Palm Calculus, Reallocatable GSMP and Insensitivity Structure

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Abstract This chapter discusses Palm calculus and its applications to various processes including queues and their networks. We aim to explain basic ideas behind this calculus. Since it differs from the classical approach using Markov processes, we scratch from very fundamental facts. The main target of Palm calculus is stationary processes, but we are also interested in its applications to Markov processes. For this, we consider piece-wise deterministic processes and reallocatable generalized Markov processes, RGSMP for short, and characterize their stationary distributions using Palm calculus. In particular, the insensitive structure of RGSMP with respect to the lifetime distributions of its clocks is detailed. Those results are applied to study the insensitive structure of product form queueing networks with respect to service requirement distributions.

1 Introduction

In queues and their networks, it is typical that their time evolutions substantially change only when customers arrive or complete service. That is, essential changes occur only at embedded instants on the continuous time axis. This is a prominent feature of stochastic models for those systems. Such time instants are caused typically by arrivals and departures of customers, and often called discrete events. As is well known, it motivates to use discrete time stochastic processes embedded at those time instants. However, sample paths of such embedded processes may lose key information on the system evolution. Thus, they may be only useful in limited situations. Of course, those sample paths can retain full information of the system if we supplement them with all information between the embedded instants. However, it causes their descriptions to be complicated, and therefore analytical tractability may be lost.

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In this chapter, we introduce a stochastic model to capture those discrete time nature in the continuous time setting. We aim to avoid to simultaneously use different processes for describing discrete events under the continuous time setting. Instead of doing so, we introduce different probability measures on the same sample space. They describe observations at times of interests. That is, they are used to compute characteristics of system states at continuous time or various embedded instants. When such characteristics are expectations of some random quantities, they are called time or event averages, respectively. Under certain stationary assumptions, it is shown that those probability measures are nicely related. This leads to useful relationship among time and event averages. It is referred to as Palm calculus since the probability measures concerning embedded epochs are called Palm distributions.

We apply this Palm calculus to stochastic processes arisen in queues and their networks. However, the Palm calculus itself may not be convenient since it usually involves integrations over the time axis. To ease this, we introduce a rate conservation law, which may be considered as a differential form of the Palm calculus.

Those results by the Palm calculus are very general in the sense that they only require the stationary assumption. However, we may need more specific models to compute characteristics in closed form. For this, we consider a piece-wise deterministic Markov process, PDMP for short. We then specialize it as a reallocatable generalized semi-Markov process, RGSMP for short. We are interested in when RGSMP has a certain nice form of the stationary distribution. It turns out that conditions for this form are closely related to those for a queueing network to have a product form stationary distribution. Under the same conditions, we also consider the conditional mean sojourn time of a customer in a queue or in a network given total amount of his/her work.

In this chapter, we consider analytical tools for studying queueing models rather than just to collect results for applications. However, we divide this chapter into small sections to highlight each topic. We expect the reader has some background in introductory levels of the probability and measure theories. Some descriptions particularly in the first few sections may look too formal since they are different from those in the standard queueing literature. However, arguments are essentially elementary. A problem is probably in the language that we use. So, we provide its details.

2 Shift operator group

When we consider a stochastic process for queueing models, we usually do not explain the probability space, i.e., the triplet of a sample space, a σ-field and a probability measure on it under which the process is defined. This is because the probability space is obviously identified. However, we here start from this very basic description since we will consider different probability measures on the same sample space and σ-field. We also play with different stochastic processes which have a common time axis. For this, it is convenient to implement a time axis in the sample space. Thus, we introduce a time-shift operator on it.

Let $(\Omega,\mathscr{F})$ be a measurable space, and let θ_t be an operator on Ω, i.e., a mapping from Ω to Ω for each real number $t \in \mathbb{R}$. Since we consider a function of time t and analytic operations on it, we need conditions to well define them. Thus, we formally define the operator θ_t in the following way.

Definition 1. For the operator θ_t on Ω for each t, define function φ from $\mathbb{R}\times\Omega$ to Ω by $\varphi(t,\omega) = \theta_t(\omega)$, and let $\mathscr{B}(\mathbb{R})$ be the Borel σ-field on $\mathbb{R}$. If the condition:

(2a) φ is $\mathscr{B}\times\mathscr{F}/\mathscr{F}$-measurable, i.e., $\varphi^{-1}(A) \in \mathscr{B}\times\mathscr{F}$ for all $A \in \mathscr{F}$,

is satisfied, then $\{\theta_t; t \in \mathbb{R}\}$ is said to be measurable. In addition to this, if

(2b) For any $s,t \in \mathbb{R}$, $\theta_s \circ \theta_t = \theta_{s+t}$, namely,

$$\theta_s(\theta_t(\omega)) = \theta_{s+t}(\omega), \qquad \omega \in \Omega,$$

is satisfied, then $\{\theta_t\}$ is said to be a shift operator group. □

Example 1. A natural candidate for the sample space Ω for θ_t to be defined is the set of functions on $\mathbb{R}$, which represents the time axis. For example, let S be a complete, separable metric space, which is called Polish space, and let $\mathscr{B}(S)$ be the Borel σ-field on S. Thus, we have measurable space $(S,\mathscr{B}(S))$. Let Ω be the set of S-valued functions on $\mathbb{R}$ whose discontinuous points are countable at most and are right continuous. Since $\omega \in \Omega$ is a function of time, it can be written as $\{\omega(t)\}$. The σ-field $\mathscr{F}$ can be generated from all the following subsets of Ω for all $n \geq 1, t_i \in \mathbb{R}, B_i \in \mathscr{B}(S)$ for $i = 1,2,\ldots,n$.

$$\{\omega \in \Omega; \omega(t_i) \in B_i, i = 1,2,\ldots,n\}.$$

Then, we can define the shift operator group θ_t through

$$\theta_t(\omega)(s) = \omega(s+t), \qquad s,t \in \mathbb{R}.$$

We refer to this θ_t as a natural shift operator. □

We next define stationarity with respect to the shift operator.

Definition 2. Let $\{\theta_t\}$ be an operator group on a measurable space $(\Omega,\mathscr{F})$. If a probability measure on $(\Omega,\mathscr{F})$ satisfies

$$P(\theta_t^{-1}(A)) = P(A), \qquad t \in \mathbb{R}, A \in \mathscr{F},$$

then P is said to be θ_t-stationary, or stationary with respect to $\{\theta_t\}$. □

Up to now, we have only considered the time to be real valued, i.e., continuous. We are also interested in discrete time. In this case, the shift operator on Ω is denoted by η_n for $n \in \mathbb{Z}$, where $\mathbb{Z}$ is the set of all integers. Obviously, conditions (2a) and (2b) are replaced by

(2c) For any $n \in \mathbb{Z}$, $\eta_n^{-1}(A) \in \mathscr{F}$ for $A \in \mathscr{F}$,
(2d) For any $m, n \in \mathbb{Z}$, $\eta_m \circ \eta_n = \eta_{m+n}$.

Similarly to θ_t, $\{\eta_n; n \in \mathbb{Z}\}$ is said to be measurable if (2c) is satisfied, and said to be an discrete time shift operator group if (2c) and (2d) are satisfied. Furthermore, P is said to be η_n-stationary if $P(\eta_1^{-1}(A)) = P(A)$ for all $A \in \mathscr{F}$.

We next apply the shift operators to functions on Ω, that is, random variables and sample paths. Throughout this chapter, we assume that random variables and states of stochastic processes take values in a Polish space S with the Borel σ-field $\mathscr{B}(S)$. However, in our applications, it is sufficient to assume that S is a finite dimensional Euclid space, i.e., real valued vector space. As usual, we also assume that a stochastic process is right-continuous with left limits.

Definition 3. Let $\{\theta_t\}$ be an operator group on $(\Omega, \mathscr{F})$, and let X be a random variable on this measurable space. Define random variable $X \circ \theta_t$ as

$$X \circ \theta_t(\omega) = X(\theta_t(\omega)), \qquad \omega \in \Omega.$$

With this notation, a stochastic process $\{X(t)\}$ defined on $(\Omega, \mathscr{F})$ is said to be consistent with θ_t if the following condition is satisfied.

$$X(s) \circ \theta_t = X(s+t), \qquad s, t \in \mathbb{R}$$

Similarly, we define the consistency of a discrete time process $\{X_n\}$ with respect to a discrete time shift operator η_n by

$$X_m \circ \eta_n = X_{m+n}, \qquad m, n \in \mathbb{Z}.$$ □

The following definitions of stationary processes are standard.

Definition 4. A stochastic process $\{X(t)\}$ is said to be stationary under P if , for each fixed $n \geq 1, t_i \in \mathbb{R}, B_i \in \mathscr{B}(S)$ for $i = 1, 2, \ldots, n$,

$$P(X(t_i + u) \in B_i, i = 1, 2, \ldots, n)$$

is unchanged for all $u \in \mathbb{R}$. Similarly, the stationarity of a discrete time process $\{X_n\}$ under P_0 is defined, where P_0 is another probability measure on $(\Omega, \mathscr{F})$. □

The next lemma is immediate from the definitions of the shift operators, the consistency and the stationarity.

Lemma 1. If P is a θ_t-stationary probability measure and if $\{X(t)\}$ is consistent with $\{\theta_t\}$, then $\{X(t)\}$ is a continuous time stationary process under P. Similarly, if P_0 is η_n-stationary and if $\{X_n\}$ is consistent with $\{\eta_n\}$, then $\{X_n\}$ is a discrete time stationary process under P_0.

It should be noted that we are concerned with different probability measures in Lemma 1, but the underlying measurable spaces, i.e., the sample space and the set of all events, are the same. This allows us to directly relate $X(t)$ to X_n through $\omega \in \Omega$.

Example 2. How one can create a sample space Ω with operations θ_t and η_n for a queueing model ? Let us consider this problem by a small example. Since an actual system usually starts at some fixed time, we assume that a queueing system starts with no customer at time $c_0 \equiv 0$. Assume that this system is closed with no customer just before time c_1. We represents the evolution of this system on the time interval by a function f from $[c_0, c_1)$ to S, where S is a finite dimensional real vector space. At time c_1, the system restarts and repeats the same trajectory until time $c_2 \equiv 2c_1$. If the system operates in this manner continuously, then we have a trajectory ω_0^+:

$$\omega_0^+(t) = \sum_{n=1}^{\infty} f(t - c_{n-1}) 1(c_{n-1} \leq t < c_n), \qquad t \geq 0,$$

where $c_n = nc_1$, and $1(\cdot)$ is the indicator function of the statement "$\cdot$", i.e., it takes 1 (or 0) if the statement is true (or false). We next shift the starting time c_0 to $-kc_1$ for positive integer k, and letting k to infinity, we have the double sided trajectory ω_0:

$$\omega_0(t) = \sum_{n=-\infty}^{+\infty} f(t - c_{n-1}) 1(c_{n-1} \leq t < c_n), \qquad t \geq 0.$$

Let $\omega_0^{(u)}(t) = \omega_0(t-u)$ for $u \in [0, c_1)$, and define the sample space Ω as

$$\Omega = \{\omega_0^{(u)}; u \in [0, c_1)\}.$$

Since this sample space is the set of functions on $\mathbb{R}$ and closed under time shift, we have a natural shift operator θ_t. Furthermore, let

$$\eta_n \circ \omega_0^{(u)}(t) = \omega_0(t - c_n), \qquad u \in [0, c_1), t \in \mathbb{R}, n \in \mathbb{Z}.$$

Then, $\{\eta_n\}$ is a discrete time shift operator group. Obviously, this operator group is stationary for any probability measure. Since f is a deterministic function, a probability measure on $(\Omega, \mathscr{F})$ can be determined by that on $[0, c_1) \times \mathscr{B}([0, c_1))$. In particular, if this distribution is uniform on $[0, c_1)$, then P is stationary with respect to the natural shift operator group $\{\theta_t\}$. □

This example is trivial in the sense that all sample paths are generated by a single function $\{\omega_0(t)\}$. Nevertheless, it can be used a prototype of the probability space for shift operators. For example, if we change $c_n - c_{n-1}$ to be *i.i.d.* (that is, independently and identically distributed) random variables and functions on the intervals $[c_{n-1}, c_n)$ to be also *i.i.d.* random functions, then, using the same uniform distribution, we can construct the probability measure which is stationary with respect to the natural shift operator group $\{\theta_t\}$. This construction will be systematically studied in the following two sections.

3 Point processes

We introduce a process for randomly chosen discrete time instants on the time axis. This process is called a point process, and will be used to generate a discrete time process, called embedded process, from a continuous time process. Thus, the point process will make a bridge between continuous time and discrete time embedded processes.

Definition 5. N is called a point process on the line if it satisfies the following two conditions.

