Research interests and results

Satoh, Takao (takao@rs.tus.ac.jp)

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1. Interests

My research interests lie in Algebraic Topology and Combinatorial Group Theory, in particular the automorphism groups of free groups and the mapping class groups of surfaces. The main interests of my study are focused on their *Johnson homomorphisms* and *twisted cohomology groups*. In my works for the Johnson homomorphisms, I have determined the rational cokernels of the Johnson homomorphisms associated to the lower central series of the IA-automorphism group by using purely algebraic tools, including Combinatorial group theory, Lie algebra theory, Representation theory and so on. This induced a break-through in the study of the Johnson cokernels of the mapping class groups of surfaces. In particular, Enomoto and I have detected new series in the Johnson cokernels, and this works are used and generalized by many authors. In my works for the twisted cohomology groups, on the study of the extension of the first Johnson homomorphisms, I computed several first cohomology groups of the automorphism groups of free groups by using its presentation in a simple way. Recently, twisted cohomology groups of the automorphism groups of free groups are studied by several ways. My works gave rise to one of motivations for their studies. To put it briefly, I will study the following two problems in the next period.

- To give the answer to *the Andreadakis conjecture* from a viewpoint of combinatorial group theory.
- To find *non-trivial high dimensional twisted cohomology classes* of $\mathrm{Aut}\,F_n$, and to give them explicit geometric descriptions by using the Outer space.

2. Background and Results

2.1. **Johnson homomorphisms.** Let F_n be a free group of rank $n \geq 2$, and $\mathrm{Aut}\,F_n$ the automrophism group of F_n. Historically, the automorphism groups of free groups were originally begun to study by Dehn and Nielsen in the 1920s who showed that the mapping class group $\mathcal{M}_{g,1}$ of a compact oriented surface $\Sigma_{g,1}$ with one boundary component can be embedded into $\mathrm{Aut}\,F_{2g}$ through the embedding induced from the action of $\mathcal{M}_{g,1}$ on the fundamental group of $\Sigma_{g,1}$. So far, a large number of theories and research techniques to study the mapping class groups have been applied to investigate the automorphism groups of free groups as a comparative study. The group cohomology theory and the Johnson homomorphisms are such typical examples.

In 1965, Andreadakis [1] introduced a certain descending central filtration $\mathcal{A}_n(1) \supset \mathcal{A}_n(2) \supset \cdots$ consisting of normal subgroups of $\mathrm{Aut}\,F_n$. Let H be the abelianization of F_n. The k-th Johnson homomorphism is an embedding of the k-th graded quotient $\mathcal{A}_n(k)/\mathcal{A}_n(k+1)$ of the filtration into the degree k part of the derivation algebra of the free Lie algebra on H. The name "Johnson" comes from Dennis Johnson who studied the group structure of the Torelli group by using the first Johnson homomorphism in the 1980s in a series of his works [20], [21], [22] and [23]. Over the last three decades, good progress was made in the study of the Johnson homomorphisms of mapping class groups through the works of many authors including Morita [33], Hain [17] and many others.

Since each of the Johnson homomorphisms is injective, to determine its image is a basic and important problem. In [33], Morita constructed the trace maps which detect the symmetric tensor products of H in the Johnson cokernels. We expanded the Morita traces in [42], and determined the cokernel of the rational Johnson homomorphsims of $\mathrm{Aut}\,F_n$ associated to the lower central series of IA_n in [44]. Then we apply these results to the study of the Johnson cokernels of the mapping class groups. Enomoto and I gave a representation theoretical description of the Johnson cokernels of $\mathrm{Aut}\,F_n$ in [13], and detect new series in the Johnson cokernels of $\mathcal{M}_{g,1}$ in [14]. These works gave rise to a break-through in the study of the Johnson homomorphisms of $\mathcal{M}_{g,1}$. Recently, our results are studied and extended by several authors, including Morita-Sakasai-Suzuki [37], Conant [8] and Kawazumi-Kuno [26].

