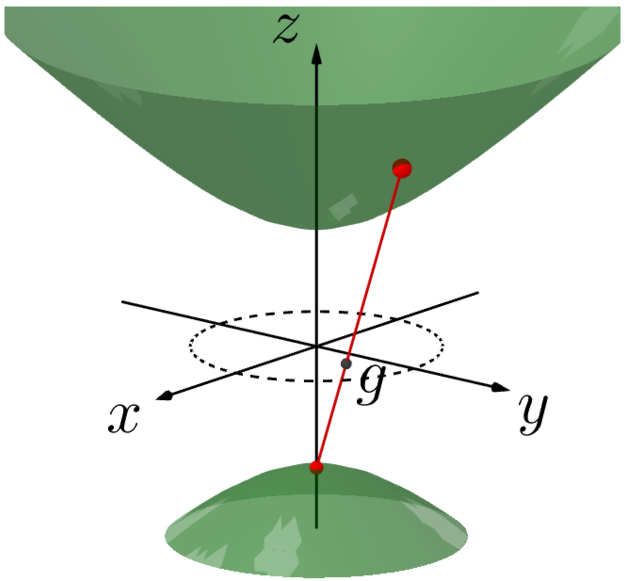


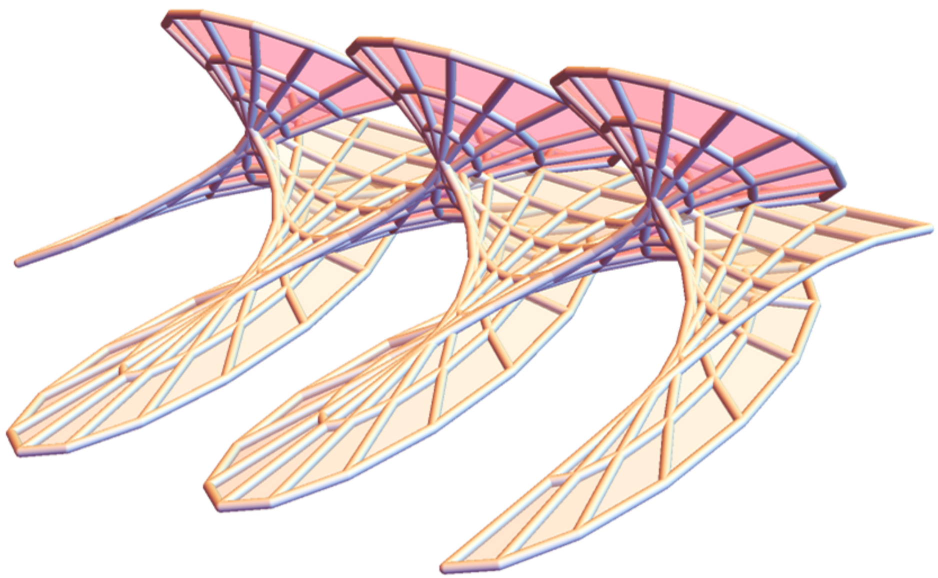


Similar recipe

1. Choose a holomorphic function $g \in \mathbb{C} \cong \mathbb{R}^2$.
2. Take the inverse image of the stereographic projection of g to $\mathbb{H}^2 \subset \mathbb{R}^{2,1}$.
3. Take the C-transform of item 2, and we have