(3a) N is an integer-valued and locally finite random measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$, that is, each $\omega \in \Omega$, $N(\cdot)(\omega)$ is an integer-valued measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ such that $N(B)(\omega) < \infty$ for any bounded $B \in \mathscr{B}(\mathbb{R})$ and $\omega \in \Omega$.
(3b) For all $n \geq 1$, $B_i \in \mathscr{B}(\mathbb{R})$ and $n_i \in \mathbb{Z}_+ \equiv \{0, 1, \ldots\}$ for $i = 1, 2, \ldots, n$,

$$\{N(B_i) = n_i, i = 1, 2, \ldots, n\} \in \mathscr{F}.$$

Furthermore, if $N(\{t\}) \leq 1$ for all $t \in \mathbb{R}$, then N is said to be simple.

If we remove the assumption that $N(B)$ is integer-valued, we can similarly define a random measure, but we do not need this generality in this chapter except for Section 11.

Similar to the case of a stochastic process, we define the operation of θ_t to point process N as

$$N(B) \circ \theta_t(\omega) = N(B)(\theta_t(\omega)), \qquad t \in \mathbb{R}, \omega \in \Omega, B \in \mathscr{B}(\mathbb{R}).$$

In what follows, we assume

$$N(B) \circ \theta_t = N(t + B), \qquad t \in \mathbb{R}, B \in \mathscr{B}(\mathbb{R}),$$

where $t + B = \{t + u; u \in \mathbb{R}\}$. In this case, N is said to be consistent with θ_t. This is meant that N and θ_t have a common time axis similar to the case of a stochastic process.

The stationarity of point process N is defined similar to that of a stationary process. That is, N is said to be stationary if, for all $n \geq 1$, $k_1, \ldots, k_n \in \mathbb{Z}_+$ and $B_1, \ldots, B_n \in \mathscr{B}(\mathbb{R})$

$$P(N(t + B_1) = k_1, N(t + B_2) = k_2, \ldots, N(t + B_n) = k_n)$$

is unchanged for all $t \in \mathbb{R}$.

The next lemma is a point process version of Lemma 1.

Lemma 2. If P is θ_t-stationary and if N is consistent with θ_t, then N is stationary under P.

Note that $N((0,t])$ with $t > 0$ can be considered to be a counter for random events that occur in the time interval $(0,t]$. Because of this, a point process is also called a counting process. From this viewpoint, it may be convenient to define the time when the discrete events occur. Let

$$T_n = \begin{cases} \inf\{t > 0; N((0,t]) \geq n\}, & n \geq 1, \\ \sup\{t \leq 0; N((t,0]) \geq 1-n\}, & n \leq 0. \end{cases}$$

This T_n is said to be the n-th counting time of N. Since $T_n = T_{n+1}$ may occur for $n \neq 0$, T_n may not be strictly increasing in n. We thus have

$$\ldots \leq T_0 \leq 0 < T_1 \leq \ldots \tag{1}$$

From the definition of T_n, we have

$$N(B) = \sum_{n=-\infty}^{+\infty} 1(T_n \in B), \qquad B \in \mathscr{B}(\mathbb{R}),$$

and the right-hand side of this equation can be written as

$$\int_{-\infty}^{+\infty} 1(u \in B) N(du).$$

Remind that $1(\cdot)$ is the indicator function of the statement "$\cdot$" (see its definition in Example 2).

In the remaining part of this section, we assume that N is a simple point process. In this case, T_n is strictly increasing in n. For convenience, let

$$N(t) = \begin{cases} N((0,t]), & t > 0, \\ -N((t,0]), & t \leq 0. \end{cases}$$

Since N is simple, $N(T_n) = n$ for $n \geq 1$. Note that N is assumed to be consistent with the shift operator θ_t. This yields, for $n \geq 1$ and $s > 0$,

$$\begin{aligned} T_n \circ \theta_s &= \inf\{t > 0; N \circ \theta_s((0,t]) = n\} \\ &= \inf\{t > 0; N((s, s+t]) = n\} \\ &= T_{N(s)+n} - s. \end{aligned} \tag{2}$$

For $s \leq 0$ and $n \leq 0$, we can get the same formula (2). In particular, letting $s = T_m$ in (2), $N(s) = m$ yields

$$T_n \circ \theta_{T_m} = T_{m+n} - T_m.$$

Thus, θ_{T_n} shifts the counting number. From this observation, we define η_n for $n \geq 1$ as

$$\eta_n(\omega) = \theta_{T_n(\omega)}(\omega), \qquad \omega \in \Omega. \tag{3}$$

Lemma 3. For simple point process N, $\{\eta_n; n \in \mathbb{Z}\}$ is a discrete time shift operator group on $(\Omega, \mathscr{F})$.

Proof. From (3), we have

$$\begin{aligned}\eta_n \circ \eta_m &= \theta_{T_n \circ \eta_m} \circ \eta_m \\ &= \theta_{T_{m+n} - T_m} \circ \theta_{T_m} \\ &= \theta_{T_{m+n}} = \eta_{m+n}.\end{aligned}$$

Hence, η_n satisfies (2d), which corresponds with (ii) of Definition 1. To see condition (2c), let $\Phi(\omega) = (T_n(\omega), \omega)$ for $\omega \in \Omega$, which is a function from Ω to $\mathbb{R} \times \Omega$. This function is $\mathscr{F}/(\mathscr{B}(\mathbb{R}) \times \mathscr{F})$-measurable. We next let $\varphi((t,\omega)) = \theta_t(\omega)$, which is a $(\mathscr{B}(\mathbb{R}) \times \mathscr{F})/\mathscr{F}$ measurable function from $\mathbb{R} \times \Omega$ to Ω. Hence, $\eta_n = \varphi \circ \Phi$ is $\mathscr{F}/\mathscr{F}$-measurable, which completes the proof. □

Thus, we get the discrete time shift operator η_n from the continuous time shift operator θ_t. The η_n describes the time shift concerning the point process N. The following observation is intuitively clear, but we give a proof since it is a key of our arguments.

Lemma 4. Suppose that stochastic process $\{X(t); t \in \mathbb{R}\}$ and simple point process N are consistent with θ_t. Define discrete time process $\{Y_n; n \in \mathbb{Z}\}$ by $Y_n = X(T_n)$ for the counting times $\{T_n\}$ of N. Then, $\{Y_n\}$ is consistent with η_n.

Proof. From (3) and the fact that $X(t)$ is consistent with θ_t, we have

$$\begin{aligned}Y_n \circ \eta_m &= X \circ \eta_m (T_n \circ \eta_m) \\ &= X \circ \theta_{T_m}(T_{m+n} - T_m) \\ &= X(T_{m+n} - T_m + T_m) = Y_{m+n}.\end{aligned}$$

Thus, Y_n is indeed consistent with η_n. □

We next add information to the counting times T_n of N. This information is called mark, and the resulted process is called a marked point process. This process is formally defined in the following way. Let N be a simple point process which is consistent with θ_t, and let $\{T_n\}$ be its counting times. Further, let $\{Y_n\}$ be a discrete time process with state space by $\mathscr{K}$, where $\mathscr{K}$ is assumed to be a Polish space. Then, $\Psi \equiv \{(T_n, Y_n)\}$ is called a marked point process, and Y_n is said to be a mark at the n-th point T_n.

Define a random measure M_Ψ on $(\mathbb{R} \times \mathscr{K}, \mathscr{B}(\mathbb{R}) \times \mathscr{B}(\mathscr{K}))$ as

$$M_\Psi(B,C) = \sum_{n=-\infty}^{+\infty} 1(T_n \in B, Y_n \in C), \qquad B \in \mathscr{B}(\mathbb{R}), C \in \mathscr{B}(\mathscr{K})$$

where $\mathscr{B}(\mathscr{K})$ is the Borel σ-field on $\mathscr{K}$. If we have, for all $t \in \mathbb{R}$,

$$M_\Psi(B,C) \circ \theta_t = M_\Psi(B+t, C), \qquad B \in \mathscr{B}(\mathbb{R}), C \in \mathscr{B}(\mathscr{K}),$$

then Ψ is said to be consistent with θ_t. In particular, if $\{Y_n\}$ is consistent with $\eta_n \equiv \theta_{T_n}$, then M_Ψ is consistent with θ_t. In fact, from (2), we have

$$\{T_n \circ \theta_t \in B\} = \{T_{N(t)+n} \in B+t\}.$$

On the other hand, $Y_n = Y_0 \circ \eta_n$ yields

$$\begin{aligned} Y_n \circ \theta_t(\omega) &= Y_0(\theta_{T_n \circ \theta_t(\omega)}(\theta_t(\omega)) \\ &= Y_0(\theta_{T_{N(t)(\omega)+n}(\omega)}(\omega)) = Y_{N(t)(\omega)+n}(\omega). \end{aligned}$$

Hence, the claim is proved by

$$M_\Psi(B,C) \circ \theta_t = \sum_{n=-\infty}^{+\infty} 1(T_{N(t)+n} \in B+t, Y_{N(t)+n} \in C) = M_\Psi(B+t,C).$$

We also define the stationarity of Ψ similar to N. Namely, if, for all n and $B_i \in \mathscr{B}(\mathbb{R}), C_i \in \mathscr{B}(\mathscr{K})$ $(i = 1,2,\ldots,n)$

$$P(M_\Psi(B_1+t,C_1), M_\Psi(B_2+t,C_2), \ldots, M_\Psi(B_n+t,C_n))$$

is unchanged for all $t \in \mathbb{R}$, then Ψ is said to be stationary under P. Similarly to Lemma 2, we have the following fact, whose proof is left to the reader.

Lemma 5. If P is θ_t-stationary and if marked point process Ψ is consistent with θ_t, then Ψ is stationary under P.

4 Palm distribution

One may wonder whether P can be θ_t-stationary and η_n-stationary simultaneously. This may look possible, but it is not true. To see this, we consider time shift operations under P assuming that it is θ_t-stationary.

We first note that the distribution of $\{T_n + t; n \in \mathbb{Z}\}$ is unchanged under P for any $t \in \mathbb{R}$ by the θ_t-stationarity. Hence, shifting the time axis does not change the probability measure. We now shift the time axis subject to the uniform distribution on the unit interval $[0,u]$ independently of everything else for a large but fixed number $u > 0$. The choice of this u is not essential in the subsequent arguments, but thinking of the large u may be more appearing. The renumbered $\{T_n\}$ still has the same distribution because P is unchanged. Under such time shifting, T_n is changed to T_1 if the time interval $(T_{n-1}, T_n]$ contains the origin. Because the longer time interval has more chance to include the origin, $T_1 - T_0$ would be differently distributed from $T_{n+1} - T_n$ for $n \neq 0$, where the numbers n of T_n are redefined after the time shifting. On the other hand, if P is η_n-stationary, then we have

$$(T_1 - T_0) \circ \eta_n = T_{n+1} - T_n,$$

which implies that $T_1 - T_0$ and $T_{n+1} - T_n$ are identically distributed. Hence, it is impossible that P is θ_t-stationary and η_n-stationary simultaneously.

This observation motivates us to introduce a convenient probability measure for η_n.

Definition 6. Suppose that P is θ_t-stationary, point process N is consistent with θ_t and has a finite intensity $\lambda \equiv N((0,1)$. Define nonnegative valued set function P_0 on $\mathscr{F}$ as

$$P_0(A) = \lambda^{-1} E\Big(\int_0^1 1_{\theta_u^{-1}(A)} N(du)\Big), \qquad A \in \mathscr{F}, \tag{4}$$

where 1_A is the indicator function of set A, i.e., $1_A(\omega) = 1(\omega \in A)$. Note that $1_{\theta_u^{-1}(A)}(\omega) = 1_A(\theta_u(\omega)) = (1_A \circ \theta_u)(\omega)$. Then, it is easy to see that P_0 is a probability measure on $(\Omega, \mathscr{F})$, which is referred to as a Palm distribution concerning N. Note that N is not necessarily simple in this definition. □

Remark 1. (4) is equivalent to that, for any function f from Ω to $\mathbb{R}$ which is $\mathscr{F}/\mathscr{B}(\mathbb{R})$-measurable and either bounded or nonnegative, the following equation holds.

$$E_0(f) = \lambda^{-1} E\Big(\int_0^1 f \circ \theta_u N(du)\Big), \tag{5}$$

where E_0 represents the expectation concerning P_0. □

Let $A = \{T_0 = 0\}$ in (2), then, for any $u \in \mathbb{R}$,

$$\begin{aligned}\theta_u^{-1}(A) &= \{\omega \in \Omega; \theta_u(\omega) \in A\}\\ &= \{T_0 \circ \theta_u = 0\} = \{T_{N(u)} = u\}.\end{aligned}$$

Furthermore, since $N(\{u\}) \geq 1$ implies $T_{N(u)} = u$, we have, from (4),

$$P_0(T_0 = 0) = \lambda^{-1} E(N((0,1])) = 1.$$

Hence, N has a mass at the origin under P_0. This means that P_0 is a conditional probability measure given $N(\{0\}) \geq 1$.

The following result is a key to relate P_0 to P when P is θ_t-stationary, where P_0 is the Palm distribution concerning N. The formula (6) below is referred to as either Campbell's or Mecke's formula in the literature.