Now, we call the filtration $\{\mathcal{A}_n(k)\}$ the Andreadakis-Johnson filtration of $\mathrm{Aut}\,F_n$. Let $\mathcal{A}'_n(1) \supset \mathcal{A}'_n(2) \supset \cdots$ be the lower central series of IA_n. Since the Andreadakis-Johnson filtration is central, $\mathcal{A}_n(k)$ contains $\mathcal{A}'_n(k)$ for each $k \geq 1$. Andreadakis [1] showed that $\mathcal{A}_2(k) = \mathcal{A}'_2(k)$ for any $k \geq 1$, and $\mathcal{A}_3(k) = \mathcal{A}'_3(k)$ for $1 \leq k \leq 3$. It has been conjectured that $\mathcal{A}_n(k) = \mathcal{A}'_n(k)$ for any $n \geq 3$ and $k \geq 1$ by Andreadakis. Today, this conjecture is called the Andreadakis conjecture. For any $n \geq 2$, it is known that $\mathcal{A}_n(2) = \mathcal{A}'_n(2)$ by Bachmuth [4], and that $\mathcal{A}'_n(3)$ has at most finite index in $\mathcal{A}_n(3)$ by Pettet [39]. Recently, for $n = 3$, by using Day and Putman's result in [11], Bartholdi [3, 4] showed that two series differ rationally by computer calculation. Now, the stable Andreadakis conjecture, namely for the case where $n \geq 4$, seems to be open. In [46], we gave the affirmative answer to the Andreadakis conjecture restricted to a certain subgroup of IA_n by showing the injectivity of the restricted Johnson homomorphisms. We will try to consider such approach to attack the original Andreadakis conjecture.

2.2. **Twisted cohomology groups.** To begin with, we briefly recall some facts about the cohomology groups with trivial quotients. In 1986, Culler and Vogtmann [10] introduced a contractible space on which the outer automorphism group of F_n naturally acts properly with finite stabilizers. It is called the Outer space, and is an analogue of the Teichm*ü*ller space for the mapping class group. By using the Outer space, Hatcher and Vogtmann [18] computed $H_4(\mathrm{Aut}\,F_4, \mathbf{Q}) = \mathbf{Q}$. Ohashi [38] computed $H_8(\mathrm{Aut}\,F_6, \mathbf{Q}) = \mathbf{Q}$. Furthermore, Galatius [16] showed that the stable rational cohomology groups of $\mathrm{Aut}\,F_n$ are trivial by using sophisticated homotopy theory. By using Kontsevich's results [27] and [28], Morita [35] constructed a series of unstable homology classes of $\mathrm{Out}\,F_n$ (See also [36].) These homology classes are called the Morita classes. It is known that the first and the second one are non-trivial, and are generators of $H_4(\mathrm{Aut}\,F_4, \mathbf{Q})$ and $H_8(\mathrm{Aut}\,F_6, \mathbf{Q})$ by [36] and [9] respectively. Today, the non-triviality of the higher Morita classes is under intense study by many authors.

My research for twisted cohomology groups is strongly inspired by Morita's and Kawazumi's works. By abuse of language, let H be the integral first homology group of $\Sigma_{g,1}$. Morita [31] and [32] computed the twisted first cohomology groups $H^1(\mathcal{M}_{g,1}, H) = \mathbf{Z}$ and $H^1(\mathcal{M}_{g,1}, \Lambda^3 H) = \mathbf{Z}^{\oplus 2}$, and gave intrinsic meanings to its generators by using the Magnus representations and the first Johnson homomorphisms. With Morita's cocycles and cup products, Kawazumi [24] showed that the Morita-Mumford-Miller classes split into twisted cohomology classes of lower degrees. In other words, the Morita-Mumford-Miller classes can be constructed from twisted cohomology classes. By using the Magnus expansions of free groups, Kawazumi [25] constructed a non-trivial cohomology class in $H^p(\mathrm{Aut}\,F_n, H^*_{\mathbf{Q}} \otimes_{\mathbf{Q}} H_{\mathbf{Q}}^{\otimes p+1})$. In particular, the cocycle for $p = 1$ is the extension of the first Johnson homomorphism.