$$x = \mathrm{Re} \int^z (1 + g^2, \sqrt{-1}(1 - g^2), 2g) \frac{dz}{g'} \ .$$

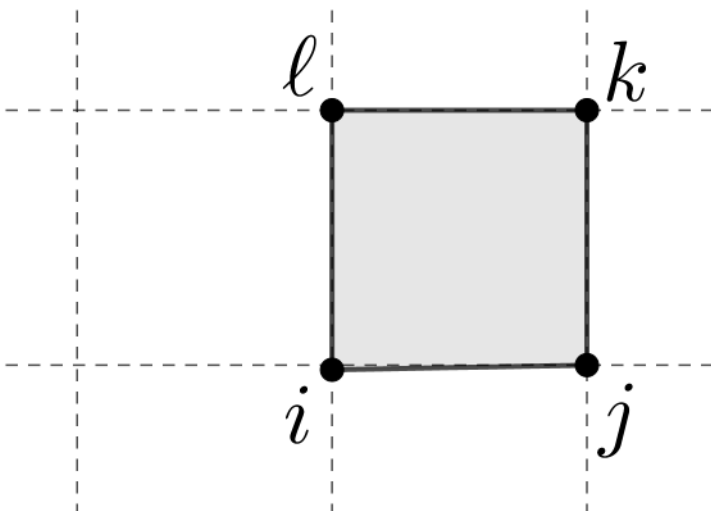


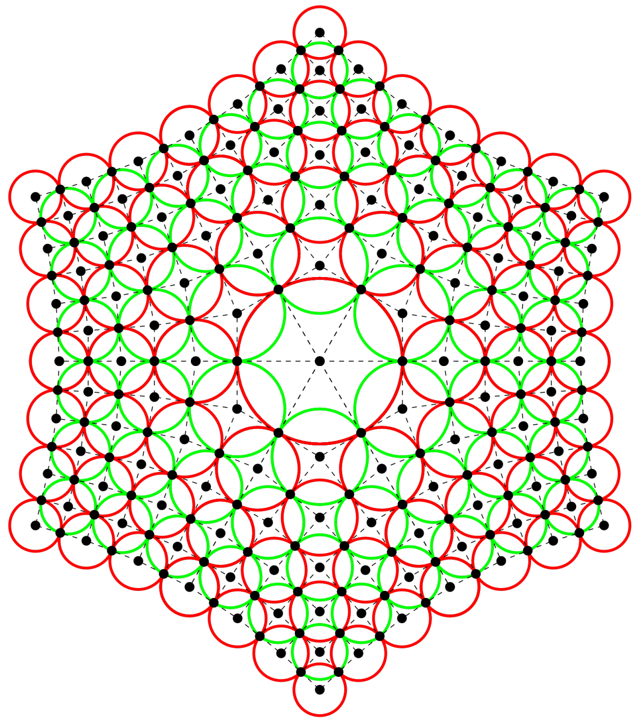


Discrete isothermic surfaces

Def 2 ([BP])**.** Let $f : \mathbb{Z}^2 \to \mathbb{R}^3$ a discrete surface. Then