Lemma 6. Let $\{X(t)\}$ be a nonnegative valued stochastic process, then we have

$$E\Big(\int_{-\infty}^{+\infty} X(u) \circ \theta_u N(du)\Big) = \lambda E_0\Big(\int_{-\infty}^{+\infty} X(u) du\Big). \tag{6}$$

Proof. Define a nonnegative random variable f as

$$f = \int_{-\infty}^{+\infty} X(s) ds = \int_{-\infty}^{+\infty} X(s+u) ds.$$

Substituting this into (5), we obtain (6) through the following computations.

$$\begin{aligned}\lambda E_0\Big(\int_{-\infty}^{+\infty} X(s)ds\Big) &= E\Big(\int_0^1 \Big(\int_{-\infty}^{+\infty} X(s+u)\circ\theta_u ds\Big)N(du)\Big)\\ &= \int_{-\infty}^{+\infty} E\Big(\int_{-\infty}^{+\infty} 1(0<u<1)X(s+u)\circ\theta_u N(du)\Big)ds\\ &= \int_{-\infty}^{+\infty} E\Big(\int_{-\infty}^{+\infty} 1(0<u<1)X(s+u)\circ\theta_{u+s} N(du+s)\Big)ds\\ &= \int_{-\infty}^{+\infty} E\Big(\int_{-\infty}^{+\infty} 1(0<u-s<1)X(u)\circ\theta_u N(du)\Big)ds\\ &= E\Big(\int_{-\infty}^{+\infty}\int_{u-1}^{u} dsX(u)\circ\theta_u N(du)\Big)\\ &= E\Big(\int_{-\infty}^{+\infty} X(u)\circ\theta_u N(du)\Big),\end{aligned}$$

where the third equation is obtained using the fact that P is θ_t-stationary. □

It is notable that $\{X(t)\}$ in Lemma 6 is not necessarily consistent with θ_t, and therefore it is not necessary stationary under P. The essence of (6) lies in the shift invariance of P and Lebesgue measure on $\mathbb{R}$.

Example 3 (Little's formula). We derive a famous formula due to Little [20] using Lemma 6. Consider a service system, where arriving customers get service and leave. Let T_n be the n-th arrival time, where T_n is also defined for $n \le 0$. Let N be a point process generated by these T_n, and let θ_t be a shift operator on Ω. We assume that N is consistent with θ_t. Let U_n be the sojourn time of n-th customer in system. We also assume that $\{U_n; n \in \mathbb{Z}\}$ is consistent with η_n defined by (3).

Then, the number of customers $L(t)$ in system at time t is obtained as

$$L(t) = \sum_{n=-\infty}^{+\infty} 1(T_n \le t < T_n + U_n).$$

Assume that $L(t)$ is finite for all $t \in \mathbb{Z}$. Let $N(s) = N((0,s])$, then $T_n \circ \theta_s = T_{N(s)+n} - s$, $U_n = U_0 \circ \eta_n$ and $\eta_n \circ \theta_s = \theta_{T_{N(s)+n}}$. Hence,

$$L(t)\circ\theta_s = \sum_{n=-\infty}^{+\infty} 1(T_{N(s)+n} \le s+t < T_{N(s)+n} + U_n) = L(s+t),$$

so $\{L(t)\}$ is consistent with θ_t. Assume that P is θ_t-stationary and $\lambda \equiv E(N((0,1]))$ is finite. Thus, $\{L(t)\}$ is a stationary process under P.

Let $X(u) = 1(T_0 \le -u < T_0 + U_0)$, then we have

$$\begin{aligned}&\int_{-\infty}^{+\infty} X(u)du = U_0,\\ &\int_{-\infty}^{+\infty} X(u)\circ\theta_u N(du) = \sum_{n=-\infty}^{+\infty} 1(0 \le -T_n < U_n) = L(0).\end{aligned}$$

Hence, Lemma 6 yields

$$E(L(0)) = \lambda E_0(U_0). \tag{7}$$

This is called Little's formula. □

Let $\Psi = \{(T_n, Y_n)\}$ be a marked point process which is consistent with θ_t, and let N be a point process generated by $\{T_n\}$ with a finite intensity $\lambda \equiv E(N(0,1])$. In Lemma 6, for each fixed $B \in \mathscr{B}(\mathbb{R}), C \in \mathscr{B}(\mathscr{K})$, let

$$X(u) = 1(u \in B, Y_0 \in C), \qquad t > 0.$$

Since

$$\int_{-\infty}^{+\infty} X(u) \circ \theta_u N(du) = \sum_{n=-\infty}^{+\infty} 1(T_n \in B, Y_n \in C),$$

(6) yields

$$E(M_\Psi(B,C)) = \lambda |B| E_0(Y_0 \in C), \qquad t > 0, \tag{8}$$

where $|B| = \int_B du$, that is, if B is an interval, then $|B|$ is the length of B. From this, we have known that measure $E(M_\Psi(B,C))$ on $(\mathbb{R} \times \mathscr{K}, \mathscr{B}(\mathbb{R}) \times \mathscr{B}(\mathscr{K}))$ is the product of Lebesgue measure and the distribution of Y_0 under P_0.

Another interesting conclusion of Lemma 6 is the orderliness of a simple point process.

Corollary 1. Assume N is a simple point process that is consistent with θ_t and has a finite intensity λ, then

$$\lim_{t \downarrow 0} \frac{1}{t} E(N(T_1,t]; T_1 < t) = 0, \tag{9}$$

and therefore

$$\lim_{t \downarrow 0} \frac{1}{t} P(\theta_{T_1}^{-1}(A), T_1 \le t) = \lambda P_0(A), \qquad A \in \mathscr{F}. \tag{10}$$

Proof. Since $N(T_1,t] = \int_0^t 1(u > T_1) N(du)$ for $t > T_1$ and $T_1 \circ \theta_{-u} = T_{-N(-u,0]+1} + u$ for $u > 0$, Lemma 6 yields

$$\begin{aligned} E(N(T_1,t]; t > T_1) &= E\left(\int_0^t (1(u > T_1) \circ \theta_{-u}) \circ \theta_u N(du) \right) \\ &= E\left(\int_0^t 1(T_{-N(-u,0]+1} < 0) \circ \theta_u N(du) \right) \\ &= \lambda E_0 \left(\int_0^t 1(T_{-N(-u,0]+1} < 0) du \right). \end{aligned}$$

Since $P_0(T_0 = 0) = 1$, the indicator function $1(T_{-N(-u,0]+1} < 0)$ vanishes as $u \downarrow 0$ (see also (1)). Hence, dividing both sides of the above equation by t and letting $t \downarrow 0$, we have (9) by the mean value theorem of an elementary calculus. To get (10), we apply Lemma 6 for $X(u) = 1(0 < u \le t)1_A$ for $t \ge 0$ and $A \in \mathscr{F}$, then

$$\begin{aligned}\lambda t P_0(A) &= \left(\int_0^t 1_A \circ \theta_u N(du)\right)\\ &= P(\theta_{T_1}^{-1}(A), T_1 \le t) + E\left(\int_{T_1+}^t 1_A \circ \theta_u N(du)\right).\end{aligned}$$

Dividing both sides by t and letting $t \downarrow 0$, (9) yields (10), where the plus sign at T_1 in the integral indicates that the lower point T_1 is not included in the integral region. □

Note that (10) gives another way to define the Palm distribution P_0. This may be more intuitive, but the limiting operation may not be convenient in addition to the restriction to a simple point process.

5 Inversion formula

We next present basic properties of Palm distribution P_0 and to give a formula to get back P from P_0 directly.

Theorem 1. Suppose that N is a simple point process which is consistent with θ_t and has a finite and non-zero intensity $\lambda = E(N((0,1]))$. If P is θ_t-stationary, then P_0 is η_n-stationary. Hence, $\{Y_n\}$ of Lemma 4 is a discrete time stationary process under P_0. Furthermore, P is obtained from P_0 by

$$P(A) = \lambda E_0\Big(\int_0^{T_1} 1_{\theta_u^{-1}(A)} du\Big), \qquad A \in \mathscr{F}. \tag{11}$$

Proof. The first half is obtained if $P_0(\eta_1^{-1}(A)) = P_0(A)$ holds. We prove this using the definition of the Palm distribution (4). Because $\eta_1 = \theta_{T_1}$ and (2) implies

$$\theta_{T_1} \circ \theta_u(\omega) = \theta_{T_{N(u)+1}(\omega)-u}(\theta_u(\omega)) = \theta_{T_{N(u)+1}}(\omega),$$

we have

$$\theta_u^{-1}(\eta_1^{-1}(A)) = \{\eta_1 \circ \theta_u \in A\} = \{\theta_{T_{N(u)+1}} \in A\}.$$

Applying this to (4), we have

$$
\begin{aligned}
P_0(\eta_1^{-1}(A)) &= \lambda^{-1} E\Big(\sum_{n=1}^{N(1)} 1(\theta_{T_{n+1}} \in A)\Big) \\
&= \lambda^{-1} \left(E\Big(\sum_{n=1}^{N(1)} 1(\theta_{T_n} \in A)\Big) + P(\theta_{T_{N(1)+1}} \in A) - P(\theta_{T_1} \in A) \right).
\end{aligned}
$$

Since $\theta_{T_1} \circ \theta_1 = \theta_{T_{N(1)+1}}$ and P is θ_t-stationary, we have

$$
P(\theta_{T_{N(1)+1}} \in A) = P(\theta_{T_1} \in A).
$$

Thus, we get $P_0(\eta_1^{-1}(A)) = P_0(A)$. We next prove (11). For this, let

$$
X(u) = 1(N((-u,0)) = 0, u > 0) 1_{\theta_u^{-1}(A)},
$$

then

$$
X(u) \circ \theta_u = 1(N((0,u)) = 0, u > 0) 1_A = 1(0 < u \le T_1) 1_A.
$$

Substituting this in the left-hand side of (6), we have

$$
E\Big(\int_{-\infty}^{+\infty} X(u) \circ \theta_u N(du)\Big) = E(1_A) = P(A),
$$

since $N(du)$ has a unit mass at $u = T_1$. On the other hand, the right-hand side of (6) becomes

$$
\begin{aligned}
E_0\Big(\int_{-\infty}^{+\infty} X(u) du\Big) &= E_0\Big(\int_{T_{-1}}^{0} 1_{\theta_u^{-1}(A)} du\Big) \\
&= E_0\Big(\int_{T_{-1}\circ\eta_1}^{0} 1_{(\theta_u \circ \eta_1)^{-1}(A)} du\Big)
\end{aligned}
$$

since P_0 is η_n-stationary. Note that $T_{-1} \circ \eta_1 = T_0 - T_1$ and $\theta_u \circ \eta_1 = \theta_{u+T_1}$. Hence, changing the integration variable from u to $u + T_1$ in the last term and using the fact that $P_0(T_0 = 0) = 1$, we have (11). □

An excellent feature of the definition (4) of Palm distribution P_0 is that it computes the conditional distribution given the event with probability zero, using neither limiting operations nor conditional expectation as a Radon-Nikodym derivative of the measure theory.

From (11), P is obtained from P_0. In this sense, it is called an inversion formula. Another interpretation of (11) is that it represents the time average of the indicator function of A from $T_0 = 0$ to T_1. Since $\{T_n - T_{n-1}\}$ is stationary under P_0, (11) is also called a cycle formula.

The next result shows that the inverse of Theorem 1 holds.

Theorem 2. Suppose that a simple point process N is consistent with θ_t, a measure P_0 on $(\Omega, \mathscr{F})$ satisfies that $0 < E_0(T_1) < \infty$ for $T_1 \equiv \sup\{u > 0; N(0,u) = 0\}$. Let $\lambda =$

$1/E_0(T_1)$. If P_0 is η_n-stationary, then P defined by (11) is a θ_t-stationary probability measure. Furthermore, $E(N((0,1]) = \lambda$ and (4) holds for these P_0 and P.