We [41, 45] computed $H^1(\mathrm{Aut}\,F_n, H_{\mathbf{Q}}) = \mathbf{Q}$ and $H^1(\mathrm{Aut}\,F_n, H^*_{\mathbf{Q}} \otimes_{\mathbf{Q}} \Lambda^2 H_{\mathbf{Q}}) = \mathbf{Q}^{\oplus 2}$ for $n \geq 5$, and showed that the generators are constructed by the Magnus representation and the first Johnson homomorphism respectively. Namely, these are free group analogues of Morita's results for the mapping class groups. Hatcher and Wahl showed that $H_q(\mathrm{Aut}\,F_n, H) = 0$ for $n \geq 3q+9$ in [19]. Recently, Djament and Vespa [12] established a remarkable method to compute the stable (co)homology groups of $\mathrm{Aut}\,F_n$ with twisted coefficients with category theory. By using their method, we can also compute the above results. We also remark that Randal-Williams [40] computed stable twisted cohomology groups of $\mathrm{Aut}\,F_n$ with coefficients in the module obtained by applying a Schur functor to $H_{\mathbf{Q}}$ and $\mathrm{Hom}_{\mathbf{Q}}(H_{\mathbf{Q}}, \mathbf{Q})$. While we can compute the stable cohomology groups of $\mathrm{Aut}\,F_n$ systematically, the problem to give an explicit description of the generators of non-trivial cohomology groups is still open.

Set $H_{\mathbf{Q}} := H \otimes_{\mathbf{Z}} \mathbf{Q}$, and identify $\mathrm{Aut}\,H_{\mathbf{Q}}$ with the general linear group $\mathrm{GL}(n, \mathbf{Q})$ by fixing the standard basis of $H_{\mathbf{Q}}$. We denote by IA_n the kernel of the homomorphism $\mathrm{Aut}\,F_n \to \mathrm{Aut}\,H$ induced from the abelianization map $F_n \to H$. The group IA_n is a free group analogue of the Torelli subgroup of the mapping class group. From a spectral sequence argument, it turns out that the study of the twisted cohomology groups of $\mathrm{Aut}\,F_n$ with coefficients in polynomial representations of $\mathrm{GL}(n, \mathbf{Q})$ is closely related to that of the rational (co)homology groups of IA_n. By the independent works of Cohen-Pakianathan [6, 7], Farb [15] and Kawazumi [25], the first homology group of IA_n is completely determined. However the structure of the higher cohomology groups of IA_n are quite complicated and not determined completely. For example, Bestvina-Bux-Margalit [5] showed that $H_{2n-3}(\mathrm{IA}_n, \mathbf{Q})$ is not finitely generated.

Here we explain the present situation for the second (co)homology group of IA_n briefly. Pettet [39] determined the image of the cup product map $\cup^2 : \Lambda^2 H^1(\mathrm{IA}_n, \mathbf{Q}) \to H^2(\mathrm{IA}_n, \mathbf{Q})$ for any $n \geq 3$. For $n \geq 4$, it is not known whether $H_2(\mathrm{IA}_n, \mathbf{Z})$ is finitely generated or not for any $n \geq 4$. We remark that Day and Putman [11] recently gave a finitely many generating set of $H_2(\mathrm{IA}_n, \mathbf{Z})$ as a $\mathrm{GL}(n, \mathbf{Z})$-module. For the case where $n = 3$, $H_2(\mathrm{IA}_3, \mathbf{Z})$ is not finitely generated by work of Krstić and McCool [29]. Recently, by using the Johnson homomorphisms, we [47] detected a non-trivial GL-irreducible component in $H^2(\mathrm{IA}_3, \mathbf{Q})$ which is not contained in $\mathrm{Im}(\cup^2)$.

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