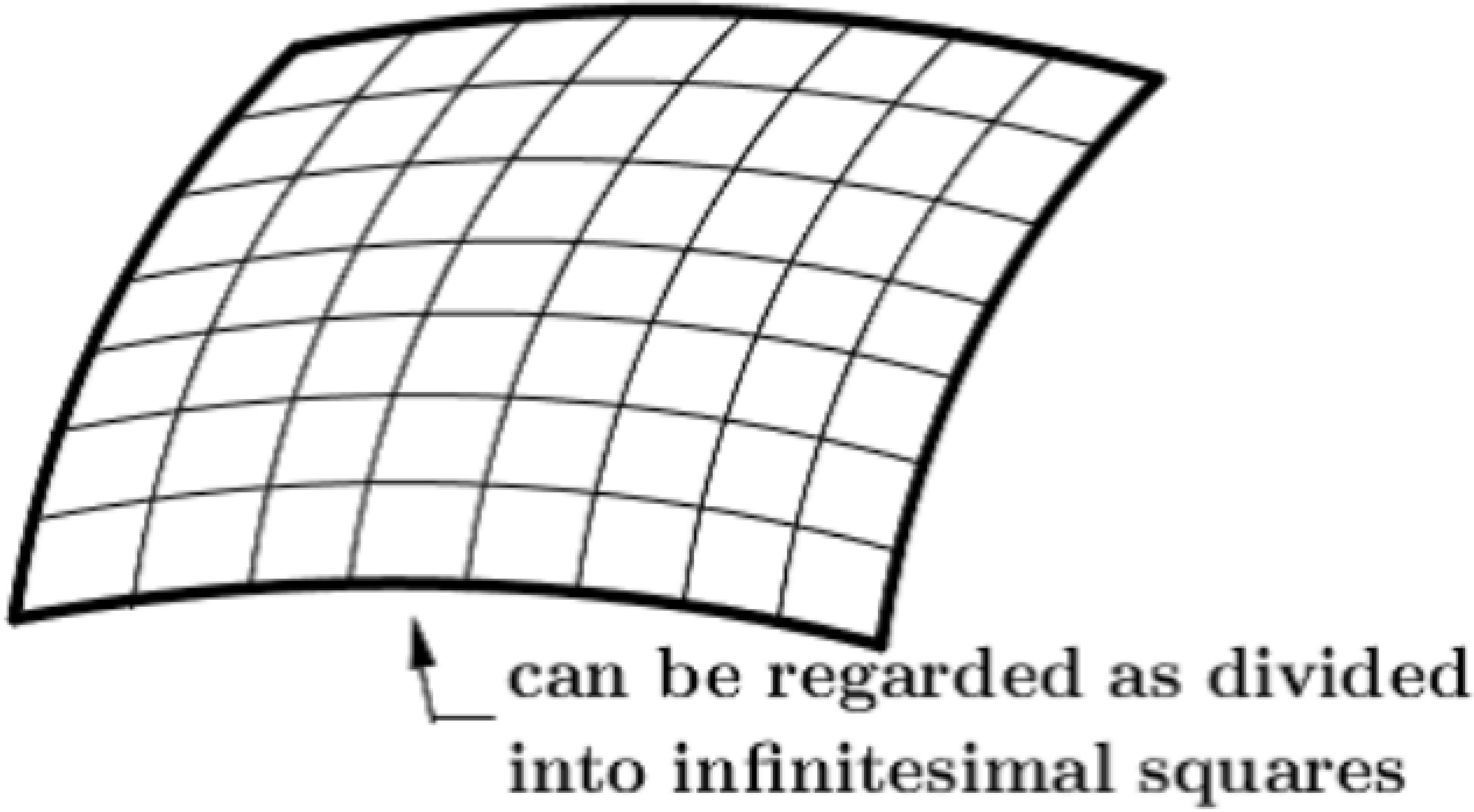
1. f is discrete isothermic if $cr(f_i, f_j, f_k, f_\ell) = \dfrac{\alpha_{ij}}{\alpha_{i\ell}} < 0$ for all $(m, n) \in \mathbb{Z}^2$. In particular, a discrete isothermic surface $g : \mathbb{Z}^2 \to \mathbb{R}^2 \cong \mathbb{C}$ is called discrete holomorphic.





Motivations

1. Cayley's interpretation (Cayley, 1872)



2. Möbius invariance i.e. isothermicity is preserved under Möbius transformations.

3. One remarkable feature in integrable systems

$\cdots$ Bianchi permutability $\Longrightarrow$ discrete symmetry!