Proof. It is easy to see that P is a probability measure. Let us show $P(\theta_t^{-1}(A)) = P(A)$ for $A \in \mathscr{F}$ and for all $t \in \mathbb{R}$. From the definition (11) of P, we have

$$\begin{aligned} P(\theta_t^{-1}(A)) &= \frac{1}{E_0(T_1)} E_0\Big(\int_0^{T_1} 1_{\theta_{t+u}^{-1}(A)} du\Big) \\ &= \frac{1}{E_0(T_1)} E_0\Big(\int_t^{t+T_1} 1_{\theta_u^{-1}(A)} du\Big) \\ &= \frac{1}{E_0(T_1)} E_0\Big(\int_0^{T_1} 1_{\theta_u^{-1}(A)} du + \int_{T_1}^{t+T_1} 1_{\theta_u^{-1}(A)} du - \int_0^t 1_{\theta_u^{-1}(A)} du\Big). \end{aligned}$$

Since P_0 is η_n-stationary, we have $E_0(X) = E_0(X \circ \eta_1)$ for a nonnegative random variable X. Hence, we have

$$\begin{aligned} E_0\Big(\int_0^t 1_{\theta_u^{-1}(A)} du\Big) &= E_0\Big(\int_0^t 1_{(\theta_u \circ \eta_1)^{-1}(A)} du\Big) \\ &= E_0\Big(\int_0^t 1_{\theta_{u+T_1}^{-1}(A)} du\Big) \\ &= E_0\Big(\int_{T_1}^{t+T_1} 1_{\theta_u^{-1}(A)} du\Big). \end{aligned}$$

Thus, we get $P(\theta_t^{-1}(A)) = P(A)$, so P is θ_t-stationary. It remains to prove (4), where P is defined by (11). Using this P, define $P_0^\dagger$ as

$$P_0^\dagger(A) = \frac{1}{E(N((0,1]))} E\Big(\int_0^1 1_{\theta_u^{-1}(A)} N(du)\Big), \qquad A \in \mathscr{F}.$$

Thus, the proof is completed if we show $P_0 = P_0^\dagger$. This equality is equivalent to that, for nonnegative and bounded random variable f,

$$E_0(f) = \frac{1}{E(N((0,1]))} E\Big(\int_0^1 f \circ \theta_u N(du)\Big).$$

From (11),

$$E\Big(\int_0^1 f \circ \theta_u N(du)\Big) = \lambda E_0\Big(\int_0^{T_1} \Big(\int_0^1 f \circ \theta_u N(du)\Big) \circ \theta_s ds\Big). \tag{12}$$

Hence, we need to verify that

$$E_0(f) = \frac{\lambda}{E(N((0,1]))} E_0\Big(\int_0^{T_1} \Big(\int_0^1 f \circ \theta_u N(du)\Big) \circ \theta_s ds\Big). \tag{13}$$

Let us prove (13). We first compute the inside of the expectation in the right-hand side of (13). Since Lebesgue integration is unchanged by shifting the integration variable, we have

$$\begin{aligned}\int_0^{T_1}\Big(\int_0^1 f\circ\theta_u N(du)\Big)\circ\theta_s ds &= \int_0^{T_1}\Big(\int_0^1 f\circ\theta_{u+s} N(du+s)\Big)ds\\ &= \int_0^{T_1}\Big(\int_s^{s+1} f\circ\theta_u N(du)\Big)ds\\ &= \int_{-\infty}^{+\infty}\int_0^{T_1} 1(s<u<s+1)ds(f\circ\theta_u)N(du)\\ &= \sum_{n=-\infty}^{+\infty}\int_0^{T_1} 1(s<T_n<s+1)ds(f\circ\theta_{T_n}).\end{aligned}$$

Since P_0 is stationary with respect to $\eta_n = \theta_{T_n}$, the expectation concerning P_0 in the left-hand side of (13) becomes

$$\begin{aligned}&\sum_{n=-\infty}^{+\infty} E_0\Big(\int_0^{T_1} 1(s<T_n<s+1)ds(f\circ\theta_{T_n})\Big)\\ &\qquad = \sum_{n=-\infty}^{+\infty} E_0\Big(\Big(\int_0^{T_1} 1(s<T_n<s+1)dsf\circ\eta_n\Big)\circ\eta_{-n}\Big)\\ &\qquad = \sum_{n=-\infty}^{+\infty} E_0\Big(\int_0^{T_{1-n}-T_{-n}} 1(s<T_0-T_{-n}<s+1)dsf\Big)\\ &\qquad = E_0\Big(\sum_{n=-\infty}^{+\infty}\int_{T_{-n}}^{T_{1-n}} 1(s<T_0<s+1)dsf\Big)\\ &\qquad = E_0\Big(\int_{-\infty}^{+\infty} 1(T_0-1<s<T_0)dsf\Big) = E_0(f).\end{aligned}$$

Thus, (13) is obtained if $\lambda = E(N((0,1]))$. The latter is obtained from (12) with $f\equiv 1$ and the above computations. This completes the proof. □

In applications, particularly in queueing networks, we frequently meet the situation that point process N is the superposition of m point processes $N_1,\ldots,N_m$ all of which are consistent with θ_t for some $m\geq 2$. Namely,

$$N(B) = N_1(B)+\ldots+N_m(B), \qquad B\in\mathscr{B}(\mathbb{R}).$$

Assume that P is θ_t-stationary, and $\lambda\equiv E(N((0,1])<\infty$ with $\lambda\neq 0$. For $i=1,2,\ldots,m$, let $\lambda_i = E(N_i((0,1]))$, and denote Palm distribution concerning N_i by P_i. Then, from the definition of Palm distribution, it is easy to see that

$$\lambda P_0(A) = \sum_{i=1}^m \lambda_i P_i(A), \qquad A\in\mathscr{F}. \tag{14}$$

This decomposition of the Palm measure is shown to be useful in applications (see Section 10).

6 Detailed Palm distribution

We have mainly considered Palm distribution when point process N is simple. The definition of Palm distribution itself does not need for N to be simple. However, if N is not simple, η_n defined by $\eta_n = \theta_{T_n}$ can not properly handle events that simultaneously occur in time. We need to differently define Palm distribution for this case. To this end, we consider a pair of T_n and η_n, where η_n is a discrete time operator group. In what follows, point process N is assumed to be generated by $\{T_n\}$, and N is not necessarily simple.

Definition 7. Let $\{T_n; n \in \mathbb{Z}\}$ be a nondecreasing sequence of random variables, which generate point process N, and let $\{\eta_n\}$ be the discrete time shift operator group. If

$$\{(T_n, \eta_n)\} \circ \theta_t = \{(T_n - t, \eta_n)\}, \qquad n \in \mathbb{Z}, t \in \mathbb{R}$$

holds, then $\{(T_n, \eta_n)\}$ is said to be a θ_t-consistent marked point process with shift operator η_n.

Definition 8. Suppose that P be θ_t-stationary and $\{(T_n, \eta_n)\}$ is a θ_t-consistent marked point process with η_n, where $\ldots \le T_{-1} \le T_0 \le 0 < T_1 \le T_2 \le \ldots$, and $\lambda \equiv E(N((0,1])) < \infty$. Then, we define $\overline{P}_0$ as

$$\overline{P}_0(A) = \lambda^{-1} E\Big(\sum_{n=1}^{N((0,1])} 1_{\eta_n^{-1}(A)} \Big), \qquad A \in \mathscr{F}. \tag{15}$$

This $\overline{P}_0$ is a probability measure on $(\Omega, \mathscr{F})$, and called a detailed Palm distribution concerning $\{T_n\}$.

Detailed Palm distribution $\overline{P}_0$ is different from Palm distribution P_0 concerning N if N is not simple. Nevertheless, we can extend the results in the previous two sections to detailed Palm distribution. Since their proofs are similar to the previous ones, we present their versions for Theorems 1 and 2 without proof.

Theorem 3. Suppose that P is θ_t-stationary, $\{(T_n, \eta_n)\}$ is a θ_t consistent marked point process, and $\lambda \equiv E(N((0,1])) < \infty$. Then, detailed Palm distribution $\overline{P}_0$ is η_n-stationary. Furthermore, P is recovered from $\overline{P}_0$ by

$$P(A) = \lambda \overline{E}_0 \Big(\int_0^{T_1} 1_{\theta_u^{-1}(A)} du \Big), \qquad A \in \mathscr{F}. \tag{16}$$

where $\overline{E}_0$ represents the expectation concerning $\overline{P}_0$. Conversely, suppose that probability measure $\overline{P}_0$ on $(\Omega, \mathscr{F})$ satisfies $0 < E_0(T_1) < \infty$ and $\overline{P}_0$ is η_n-stationary for a given discrete time shift operator group $\{\eta_n\}$. Let $\lambda = 1/\overline{E}_0(T_1)$ and define P by (16), Then, P is a probability measure on $(\Omega, \mathscr{F})$ which is θ_t-stationary, and we have $E(N((0,1]) = 1/E_0(T_1) = \lambda$. For these $\overline{P}_0$ and P, we have (15).

We next consider another way to define the detailed Palm distribution. For this, we use a simple point process which have masses at the same time instant as N. Denote this point process by N^*. Namely, N^* is defined as

$$N^*(B) = \int_B \frac{1}{N(\{u\})} N(du), \qquad B \in \mathscr{B}(\mathbb{R}). \tag{17}$$

This point process is said to be a simple version of N. Let T_n^* be the n-th counting point of N^*.

Lemma 7. Under the same assumptions of Definition 8, let $\lambda^* = E(N^*((0,1]))$ and denote the Palm distribution concerning N^* by P_0^*, then we have

$$\overline{P}_0(A) = \frac{\lambda^*}{\lambda} E_0^* \Big(\sum_{n=1}^{N((0,T_1^*])} 1_{\eta_n^{-1}(A)} \Big), \qquad A \in \mathscr{F}, \tag{18}$$

where E_0^* represents the expectation concerning P_0^*.

Proof. Let $\eta_n^* = \theta_{T_n^*}$. Then, from the definition of Palm distribution P_0^*, we have, for random variable f,

$$E_0^*(f) = (\lambda^*)^{-1} E \left(\sum_{n=1}^{N^*((0,1])} f \circ \eta_n^* \right).$$

In this equation, letting $f = \sum_{\ell=1}^{N((0,T_1^*])} 1_{\eta_\ell^{-1}(A)}$, the expectation in the right-hand side becomes

$$\begin{aligned} E & \left(\sum_{n=1}^{N^*((0,1])} \left(\sum_{\ell=1}^{N((0,T_1^*])} 1_{\eta_\ell^{-1}(A)} \right) \circ \eta_n^* \right) \\ &= E \left(\sum_{n=1}^{N^*((0,1])} \left(\sum_{\ell=1}^{N((T_n^*,T_{n+1}^*])} 1_{(\eta_{N((0,T_n^*])+\ell})^{-1}(A)} \right) \right) \\ &= E \left(\sum_{n=1}^{N^*((0,1])} \left(\sum_{\ell=N((0,T_n^*])+1}^{N((0,T_{n+1}^*])} 1_{(\eta_\ell)^{-1}(A)} \right) \right) \\ &= E \left(\sum_{n=1}^{N((0,1])} 1_{\eta_\ell^{-1}(A)} \right). \end{aligned}$$

Since the last term equals $\lambda \overline{P}_0(A)$ by (15), we have (18). ☐

Example 4. Let us consider batch arrival queueing system. Let T_n^* be the n-th batch arrival time. We then number all customers sequentially including those who are in the same batch. Let T_n be the n-th arrival time of a customer in this sense. Let B_n the size of the batch arriving at time T_n^*, and let J_n be the number of the n arriving customer counted in his batch. That is, $J_n = \max\{\ell \geq 1; T_n = T_{n-\ell+1}\}$. In particular, $J_0 = B_0$. Let $\eta_n = \theta_{T_n}$, then

$$J_0 \circ \eta_n = \max\{\ell \geq 1; 0 = T_{n-\ell+1} - T_n\} = J_n.$$

Hence, if $\{T_n\}$ is η_n-stationary under $\overline{P}_0$, then we have, for any $n \in \mathbb{Z}$,

$$\begin{aligned}\overline{P}_0(J_n = k) &= \overline{P}_0(J_0 = k) \\ &= \frac{\lambda^*}{\lambda} E_0^* \Big(\sum_{n=1}^{B_1} 1(J_0 \circ \eta_n = k) \Big) = \frac{1}{E_0^*(B_1)} P_0^*(B_1 \geq k),\end{aligned}$$

because $J_n = k$ for some positive $n \leq B_1$ if and only if $B_1 \geq k$. This means that a randomly chosen customer is counted in its batch subject to the so called stationary excess distribution of B_1 under P_0^*. □

7 Time and event averages

In this section, we give interpretations of stationary P and P_0 through sample averages. It will be shown that these sample averages are unchanged under both of them. This means that both probability measures can be used for computing stationary characteristics when either one of them is taken for a probability model. Furthermore, sample averages may be only a way to identify system parameters. Thus, the unchanged sample averages are particularly important in applications of Palm calculus. This is something like to use two machines for production which is originally designed for one machine. Throughout this section, we assume

(7a) Measurable space $(\Omega, \mathscr{F})$ is equipped with a shift operator group $\{\theta_t; t \in \mathbb{R}\}$.
(7b) There exists a simple point process N which is consistent with θ_t, and the discrete time shift operator group $\{\eta_n; n \in \mathbb{R}\}$ is defined by (3).
(7c) There exists a probability measure P on $(\Omega, \mathscr{F})$ which is θ_t-stationary and satisfies $\lambda \equiv E(N(0,1]) < \infty$.

By these assumptions, Palm distribution P_0 is well defined for N. Let

$$\mathscr{I} = \{A \in \mathscr{F}; \theta_t^{-1}(A) = A \text{ holds for all } t \in \mathbb{R}\},$$

then $\mathscr{I}$ is σ-field on Ω. Since $\theta_t^{-1}(\mathscr{I}) = \mathscr{I}$, this $\mathscr{I}$ is called an invariant σ-field concerning θ_t. Similarly, an invariant σ-field concerning η_n is defined.

Lemma 8. For the shift operator group $\{\eta_n; n \in \mathbb{Z}\}$, define $\mathscr{I}_0$ as

$$\mathscr{I}_0 = \{A \in \mathscr{F}; \eta_1^{-1}(A) = A\}.$$

Then, $\mathscr{I}_0 = \mathscr{I}$, and $\mathscr{I}_0$ is the invariant σ-field concerning η_n.

Proof. From the definition, $\mathscr{I}_0$ is clearly η_n-invariant, i.e., $\eta_n^{-1}(\mathscr{I}_0) = \mathscr{I}_0$, Hence, we only need to prove $\mathscr{I}_0 = \mathscr{I}$. Choose $A \in \mathscr{I}$. Since $\theta_t^{-1}(A) = A$, we have

$$\begin{aligned}\eta_1^{-1}(A) &= \{\omega \in \Omega; \theta_{T_1(\omega)}(\omega) \in A\} \\ &= \cup_{t\in\mathbb{R}}\{T_1 = t\} \cap \theta_t^{-1}(A) \\ &= \cup_{t\in\mathbb{R}}\{T_1 = t\} \cap A = A.\end{aligned}$$

Thus, we have $A \in \mathscr{I}_0$. Conversely, let $A \in \mathscr{I}_0$. Since $\eta_n \circ \eta_1 = \eta_{n+1}$, we have $\eta_n^{-1}(A) = A$ for any $n \in \mathbb{Z}$. If $T_{n-1} \le t < T_n$, then

$$\eta_1 \circ \theta_t(\omega) = \theta_{T_1(\theta_t)}(\theta_t(\omega)) = \theta_{T_n(\omega)-t}(\theta_t(\omega)) = \theta_{T_n}(\omega) = \eta_n(\omega).$$

Hence, for any $t \in \mathbb{R}$,

$$\begin{aligned}\theta_t^{-1}(A) &= \cup_{n=-\infty}^{+\infty}\{T_{n-1} \le t < T_n\} \cap \theta_t^{-1}(A) \\ &= \cup_{n=-\infty}^{+\infty}\{T_{n-1} \le t < T_n\} \cap \theta_t^{-1}(\eta_1^{-1}(A)) \\ &= \cup_{n=-\infty}^{+\infty}\{T_{n-1} \le t < T_n\} \cap (\eta_1 \circ \theta_t)^{-1}(A) \\ &= \cup_{n=-\infty}^{+\infty}\{T_{n-1} \le t < T_n\} \cap \eta_n^{-1}(A) \\ &= \cup_{n=-\infty}^{+\infty}\{T_{n-1} \le t < T_n\} \cap A = A.\end{aligned}$$

Thus, we have $A \in \mathscr{I}$, which completes the proof. □

For $A \in \mathscr{I}$, $P(A) \neq P_0(A)$ in general, but we have the following result.

Lemma 9. For $A \in \mathscr{I}$, $P_0(A) = 1$ if and only if $P(A) = 1$.

Proof. Since $\theta_u^{-1}(A) = A$ for $A \in \mathscr{I}$, from (4) and (11), it follows that

$$P_0(A) = \lambda^{-1}E(1_A N((0,1])), \qquad P(A) = \frac{1}{E_0(T_1)}E_0(1_A T_1).$$

Hence, if $P_0(A) = 1$, then $E(1_A N((0,1])) = E(N((0,1])$, which implies $P_0(A) = 1$. Conversely, if $P(A) = 1$, then $E_0(1_A T_1) = E_0(T_1)$, which implies $P(A) = 1$. □

Clearly, the equivalence in this lemma is not true for $A = \{T_0 = 0\}$. Thus, it may not be true for $A \notin \mathscr{I}$.

Definition 9. Suppose that probability measure P on $(\Omega, \mathscr{F})$ is θ_t-stationary, and let $\mathscr{I}$ be the invariant σ-field concerning θ_t. If either $P(A) = 0$ or $P(A) = 1$ for each $A \in \mathscr{I}$, then P is said to be ergodic concerning θ_t. As for P_0 and $\{\eta_n; n \in \mathbb{Z}\}$, we similarly define P_0 to be ergodic concerning η_n.

From Lemma 9, the following result is immediate.

Lemma 10. Assume that probability measure P on $(\Omega, \mathscr{F})$ is θ_t-stationary. Then, P_0 is ergodic concerning η_n if and only if P is ergodic concerning θ_t.

The next result is a version of law of large numbers, and called ergodic theorem. We omit its proof, which can be found in text books on probability theory (see, e.g., [5]).

Theorem 4. Let $\{\eta_n; n \in \mathbb{R}\}$ be the shift operator group on $(\Omega, \mathscr{F})$, and let $\{Y_n\}$ be a discrete time stochastic process which is consistent with η_n. Let P_0 be a η_n-stationary probability measure on $(\Omega, \mathscr{F})$, and denote the expectation concerning P_0 by E_0. If $E_0(|Y_0|) < \infty$, then we have, under P_0,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{\ell=1}^{n} Y_\ell = E_0(Y_0 | \mathscr{I}_0) \tag{19}$$

with probability one, where $\mathscr{I}_0$ is the invariant σ-field concerning η_n, and $E_0(Y_0|\mathscr{I}_0)$ is the conditional expectation of Y_0 given $\mathscr{I}_0$.

This theorem leads to the following results.

Corollary 2. Suppose (7a), (7b), (7c). Then, $\{Y_n\}$ is consistent with η_n. If $E_0(|Y_0|) < \infty$, then we have, under both of P_0 and P,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{\ell=1}^{n} Y_\ell = E_0(Y_0 | \mathscr{I}) \tag{20}$$

with probability one. Furthermore, $\{X(t)\}$ is consistent with θ_t, and if $E(|X_0|) < \infty$, then we have, under both of P_0 and P,

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t X(u) du = E(X(0) | \mathscr{I}) \tag{21}$$

with probability one.

Remark 2. Sample averages in (20) and (21) are referred to as event and time averages, respectively. If P or P_0 is ergodic, then $\mathscr{I}$ consists of events which have either probability zero or probability one. Hence, the conditional expectations in (20) and (21) reduce to the unconditional ones. If Y_n (or $X(t)$) is nonnegative, we do not need the condition that $E_0(Y_0) < \infty$ (or $E(X(0)) < \infty$). To see this, we first apply $\min(a, X(t))$ to (20) for fixed constant $a > 0$, then let $a \uparrow \infty$. □

Proof. Since P_0 is η_n-stationary, (20) holds under P_0 with probability one by Theorem 4. Let A be the set of all $\omega \in \Omega$ such that (20) holds. Since $P_0(A) = 1$, Lemma 9 yields $P(A) = 1$. Hence, (20) holds under P with probability one. As for (21), if it holds under P with probability one, we similarly get it under P_0. To get (21) under P, we let

$$\eta_n = \theta_n, \qquad Y_n = \int_{n-1}^{n} X(u) du.$$

Then, it can be shown that Theorem 4 yields (21) under P since P is also stationary concerning this η_n. □

Similarly to this corollary, we can prove the next result.

Corollary 3. Under the same assumptions of Corollary 2, we have, under both of P_0 and P,

$$\lim_{t\to\infty} \frac{N((0,t])}{t} = \lim_{t\to\infty} \frac{N((-t,0])}{t} = E(N((0,1])|\mathscr{I}) \tag{22}$$

holds with probability one.

Corollary 2 and Corollary 3 are convenient to compute sample averages since we can choose either P or P_0 to verify them for both of P and P_0.

Example 5 (Little's formula in sample averages). We consider the same model discussed in Example 3. Consider a service system, where arriving customers get service and leave. For simplicity, we here assume that all T_n are distinct, i.e., not more than one customers arrive at once. If either P or P_0 is ergodic, then, by Corollary 2, we can rewrite Little's formula (7) in terms of time and event averages as

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t L(u)du = \lambda \lim_{n\to\infty} \frac{1}{n} \sum_{\ell=1}^{n} U_n,$$

which holds with probability one under both of P and P_0.

The simplicity condition on N is not essential in the above arguments. We only need to replace E_0 by the expectation $\overline{E}_0$ of the detailed Palm distribution. □

8 Rate conservation law

In the previous sections, we have considered two kinds of expectations by a stationary probability measure and its Palm distributions. Computations using those distributions is called Palm calculus. This calculus gives relationship among characteristics observed at arbitrary points in time and those in embedded epochs. Typical formulas are (4), (6) and (11). They can be applied to a stochastic process. However, they are generally not so convenient for studying a complex systems such as queueing networks. Dynamics of those systems is typically driven by differential operators such as generators and transition rate matrices of Markov processes or chains while formulas in Palm calculus concern integrations over time in general.

In this section, we consider a convenient form of Palm calculus for stochastic processes. As we shall see, this form can be used to characterize the stationary distribution when they are Markov processes. However, in this section, we do not assume any Markovian assumption, but use the same framework as Palm calculus. So, our assumptions is basically of the stationary of processes. Extra assumptions that we need is the smoothness of a sample path except jump instants, which is not so restrictive in queueing applications.

Throughout this section, we assume (7a), (7b), (7c) of Section 7. Since these assumptions are important in our arguments, we restate it as follows.

(8a) There is a probability space $(\Omega, \mathscr{F}, P)$ such that shift operator group $\{\theta_t; t \in \mathbb{R}\}$ is defined on Ω and P is θ_t-stationary. There is a simple point process N which is consistent with θ_t and satisfies $\lambda \equiv E(N(0,1]) < \infty$.

We further assume the following three conditions on a stochastic process of interest.

(8b) $\{X(t)\}$ is a real valued continuous time stochastic process such that it is consistent with θ_t and right-continuous with left-limits for each $t \in \mathbb{R}$, that is, $\lim_{\varepsilon \downarrow 0} X(t+\varepsilon)(\omega) = X(t)(\omega)$, and $X(t-)(\omega) \equiv \lim_{\varepsilon \downarrow 0} X(t-\varepsilon)(\omega)$ exists for each $t \in \mathbb{R}$ and each $\omega \in \Omega$.

(8c) At all t, $X(t)$ has the right-hand derivative $X'(t)$. That is,

$$X'(t) \equiv \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}(X(t+\varepsilon) - X(t))$$

exists and finite.

(8d) N includes all the times when $X(t)$ is discontinuous in t. That is, for each $B \in \mathscr{B}(\mathbb{R})$, $N(B) = 0$ implies $\sum_{t \in B} 1(X(t) \neq X(t-)) = 0$.

All these conditions are sufficient to hold with probability one for our arguments. However, we rather prefer that they hold for all $\omega \in \Omega$ for simplicity.

Lemma 11. Under assumptions (8a), (8b), (8c) and (8d), $\{X(t)\}$ and $\{X'(t)\}$ are stationary processes, and N is a stationary simple point process. If $E(X'(0))$ and $E_0(X(0-)) - X(0))$ are finite, then

$$E(X'(0)) = \lambda E_0(X(0-) - X(0)). \tag{23}$$

Proof. From (8a) and (8b), $X(t)$ and N are clearly stationary. From the consistency on $X(t)$ and the differentiability (8c), it follows that

$$\begin{aligned} X'(t) \circ \theta_u &= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}(X(t+\varepsilon) - X(t)) \circ \theta_u \\ &= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}(X(t+u+\varepsilon) - X(t+u)) = X'(t+u). \end{aligned}$$

Hence, $X'(t)$ is also consistent with θ_t, so $\{X'(t)\}$ is a stationary process. From (8a), $\lambda < \infty$, so $N(0,1]$ is finite with probability one. Hence, from (8d), we have

$$X(t) = X(0) + \int_0^t X'(u)du + \int_0^t (X(0) - X(0-)) \circ \theta_u N(du), \qquad t > 0. \tag{24}$$

We tentatively suppose that $E(X(t))$ is finite, which implies that $E(X(t)) = E(X(0))$ due to the stationarity of $X(t)$. Since $E(X'(0))$ is finite and $X'(t)$ is stationary, we have

$$E\Big(\int_0^1 X'(u)du\Big) = \int_0^1 E(X'(u))du = E(X'(0)).$$

Hence, taking the expectations of (24) for $t = 1$, we have

$$E(X'(0)) + E\Big(\int_0^1 (X(0) - X(0-)) \circ \theta_u N(du)\Big) = 0.$$

This yields (23) by applying the Palm calculus in Definition 6.