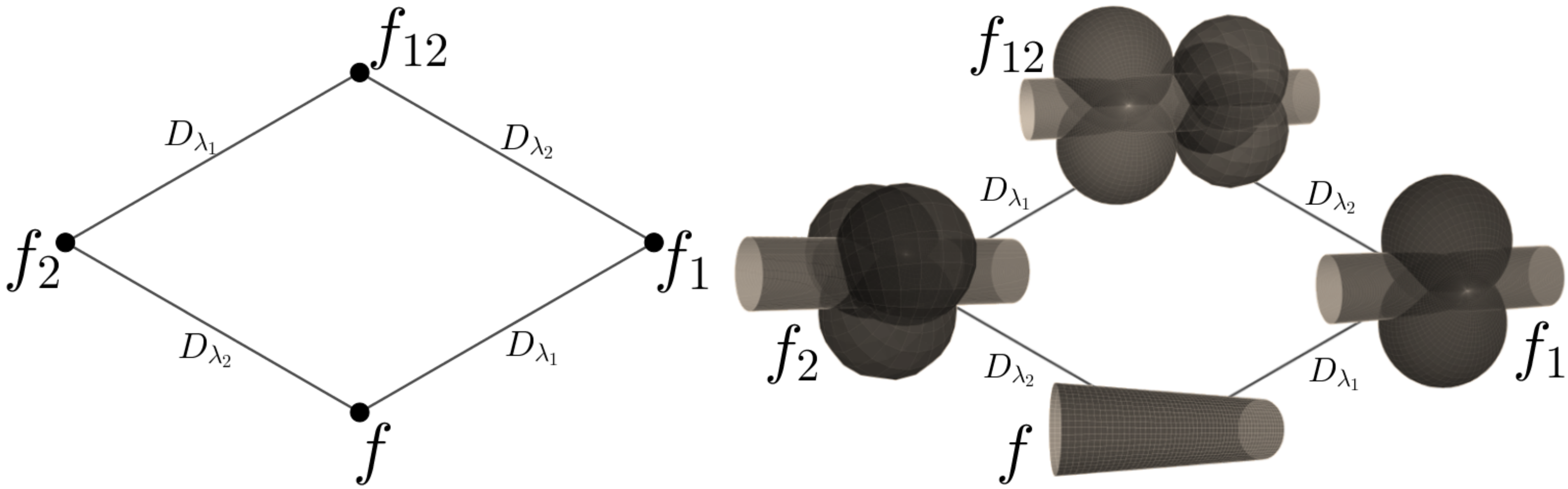


Figure (reprint): Bianchi permutability of Darboux transforms and its application to (smooth) CMC surfaces

<u>Claim</u> There exists a unique f_{12} such that it is a Darboux transform of both f_1 and f_2.

Further property

Bianchi lattice forms a discrete isothermic surface, that is, iterations of Darboux transforms of a smooth isothermic surface form a discrete isothermic surface.

$\longrightarrow$ One nature to discretize smooth geometries so that discrete integrabilities are preserved!

$\Longrightarrow$ The phrase "multidimensional consistency"

geometric object	integrability
smooth	discrete
semi-discrete	discrete
discrete	discrete

2. Let f be a discrete isothermic surface with $cr(f_i, f_j, f_k, f_\ell) = \dfrac{\alpha_{ij}}{\alpha_{i\ell}}$. Then a discrete surface f^* solving

$$f^*_\star - f^*_i = \alpha_{i\star} \frac{f_\star - f_i}{\|f_\star - f_i\|^2}$$

is called a Christoffel transform of f.

Prop 1 (Bobenko-Pinkall [BP]). Any discrete isothermic minimal surface f in $\mathbb{R}^3$ can be obtained by solving

$$f_\star - f_i = \mathrm{Re}\left(\frac{\alpha_{i\star}}{2(g_\star - g_i)} \begin{pmatrix} 1 - g_i g_\star \\ \sqrt{-1}(1 + g_i g_\star) \\ g_i + g_\star \end{pmatrix} \right)$$

with $cr(g_i, g_j, g_k, g_\ell) = \dfrac{\alpha_{ij}}{\alpha_{i\ell}}$.

3. Discrete ZMC surfaces in $\mathbb{R}^{2,1}$

Discrete maximal surfaces in $\mathbb{R}^{2,1}$ ([Y1])

Similarly to the Euclidean case, we can define discrete isothermic surfaces in $\mathbb{R}^{2,1}$, and we have the following result:

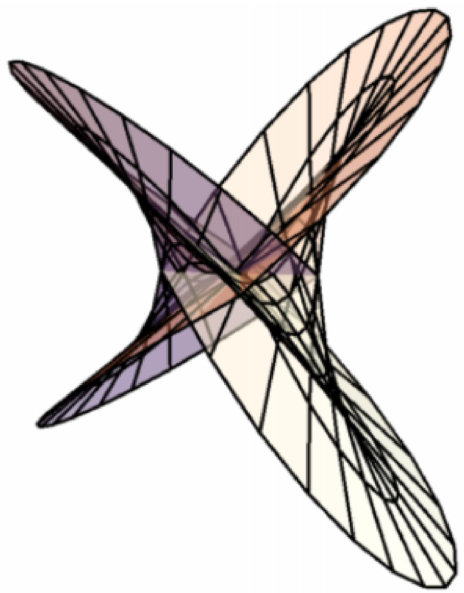
Prop 2. Any discrete isothermic maximal surface f in $\mathbb{R}^{2,1}$ can be obtained by solving

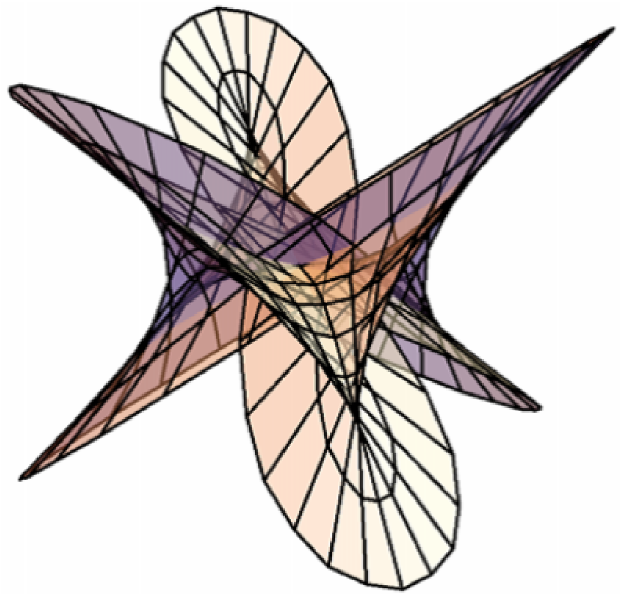
$$f_\star - f_i = \mathrm{Re}\left(\frac{\alpha_{i\star}}{2(g_\star - g_i)}\begin{pmatrix} 1 + g_i g_\star \\ \sqrt{-1}(1 - g_i g_\star) \\ g_i + g_\star \end{pmatrix}\right) .$$

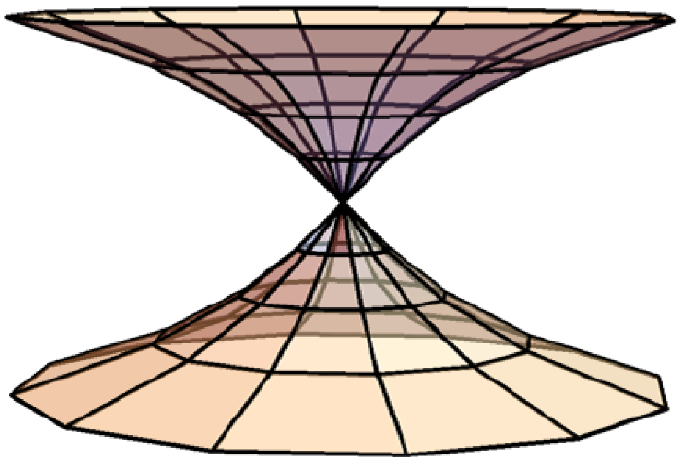
Remark. We have two remarks:

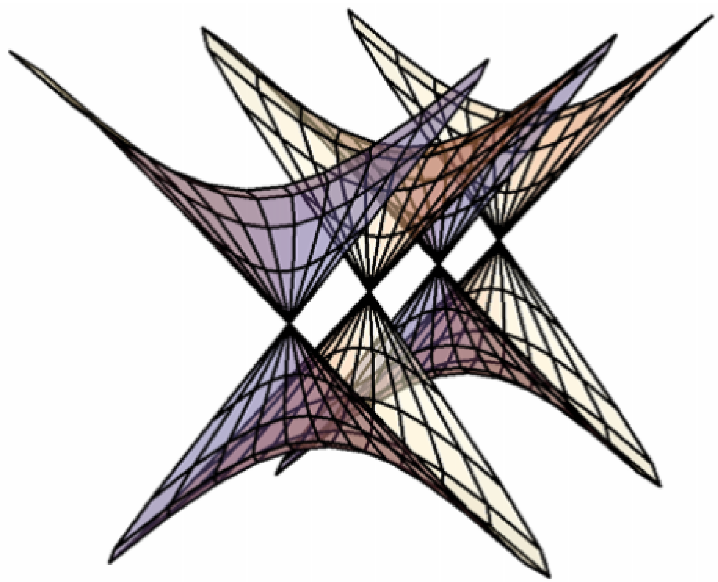
- Discrete maximal surfaces generally have "singularities".
- Multiplying $e^{\sqrt{-1}\theta}$ $(\theta \in [0, 2\pi))$ with α_{i*}, we have the associated family of discrete maximal surfaces.

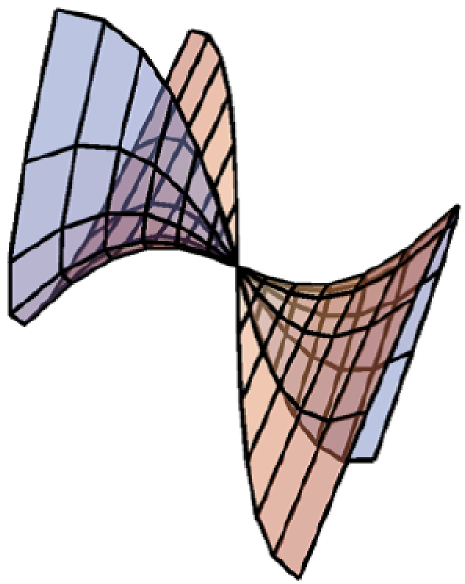
Examples











Discrete timelike minimal surfaces in $\mathbb{R}^{2,1}$ ([Y2])

First we define a discrete version of p-holomorphic functions.

Def 3. A map $G : \mathbb{Z}^2 \to \mathbb{C}'$ is called *discrete p-holomorphic* if it satisfies

$$cr(G_i, G_j, G_k, G_\ell) := \frac{(G_j - G_i)(G_k - G_\ell)}{(G_k - G_j)(G_\ell - G_i)} \equiv +1.$$

Motivation

A Lorentz version of the Cauchy-Riemann equation.