We next remove the assumption that $E(X(t))$ is finite. To this end, for each integer $n \geq 1$, define function f_n as

$$f_n(x) = \begin{cases} x, & |x| \leq n, \\ -\frac{1}{2}(\max(0, n+1-x))^2 + \frac{1}{2} + n, & x > n, \\ \frac{1}{2}(\max(0, n+1+x))^2 - \frac{1}{2} - n, & x < -n. \end{cases}$$

This function is bounded since $|f_n(x)| \leq \frac{1}{2} + n$ for all $x \in \mathbb{R}$. Furthermore, it has the right-hand derivative:

$$f_n'(x) = \begin{cases} 1, & -n \leq x < n, \\ n+1-x, & n \leq x < n+1, \\ n+1+x, & -n-1 \leq x < -n, \\ 0, & \text{otherwise}. \end{cases}$$

From this, it is easy to see that $f'(x)$ is continuous in x, and $|f_n'(x)| \leq 1$. Furthermore,

$$|f_n(x) - f_n(y)| \leq \int_y^x |f_n'(z)| dz \leq |x - y|.$$

Let $Y_n(t) = f_n(Y(t))$, then $Y_n(t)$ is bounded and

$$\begin{aligned} |Y_n'(t)| &= |Y'(t) f_n'(Y(t))| \leq |Y'(t)|, \\ |Y_n(t-) - Y_n(t)| &\leq |Y(t-) - Y(t)|, \end{aligned}$$

so $E(Y_n'(0))$ and $E_0(Y_n(0-) - Y_n(0))$ are finite by the assumptions. Hence, from the first part of this proof, (23) is obtained for $Y_n(t)$. Namely, we have

$$E(Y_n'(0)) = \lambda E_0(Y_n(0-) - Y_n(0)).$$

Let $n \to \infty$ in this equation noting that $f_n(x) \to x$ and $f_n'(x) \uparrow 1$ as $n \to \infty$. Then, the bounded convergence theorem yields (23) since $|Y_n'(t)|$ and $|Y_n(t-) - Y_n(t)|$ are uniformly bounded in n. □

Remark 3. From the proof of Lemma 11, we can see that, if $E(X(0))$ is finite, then the finiteness of either $E(X'(0))$ or $E_0(X(0-) - X(0))$ is sufficient to get (23). Here, we note that the finiteness of $E(X)$ for a random variable X is equivalent to the finiteness of $E(|X|)$ due to the definition of the expectation.

Formula (23) is referred to as a rate conservation law, RCL for short. In fact, it can be interpreted that the total rate due to continuous and discontinuous changes of $X(t)$ are kept zero. In application of the RCL, $X(t)$ is a real or complex valued function of a multidimensional process. Let $\mathbf{X}(t) = (X_1(t), \ldots, X_d(t))$ be such a process for a positive integer d, and let f be a partially differentiable function from $\mathbb{R}^d$ to $\mathbb{R}$. In this case, we put $X(t) = f(\mathbf{X}(t))$. Then, Lemma 11 yields

Corollary 4. Let d be a positive integer, and let f be a continuously partially differentiable function from $\mathbb{R}^d$ to $\mathbb{R}$. If each $X_\ell(t)$ instead of $X(t)$ satisfies conditions (8a), (8b), (8c) and (8d) for all $\ell = 1,2,\ldots,d$ and if $E(f(\mathbf{X}(0)))$ and $E(f(\mathbf{X}(0)) - f(\mathbf{X}(0-)))$ are finite, then we have

$$E\left(\mathbf{X}'(0)\nabla f(\mathbf{X}(0))\right) = \lambda E_0(f(\mathbf{X}(0-)) - f(\mathbf{X}(0))). \tag{25}$$

where $\mathbf{X}'(0) = (X_1'(0),\ldots,X_d'(0))$ and $\nabla f(\mathbf{x}) = (\frac{\partial}{\partial x_1} f(\mathbf{x}),\ldots \frac{\partial}{\partial x_d} f(\mathbf{x}))^{\mathrm{T}}$ for $\mathbf{x} = (x_1,\ldots,x_d)$.

This is the most convenient form for queueing applications even for $d = 1$ because we can choose any f as far as it is differentiable and satisfies the finiteness conditions on the expectations. We refer this type of f as a test function.

Example 6 (Workload process). We consider the workload process with a state dependent processing rate r and an input generated by a point process N and a sequence of input works $\{S_n\}$. The workload $V(t)$ at time $t \geq 0$ is defined as

$$V(t) = V(0) + \sum_{n=1}^{N(t)} S_n - \int_0^t r(V(u))1(V(u) > 0)du,$$

where $r(x)$ is a nonnegative valued right-continuous function on $[0,\infty)$. If $r(x) \equiv 1$, then $V(t)$ is the workload process of a single server queue.

Let T_n be the n-th point of N. We assume that N has a finite intensity λ, $\{(T_n,S_n)\}$ is consistent with θ_t (see Definition 7), and $\{V(t)\}$ is a stationary process under P. Let f be a bounded and continuously differentiable function on $\mathbb{R}$. Since $X(t) \equiv V(t)$ satisfies all the conditions of Corollary 4 and $X'(t) = r(t)$, we have

$$E(r(V(0))f'(V(0))1(V(0) > 0)) = \lambda E_0(f(V(0-)) - f(V(0-) + S_0)). \tag{26}$$

Using Corollary 4, we can generalize this formula for a multidimensional workload process with a multidimensional input. The virtual waiting time vector of a many server queue is such an example. □

Another useful form is obtained from decomposing the point process N.

Corollary 5. Under the assumptions of Lemma 11, suppose that the point process N is decomposed into m point processes $N_1,N_2,\ldots,N_m$ all of which are consistent with θ_t for some $m \geq 2$. Namely,

$$N(B) = N_1(B) + N_2(B) + \ldots + N_m(B), \qquad B \in \mathscr{B}(\mathbb{R}).$$

Further suppose that $\lambda_i \equiv E(N_i((0,1]))$ is finite for all $i = 1,2,\ldots,m$, and denote the expectation concerning Palm distribution with respect to N_i by E_i. If

$E_i(X(0)), E_i(X(0))$ are finite for $i = 1,2,\ldots,m$ and if $E(X'(0))$ is finite, then we have

$$E(X'(0)) = \sum_{i=1}^{m} \lambda_i E_i(X(0-) - X(0)). \tag{27}$$

Proof. From the assumption on the finiteness of λ_i, N has the finite intensity $\lambda \equiv \sum_{i=1}^{m} \lambda_i$. Denote the expectation concerning Palm distribution with resect to N by E_0, then

$$\begin{aligned}\lambda E_0(X(0-) - X(0)) &= E\left(\int_0^1 (X(u-) - X(u))N(du)\right) \\ &= \sum_{i=1}^{m} E\left(\int_0^1 (X(u-) - X(u))N_i(du)\right) \\ &= \sum_{i=1}^{m} \lambda_i E_i(X(0-) - X(0)).\end{aligned}$$

Hence, Lemma 11 concludes (27). □

We have been only concerned with the simple point process N for the rate conservation law (23). If N is not simple, we can use its simple version N^* defined by (17). However, we must be careful about the changes of $X(t)$ at atoms of N^*. That is, we need to use the detailed Palm distribution $\overline{P}_0$ instead of P_0 in (23) when we consider embedded events at T_n.

9 PASTA: a proof by the rate conservation low

In queueing problems, we frequently require to compute system characteristics observed at different points in time. In this section, we demonstrate how we can use the rate conservation law to the observation of a customer arriving subject to a Poisson process, where point process N is called a Poisson process if $\{t_{n+1} - t_n; n \in \mathbb{Z}\}$ is the sequence of independently, identically and exponential random variables for the n-th increasing instants instant t_n of N. The following result is called PASTA, which is the abbreviation of "Poisson Arrivals See Time Averages", which is coined by Wolff [35].

Theorem 5 (PASTA). Under the assumptions (8a), (8b), (8c) and (8d), let N_0 be the Poisson process which is consistent with θ_t and finite intensity λ_0, and denote the expectation of Palm distribution with respect to N_0 by E_0. If $\{X(u); u < t\}$ is independent of $\{N_0([t,t+s]); s \geq 0\}$ for all t, then we have, for all measurable function f such that $E(f(X(0)))$ and $E_0(f(X(0-)))$ are finite,

$$E(f(X(0))) = E_0(f(X(0-))). \tag{28}$$

Proof. Similarly to Lemma 11 and the standard arguments for approximation of functions in expectation, it is sufficient to prove (28) for the f such that f is differentiable and its derivative is bounded. Let $R_0(t) = \sup\{u \geq 0; N_0(t,t+u] = 0\}$. That is, $R_0(t)$ is the remaining time to count the next point of N_0 at time t. For nonnegative number s, let

$$Y(t) = f(X(t))e^{-sR_0(t)}, \qquad t \in \mathbb{R}.$$

Then, clearly $Y(t)$ is bounded and consistent with θ_t . Since $R_0'(t) = -1$, the right-hand derivative of $Y(t)$ is computed as

$$Y'(t) = (X'(t)f'(X(t)) + sf(X(0)))e^{-sR_0(t)}.$$

We next define a point process N_1 as

$$N_1(B) = N(B) - \max(N(B),N_0(B)), \qquad B \in \mathscr{B}(\mathbb{R}),$$

where it is noted that N is the point process given in the assumption (8d). Obviously, N_0 and N_1 are simple, and do not have a common point. Furthermore, N_1 is consistent with θ_t and has the intensity $\lambda_1 \equiv E(N_1(0,1]) < \lambda < \infty$. Thus, we can apply Corollary 5 (see (a) in Remark 3). Let $\varphi(s) = E(e^{-sR_0(0)})$, then

$$\begin{aligned} E\left(X'(0)f'(X(0)) + sf(X(0))\right)\right)\varphi(s) \\ &= \lambda_0\left(E_0(f(X(0-))) - E_0(f(X(0)))\varphi(s)\right) \\ &\quad + \lambda_1 E_1\left(f(X(0-)) - f(X(0))\right)\varphi(s). \end{aligned} \tag{29}$$

By the memoryless property of the exponential distribution, we have $\varphi(s) = \lambda_0/(s+\lambda_0)$. Hence, letting $s \to \infty$ in (29) and using the fact that $s\varphi(s) \to \lambda_0$ and $\varphi(s) \to 0$, we obtain (28). □

Remark 4. If the reader is familiar with a martingale and the fact that the Poisson process N of Theorem 5 can be expressed as

$$N((0,t]) = \lambda t + M(t), \qquad t \geq 0,$$

where $M(t)$ is an integrable martingale with respect to the filtration $\sigma(X(u);u \leq t)$. See Section 11 for the definition of the martingale. Then, (28) is almost immediate from the definition of the Palm distribution since

$$\int_0^t f(X(u-))dM(u)$$

is also a martingale, and therefore its expectation vanishes. This proof is less elementary than the above proof.

Let us apply Theorem 5 together with the rate conservation law to the $M/G/1$ queue, which is a single server queue with the Poisson arrivals and independently and identically distributed requirements.

Example 7 (Pollaczek-Khinchine formula). We consider the special case of the workload process in Example 6. We here further assume that the processing rate $r(x) \equiv 1$, N is the Poisson process with rate $\lambda > 0$ and $\{S_n\}$ is a sequence of $i.i.d.$ (independent and identically distributed) random variables which are independent of everything else.

Thus, we consider the workload process $V(t)$ of the $M/G/1$ queue. The service discipline of this queue can be arbitrary as long as the total service rate is always unit and the server can not be idle when there is a customer in the system. This process is known to be stable, that is, its stationary distribution exists if and only if

$$\rho \equiv \lambda E(S_1) < 1.$$

We assume this stability condition. Then, $\{V(t)\}$ is a stationary process under the stationary distribution. For nonnegative number θ, let $f(x) = e^{-\theta x}$ for $x \geq 0$, then (26) yields

$$\begin{aligned}\theta E(e^{-\theta V(0)} 1(V(0) > 0)) &= \lambda E_0(e^{-\theta V(0-)} - e^{-\theta (V(0-)+S_0)}) \\ &= \lambda E(e^{-\theta V(0-)})(1 - E(e^{-\theta S_1})),\end{aligned} \tag{30}$$

where we have used the $i.i.d.$ assumption of S_n and Theorem 5 to get the second equality. We can rewrite the left-hand side as

$$\theta E(e^{-\theta V(t)}) - \theta P(V(0) = 0).$$

Let $\varphi(\theta) = E(e^{-\theta V(t)})$ and $g(\theta) = E(e^{-\theta S_1})$, then we have, from (30),

$$\theta \varphi(\theta) - \theta P(V(0) = 0) = \lambda \varphi(\theta)(1 - g(\theta)).$$

Since $\varphi(\theta) \to 1$ and $\frac{1-g(\theta)}{\theta} \to -g'(0) = E(S_1)$ as $\theta \downarrow 0$. we have $P(V(0) = 0) = 1 - \rho$, dividing the above formula by θ and letting $\theta \downarrow 0$. Thus, we have the Laplace-transform of $V(0)$ under the stationary assumption.

$$\varphi(\theta) = \frac{\theta(1-\rho)}{\theta - \lambda(1 - g(\theta))}, \qquad \theta > 0. \tag{31}$$

This formula is independently obtained by Pollaczek and Khinchine, and called Pollaczek-Khinchine formula. □

In this example, if the processing rate $r(x)$ is not a constant, then it is generally hard to get the stationary distribution of the workload in any form. We show that there is an exceptional case in the following example.