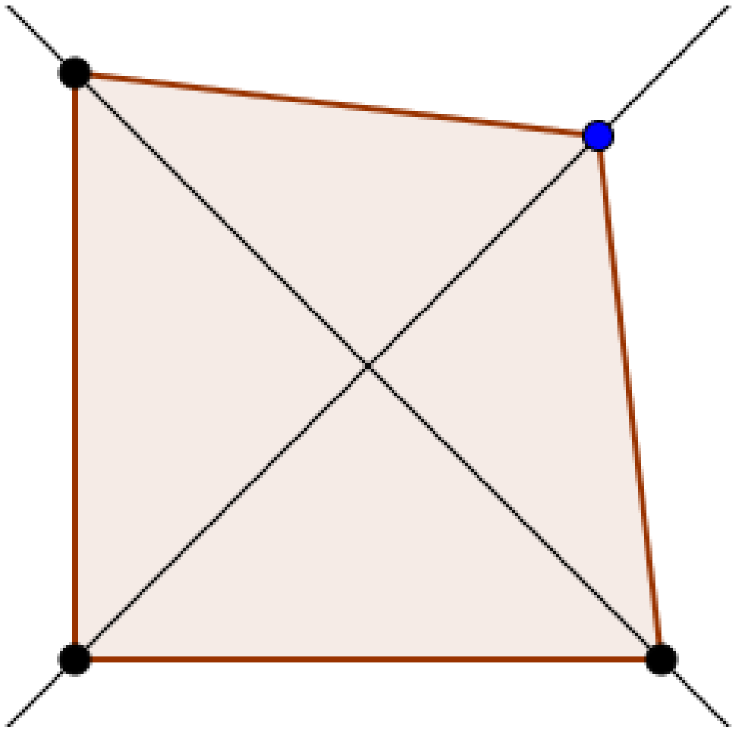
Cf. Smooth holomorphic functions

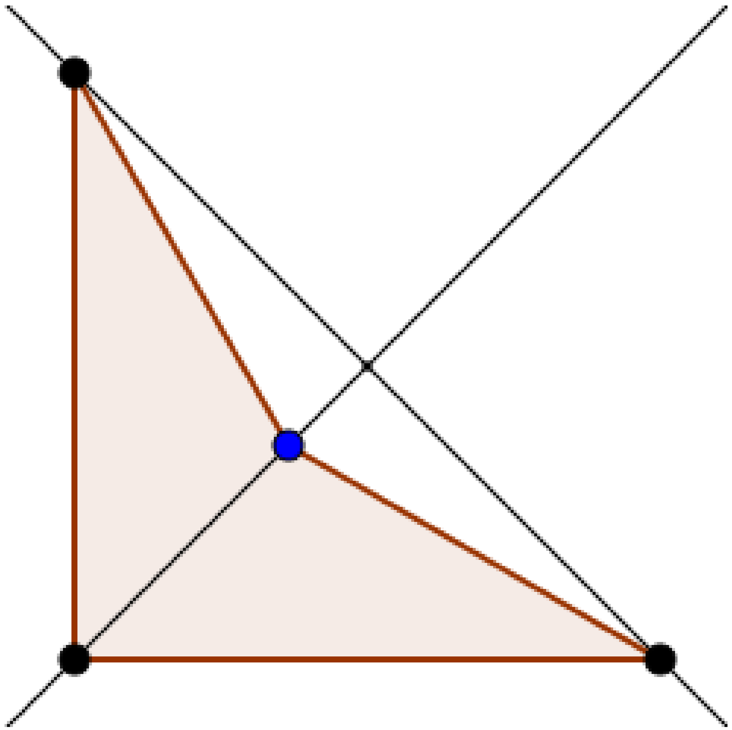
$\rightsquigarrow$ discrete holomorphic functions

Remark. The cross ratio of four points is 1 if and only if they lie on a 1-dimensional lightcone.

But...

Even if we fix three points on $\mathbb{C}'$, the cross ratio condition does not imply the unique fourth point.





On the other hand...

Discrete p-holomorphic functions have the following natural property:

Prop 3. Let g_1 and g_2 be discrete p-holomorphic functions. Then $g_1 + g_2$ and $g_1 \cdot g_2$ are also discrete p-holomorphic.

From this observation, we have the following proposition:

Lem 1. Any discrete p-holomorphic function can be written as

$$g_{m+n,m-n} = (1 \pm j')p_m + (1 \mp j')q_n.$$

Thm 1. Any discrete timelike minimal surface F can be obtained using a discrete p-holomorphic function G by solving