Example 8 (State dependent service). We again consider the workload process $V(t)$ in Example 6 under the assumptions of Example 7 except for the processing rate $r(x)$, which may be arbitrary. Thus, we consider the $M/G/1$ workload process with state dependent processing rate. In this case, the stability condition for $V(t)$ is complicated, so we here just assume that the stationary distribution exists. Similar to

(30), we have

$$\theta E(r(V(0))e^{-\theta V(0)}1(V(0)>0)) = \lambda E(e^{-\theta V(0-)})(1-E(e^{-\theta S_1})). \tag{32}$$

From this equation, it is generally hard to get $\varphi(\theta) \equiv E(e^{-\theta V(0-)})$ for a given $g(\theta) \equiv E(e^{-\theta S_1})$ except for the case that $r(x)$ is a constant.

We thus consider the special case that $r(x) = a + bx$ for nonnegative constant a and positive constant b. In this case, the left-hand side of (32) can be written as

$$\theta E((a+bV(0))e^{-\theta V(0)}1(V(0)>0)) = \theta(a\varphi(\theta) - b\varphi'(\theta)) - a\theta P(V(0)=0).$$

Hence, we have the following differential equation from (32).

$$-\frac{1}{b}\left(a - \lambda \frac{1-g(\theta)}{\theta}\right)\varphi(\theta) + \varphi'(\theta) = -\frac{a}{b}P(V(0)=0) \tag{33}$$

For $\theta \geq 0$, let

$$h(\theta) = -\frac{1}{b}\int_0^\theta \left(a - \lambda \frac{1-g(u)}{u}\right) du.$$

Then, the solution of (33) with the boundary condition $\varphi(0) = 1$ is obtained as

$$\varphi(\theta) = e^{-h(\theta)}\left(1 - \frac{a}{b}P(V(0)=0)\int_0^\theta e^{h(u)}du\right).$$

To determine $P(V(0)=0)$, Note that $h(\infty) \equiv \lim_{\theta\to\infty} h(\theta) = -\infty$ if $a > 0$ while $h(\infty) = +\infty$ if $a = 0$. Hence, if $a = 0$, then

$$\varphi(\theta) = e^{-h(\theta)}.$$

If $a > 0$, then we must have $1 = \frac{a}{b}P(V(0)=0)\int_0^\infty e^{h(u)}du$, which concludes $P(V(0)=0) = \frac{b}{a}\left(\int_0^\infty e^{h(u)}du\right)^{-1}$. Hence, we finally have, for $a > 0$,

$$\varphi(\theta) = e^{-h(\theta)}\int_\theta^\infty e^{h(u)}du \left(\int_0^\infty e^{h(u)}du\right)^{-1}.$$

It may be interesting to see the mean workload $E(V(0)) = -\varphi'(0)$, which is

$$E(V(0)) = \begin{cases} \frac{1}{b}\rho, & a = 0, \\ \frac{1}{b}(\rho - a) + \left(\int_0^\infty e^{h(u)}du\right)^{-1}, & a > 0. \end{cases}$$

Note that these computations are not valid for $b = 0$. □

10 Relationship among the queueing length processes observed at different points in time

The rate conservation is powerful for complicated systems. This is exemplified for the system queue length process, i.e., the total number of customer in system, under a very general setting. Here, the queueing system is meant a service system with arrivals and departures. Let N_a and N_d be point processes composed of arrival and departure instants, respectively. We here allow those point processes to be not simple. Then, the the system queue length $L(t)$ at time t is defined as

$$L(t) = L(0) + N_a((0,t]) - N_d((0,t]), \qquad t \geq 0.$$

Note that customers who leave the system immediately after their arrivals without any service are counted as departure.

Theorem 6. For a queue system with arrival point process N_a, departure point process N_d and the system queue length $L(t)$, assume that P is θ_t-stationary and $N_a, N_d, L(t)$ are consistent with θ_t, that is, $(N_a, N_d, \{L(t)\})$ is jointly stationary. Let N_a^* and N_d^* be the simple versions of N_a and N_d. If $\lambda_a^* \equiv E(N_a^*((0,1])$ and $\lambda_d^* \equiv E(N_d^*((0,1]))$ are finite, then, for $n \in \mathbb{Z}_+$,

$$\lambda_a^* P_a^*(n+1-\triangle L(0) \leq L(0-) \leq n) = \underline{\lambda}_d^* \underline{P}_d^*(n+1+\triangle L(0) \leq L(0) \leq n), \quad (34)$$

where $\triangle L(0) = L(0) - L(0-)$, and P_a^* and $\underline{P}_d^*$ are Palm distributions of N_a^* and the following point process, respectively.

$$\underline{N}_d^*(B) = N_d^*(B) - \min(N_a^*(B), N_d^*(B)), \qquad B \in \mathscr{B}(\mathbb{R}),$$

and $\underline{\lambda}_d^*$ is its intensity.

Proof. We can apply Corollary 5 for $X(t) = 1(L(t) \geq n+1)$ and $N = N_a^* + \underline{N}_d^*$ because $X(t)$ is bounded and $X'(t) = 0$. Hence, (27) yields

$$\begin{aligned}\lambda_a^*(P_a^*(L(0-) \geq n+1) - P_a^*(L(0-) + \triangle L(0) \geq n+1))\\ +\underline{\lambda}_d^*(\underline{P}_d^*(L(0) - \triangle L(0) \geq n+1) - \underline{P}_d^*(L(0) \geq n+1)) = 0,\end{aligned}$$

which concludes (34). □

In Theorem 6, $\underline{N}_d^*$ count instants when departures only occur.

Example 9 (Queueing model with no customer loss). In the model of Theorem 6, assume that there is no lost customer, and customers singly arrive and singly depart. Furthermore assume that arrivals and departures do not simultaneously occur. That is, $P_a(\triangle L(0) = 1) = P_d(\triangle L(0) = -1) = 1$, where P_d is the Palm distribution of N_d. From (34), it follows that

$$\lambda_a P_a(L(0-) = n) = \lambda_d P_d(L(0) = n), \qquad n = 0, 1, \ldots.$$

Summing both sides of the above equation over all n, we have $\lambda_a = \lambda_d$. Hence, we have

$$P_a(L(0-) = n) = P_d(L(0) = n), \qquad n \in \mathbb{Z}_+. \tag{35}$$

Thus, the system queue length observed by arriving customers is identical with the one observed by departing customers. This is intuitively clear, but it is also formally obtained, which is important to consider more complex situations. □

Example 10 (Loss system). For the queueing system of Example 9, assume that the system queue length is limited to M, and arriving customers who find M customers in system are lost. In this case, (35) does not hold generally. For example, its right-hand side vanishes for $n = M$, but its left-hand side may not be zero. Since both sides of (34) vanishes for $n \geq M$, we have

$$\lambda_a P_a(L(0-) = n) = \lambda_d P_d(L(0) = n), \qquad n = 0, 1, \ldots, M-1. \tag{36}$$

Since the departure reduces one customer, $P_d(L(0) \leq M-1) = 1$. Hence, summing (36) over for $n = 0, 1, \ldots, M-1$, we have

$$P_a(L(0-) = M) = \frac{\lambda_a - \lambda_d}{\lambda_a}.$$

This is the probability that customers are lost, and is again intuitively clear. We here correctly present it using Palm distributions. □

Theorem 6 does not have information on the system queue length at an arbitrary point in time. Let us include this information using supplementary variables.

Theorem 7. Under the assumptions of Theorem 6, let $R_a(t)$ be the time to the next arrival instant measure from time t, i.e., remaining arrival time, and let T_1 be the first arrival time after time 0. For a nonnegative measurable function f on $\mathbb{R}$ such that it is differentiable, $\overline{E}_a(f(T_1)) < \infty$ and $|E(f'(R_a(0)))| < \infty$, where $\overline{E}_a$ represents the expectation concerning the detailed Palm distribution of N_a, we have

$$\begin{aligned} &-E(f'(R_a(0)); L(0) \geq n+1) \\ &\quad = \lambda_a \left(\overline{E}_a(f(0); L(0-) \geq n+1) - \overline{E}_a(f(T_1); L(0) \geq n+1)\right) \\ &\qquad + \underline{\lambda}_d \underline{\overline{E}}_d(f(R_a(0)); n+1+\triangle L(0) \leq L(0) \leq n), \qquad n \in \mathbb{Z}_+, \end{aligned} \tag{37}$$

where $\underline{\overline{E}}$ represents the expectation of the detailed Palm distribution of $\underline{N}_d$, and λ_a and $\underline{\lambda}_d$ are the intensities of N_a and $\underline{N}_d$, respectively.

Proof. Let $X(t) = f(R_a(t))1(L(t) \geq n+1)$. Since $R_a'(t) = -1$, we have $X'(t) = -f'(R_a(t))1(L(t) \geq n+1)$. Since $P_a(R_a(0) = T_1) = 1$, Corollary 5 and the remark on the detailed version of the rate conservation law at the end of Section 8 concludes (37). □

Example 11 (NBUE distribution). In Example 9, assume that the interarrival times of customers are independent and identically distributed and that arrivals and departures do not simultaneously occur. Furthermore, assume that the interarrival time T_1 satisfies

$$E_a(T_1 - x | T_1 > y) \le E_a(T_1), \qquad x \ge 0. \tag{38}$$

The distribution of T_1 under P_a satisfying this condition is said to be NBUE type, where NBUE is the abbreviation of New Better than Used in Expectation. In fact, (38) represents that the conditional expectation of the remaining arrival time is not greater than the mean interarrival time. If the inequality in (38) is reversed, then the distribution of T_1 is said to be NWUE, which is the abbreviation of New Worse than Used in Expectation. Form the NBUE assumption, we have

$$E(R_a(0); L(0) \ge n+1) \le E_a(T_1) P(L(0) \ge n+1).$$

We apply Theorem 7 with $f(x) = x$. Since $f(0) = 0$, $f'(x) = 1$ and $\lambda_a = \underline{\lambda}_d = \lambda_d = 1/E_a(T_1)$, (37) yields

$$-P(L(0) \ge n+1) \le -\overline{P}_a(L(0) \ge n+1) + \underline{\overline{P}}_d(L(0) = n), \qquad n \ge 0.$$

Since $P_a = \overline{P}_a$ and $P_d = \underline{\overline{P}}_d$, this and (35) lead to

$$P_a(L(0-) \ge n+1) \le P(L(0) \ge n+1), \qquad n \ge 0. \tag{39}$$

Hence, the distribution of the system queue length at the arrival instants is greater than the one at an arbitrary point in time in stochastic order, where, for two distribution functions F and G, F is said to be greater than G in stochastic order if $1 - F(G) \ge 1 - G(x)$ for all $x \in \mathbb{R}$. □

Similarly to Theorem 7, we can take the minimum of the remaining service times of customers being served, and get relationships among the distributions of the system queue lengths at different embedded points in time.

11 An extension of the rate conservation law

In this section, we briefly discuss how the rate conservation law (23) can be generalized for other types of processes. For this, it is notable that this law is obtained from the integral representation of the time evolution (24) and the definition of Palm distribution P_0. There are two integrators, du of the Lebesgue measure and $N(du)$ of a point process, both of which are defined on the line. To closely look at this, we rewrite (24) in a slightly extended form as

$$X(t) = X(0) + \int_0^t X'(u) A(du) + \int_0^t \Delta X(u) N(du),$$

where $A(t)-A(0)$ is consistent with the shift operator θ_t and has bounded variations, and $\Delta X(u)=X(u)-X(u-)$. If $X(t)$ has either a component of unbounded variations or a continuous and singular component with respect to the Lebesgue measure, this expression breaks down. To get back the expression, we subtract this component, denoting it by $M(t)$. Thus, we have

$$X(t)-M(t)=X(0)-M(0)+\int_0^t Y'(u)A(du)+\int_0^t \Delta Y(u)N(du),$$

where $Y(u)=X(u)-M(u)$. If $M(t)$ is consistent with θ_t, then we have the rate conservation law for the process $\{Y(t)\}$. It may be reasonable to assume that $M(t)$ is continuous. However, this rate conservation law may not be useful to study $\{X(t)\}$ because $X(t)$ is not directly involved.

To get useful information, we make use of a test function, which is used in Corollary 4, and apply Itô's integration formula, assuming that $M(t)$ is a square integrable martingale. That is,

(11a) $M(t)$ is continuous in t and consistent with $\{\theta_t\}$.
(11b) $E((M(t)-M(0))^2)<\infty$ for all $t\geq 0$.
(11c) $\{M(t)-M(0);t\geq 0\}$ is a martingale with respect to $\{\mathscr{F}_t\}$, that is,

$$E(M(t)-M(0)|\mathscr{F}_s)=M(s)-M(0), \qquad 0\leq s\leq t,$$

where $\mathscr{F}_t$ is a sub σ-field of $\mathscr{F}$ which is increasing in $t\in\mathbb{R}$, and $\{\mathscr{F}_t;t\in\mathbb{R}\}$ is called a filtration.