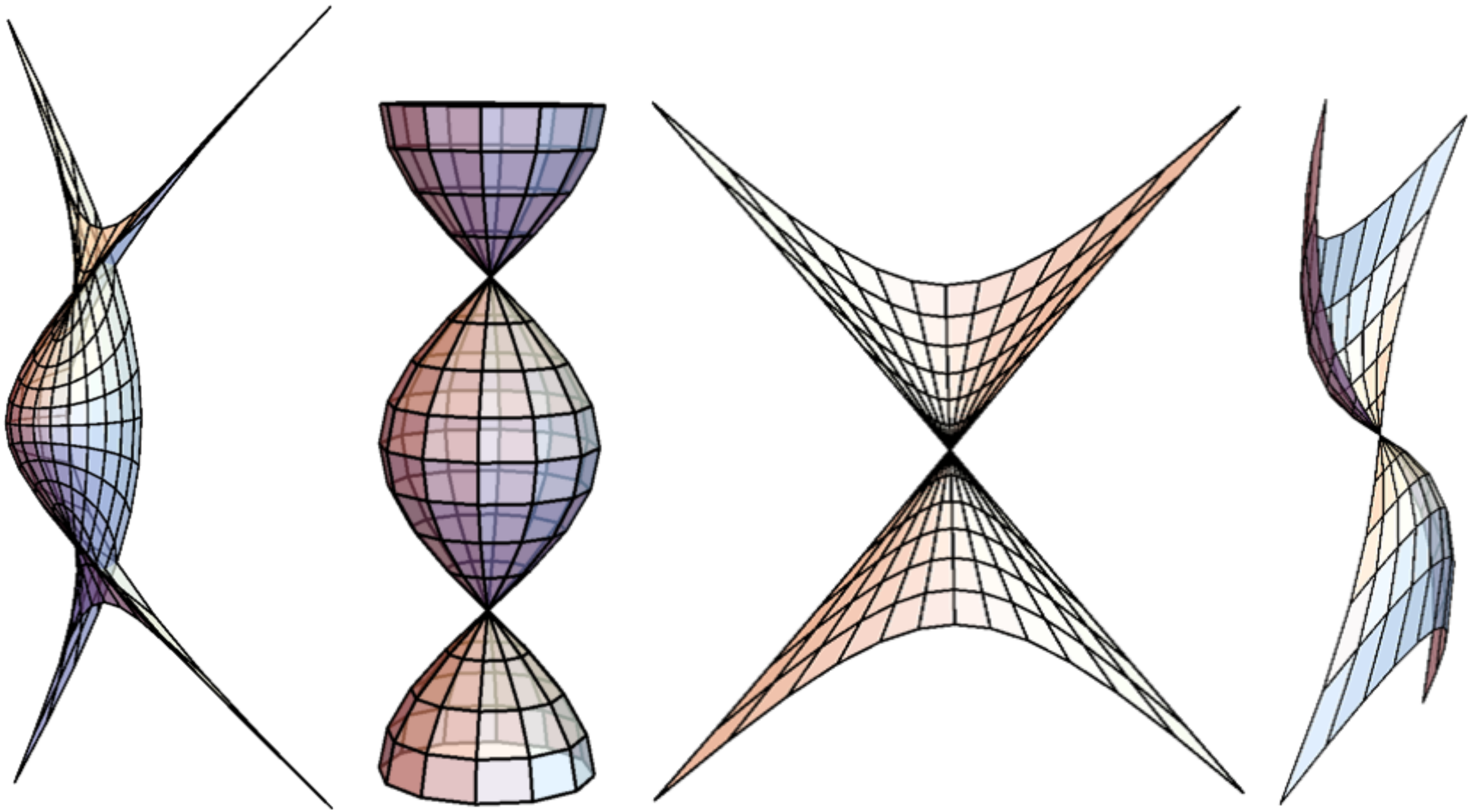
$$F_* - F_i = \frac{1}{2}\mathrm{Re}\left(\frac{G_i + G_*}{G_* - G_i}, \frac{1 - G_i G_*}{G_* - G_i}, \frac{-j'(1 + G_i G_*)}{G_* - G_i}\right)^t.$$

Furthermore, by a reparametrization, any discrete timelike minimal surface can be described by

$$F_{x+1,y} - F_{x,y} = \alpha(x)\left(-2p_{x+\frac{1}{2}}, (p_{x+\frac{1}{2}})^2 - 1, (p_{x+\frac{1}{2}})^2 + 1\right),$$
$$F_{x,y+1} - F_{x,y} = \beta(y)\left(-2q_{y+\frac{1}{2}}, (q_{y+\frac{1}{2}})^2 - 1, -(q_{y+\frac{1}{2}})^2 - 1\right),$$

where $p_x,\ q_y,\ \alpha(x),\ \beta(y)$ are real-valued functions depending only on x or y.

Examples



~~4. Future works~~

Erased, because they are in progress.

ご清聴ありがとうございました