This martingale assumption is typical for a process with unbounded variations. It is beyond our scope to fully discuss Itô's integration formula, but we like to see how it works. The reader may refer to standard text books such as [16] and [17] for more details. Assume that $X(t)$ and $M(t)$ are $\mathscr{F}_t$-measurable for all $t\in\mathbb{R}$.

For convenience, let $M_0(t)=M(t)-M(0)$ for $t\geq 0$. Under these assumptions, $M_0^2(t)$ is submartingale, that is,

$$E(M_0^2(t)|\mathscr{F}_s)\geq M_0^2(s), \qquad 0\leq s\leq t,$$

and there exists a nondecreasing process $\langle M_0(t)\rangle$ such that $M_0^2(t)-\langle M_0(t)\rangle$ is a martingale. Then, Itô's integration formula reads: for twice continuously differentiable function f,

$$\begin{aligned} f(X(t)) = f(X(0)) &+ \int_0^t f'(X(u))dM(u)+\int_0^t f'(X(u))Y'(u)A(du)\\ &+\frac{1}{2}\int_0^t f''(X(u))d\langle M_0(u)\rangle+\int_0^t \Delta f(Y(u))N(du), \end{aligned} \tag{40}$$

where the integration on the interval $[0,t]$ with respect to $dM(u)$ is defined L^2-limit of the Riemann sum, that is, $\sum_{\ell=1}^n f'(X(\frac{\ell-1}{n}))(M(\frac{\ell}{n})-M(\frac{\ell-1}{n}))$. See Theorems 17.18 and 26.6 of [16] and Theorem 3.3 of [17]. This integration is a martingale, and its expectation vanishes. Define the Palm distribution with respect to $\langle M_0(t)\rangle$ as

$$P_{\langle M\rangle}(C) = \frac{1}{\lambda_{\langle M\rangle}} E\left(\int_0^1 1_C \circ \theta_u d\langle M_0(u)\rangle\right), \qquad C \in \mathscr{F}.$$

where $\lambda_{\langle M\rangle} = E(M(1) - M(0))$. The Palm measure P_A is similarly defined for the non-decreasing process A. Thus, taking the expectation of both sides of (40), we arrive at

$$E_A(f'(X(0))Y'(0)) + \frac{1}{2}\lambda_{\langle M\rangle} E_{\langle M\rangle}(f''(X(0))) + \lambda E_0(\Delta f(Y(0))) = 0, \qquad (41)$$

assuming suitable finiteness conditions for the expectations, where E_A and $E_{\langle M\rangle}$ stand for the expectations concerning P_A and $P_{\langle M\rangle}$.

We can proceed one further step using the representation theorem for a continuous martingale by the Brownian motion. This theorem says that, for a continuous martingale $M_0(t)$ with respect filtration $\{\mathscr{F}_t\}$, there exists a progressively measurable process $Z(t)$ such that

$$M_0(t) = \int_0^t Z(u)dB(u), \qquad \langle M_0(t)\rangle = \int_0^t Z^2(u)du < \infty, \qquad t \ge 0,$$

where $\{Z(t)\}$ is said to be progressively measurable if $\{(u,\omega) \in [0,t] \times \Omega; Z(u) \in A\} \in \mathscr{B}([0,t]) \times \mathscr{F}_t$ for all $t \ge 0$, and $\{B(t); t \ge 0\}$ is called a Brownian motion if it has independent and stationary increments which are normally distributed with mean 0 and unit variance (see, e.g., Theorem 18.12 of [16] and Theorem 4.15 of [17]). Hence, (41) can be written as

$$E_A(f'(X(0))Y'(0)) + \frac{\lambda_{Z^2}}{2} E\left(f''(X(0))Z^2(0)\right) + \lambda E_0(\Delta f(Y(0))) = 0, \qquad (42)$$

where $\lambda_{Z^2} = E(\int_0^1 Z^2(u)du)$.

In queueing applications of (42), $Y(t)$ and $Z(t)$ are often identified as functions of $X(t)$ from their modeling assumptions through the expression:

$$X(t) = X(0) + \int_0^t Y'(u)A(du) + \int_0^t Z(u)dB(u) + \int_0^t \Delta Y(u)N(du). \qquad (43)$$

In this case, $Y(t) = g(X(t))$ and $Z(t) = h(X(t))$ for some functions g and h, and (42) is really useful to consider the stationary distribution of $X(t)$.

Example 12 (extended Pollaczek-Khinchine formula). Let us consider to add the Brownian motion to the workload process $V(t)$ in Example 7. That is, $V(t)$ is changed to the following $X(t)$.

$$\begin{aligned} X(t) &= X(0) + \sigma^2 B(t) + \sum_{n=1}^{N(t)} S_n - t + I(t) \\ &= X(0) + \int_0^t (I(du) - du) + \int_0^t \sigma^2 dB(u) + \int_0^t \Delta Y(u)N(du) \end{aligned}$$

where $I(t)$ is a minimum non-decreasing process for $X(t)$ to be nonnegative. That is, $I(t)$ is a regulator. Thus, if we put $A(t) = I(t) - t$, $Y(t) = t + \int_0^t \Delta Y(u)N(du)$ and $Z(t) = \sigma$, then we have (43).

Assume the stability condition that

$$\rho \equiv \lambda E(S_1) < 1.$$

Then, $\{X(t)\}$ is a stationary process under the stationary distribution. For nonnegative number θ, let $f(x) = e^{-\theta x}$. We apply (42) to $X(t)$ and this f. Since $I(t)$ is increased only when $X(t) = 0$, we have, using $\varphi(\theta) = E(e^{-\theta X(0))})$ and $g(\theta) = E(e^{-\theta S_1})$,

$$\theta(\varphi(\theta) - E_I(1)) + \frac{\sigma^2\theta^2}{2}\varphi(\theta) = \lambda\varphi(\theta)(1 - g(\theta)).$$

Similar to Example 7, we have $E_I(1) = 1 - \rho$, dividing the above formula by θ and letting $\theta \downarrow 0$. Thus, we have the Laplace-transform of $X(0)$ under the stationary assumption.

$$\varphi(\theta) = \frac{\theta(1-\rho)}{\theta + \frac{1}{2}\sigma^2\theta^2 - \lambda(1 - g(\theta))}, \qquad \theta > 0. \tag{44}$$

This is an extension of the Pollaczek-Khinchine formula (31). □

The results of the present section can be obtained under weaker assumptions and for a multidimensional process. The latter is in a similar line to Corollary 4 with a multidimensional version of the Itô integration formula, while the continuous martingale can be weakened to a local martingale with unbounded variational discontinuity. Of course, we need to carefully consider the integration under such discontinuity.

12 Piece-wise deterministic Markov process (PDMP)

As we discussed in Section 1, many queueing models can be described by stochastic processes whose major changes occur in embedded points in time. In this section, we introduce a typical Markov process having such structure. The sample path of this Markov process is assumed to satisfy the integral representation (24) and to have discontinuous points only on a set, called boundary. It will be shown that this process is flexible and has a wide range of applications.

We first introduce notation for state spaces. Let $\mathscr{X}$ be a countable set. An element $\mathbf{x} \in \mathscr{X}$ is referred to as a macro state. For each $\mathbf{x} \in \mathscr{X}$, let $K_{\mathbf{x}}$ be a closed subset of $\mathbb{R}^{m(\mathbf{x})}$, where $m(\mathbf{x})$ be a positive integer determined by $\mathbf{x}$ and $\mathbb{R}^n$ is the n-dimensional Euclid space, i.e., vector space with the Euclidean metric. Define sets K and $J(\mathbf{x})$ as

$$K = \{(\mathbf{x},\mathbf{y}); \mathbf{x} \in \mathscr{X}, \mathbf{y} \in K_{\mathbf{x}}\}, \qquad J(\mathbf{x}) = \{1, 2, \ldots, m(\mathbf{x})\}.$$

For $(\mathbf{x},\mathbf{y}) \in K$, $\mathbf{y}$ is referred to as a continuous component or supplementary variable under macro state $\mathbf{x}$.

On this K, we introduce a natural topology induced from those on $K_{\mathbf{x}}$. For each $\mathbf{z} \equiv (\mathbf{x},\mathbf{y}) \in K$, the family of its neighborhoods is generated by all the sets of the form $\{\mathbf{x}\} \times (V_{\mathbf{y}} \cap K_{\mathbf{x}})$, where $V_{\mathbf{y}}$ is a neighborhood of $\mathbf{y} \in \mathbb{R}^{m(\mathbf{x})}$. Let $\mathscr{B}(K)$ be the Borel σ-field on K, i.e., the σ-field generated by all open sets of K. Thus, $(K,\mathscr{B}(K))$ is measurable space and we can define a probability measure on it.

We further need notation on boundary. Let $K_{\mathbf{x}+}$ be an open subset of $K_{\mathbf{x}}$, and let $K_{\mathbf{x}0} \equiv K_{\mathbf{x}} \setminus K_{\mathbf{x}+}$, which is called a boundary. For K, we define its inside K_+ and its boundary K_0 as

$$K_+ = \{(\mathbf{x},\mathbf{y}); \mathbf{x} \in \mathscr{X}, \mathbf{y} \in K_{\mathbf{x}+}\}, \qquad K_0 = K \setminus K_+.$$

Definition 10 (PDMP). Let $\mathbf{Z}(t) \equiv (\mathbf{X}(t),\mathbf{Y}(t))$ be a stochastic process with state space K defined above, and assume that $\mathbf{Z}(t)$ is right-continuous with left-limits. This $\{\mathbf{Z}(t)\}$ is said to be a piece-wise deterministic Markov process, PDMP for short, if the following three conditions are satisfied.

(12a) $\mathbf{X}(t)$ is unchanged as long as $\mathbf{Y}(t) \equiv (Y_1(t),\ldots,Y_{m(\mathbf{x})}(t)) \in K_{\mathbf{x}+}$, which changes according to the following differential equation when $\mathbf{X}(t) = \mathbf{x}$.

$$\frac{dY_\ell(t)}{dt} = g_{\mathbf{x}\ell}(\mathbf{Y}(t)), \qquad \ell \in J(\mathbf{x}),$$

where $g_{\mathbf{x}\ell}$ is a bounded measurable function from $\mathbb{R}^{m(\mathbf{x})}$ to $\mathbb{R}$ for each $\mathbf{x} \in \mathscr{X}$, and $\mathbf{Y}(t)$ hits boundary $K_{\mathbf{x}0}$ in a finite time with probability one. We refer to $\mathbf{X}(t)$ and $\mathbf{Y}(t)$ as macro state and continuous component, respectively.

(12b) At the moment when $\mathbf{Z}(t)$ hits the boundary K_0, that is, $\mathbf{Z}(t-) \in K_0$, it instantaneously returns to the inside, that is, $\mathbf{Z}(t-) \in K_0$ is changed to $\mathbf{Z}(t) \in K_+$ subject to the transition kernel Q from the boundary K_0 to the inside K_+. That is, for each $(\mathbf{x},\mathbf{y}) \in K_0$,

$$P(\mathbf{Z}(t) \in A | \mathbf{Z}(t-) = (\mathbf{x},\mathbf{y})) = Q((\mathbf{x},\mathbf{y}),A), \qquad A \in \mathscr{X} \times \mathscr{B}(K).$$

Q is referred to as a jump transition kernel.

(12c) For each finite time interval, the number of the hitting times at the boundary, i.e., the number of t such that $Y(t-) \in K_0$ is finite. We denote the point process generated by such hitting times by N.

Remark 5. The PDMP was introduced by Davis [10] (see also [11]). However, our definition of PDMP is slightly different from his definition. They use the attained lifetimes for the supplementary variables $\mathbf{Y}(t)$. Thus, the macro state transitions randomly occur subject to intensity depending on $\mathbf{Y}(t)$, which may hit the boundary. However, if the time is reversed, then their PDMP becomes ours. A minor advantage of ours is that the existence of the intensity is not necessary. This means that we do not need to assume the existence of densities of lifetime distributions for macro state transitions, which will be discussed below. □

The PDMP (piece-wise deterministic Markov process) looks complicated, but it has simple structure when we only observe the embedded epochs due to the state transition by Q. Let $\{t_n; n \in \mathbb{Z}\}$ be the set of such epochs numbered in increasing order. Then, $\{\mathbf{Z}(t_n-)\}$ is a discrete time embedded Markov chain. Let us consider the transition kernel of this embedded Markov chain.

For each state $\mathbf{z} \equiv (\mathbf{x},\mathbf{y}) \in K_+$, denote the time to the next transition starting from this state by $\zeta(\mathbf{x},\mathbf{y})$, which is uniquely determined by (12a). We let $\zeta(\mathbf{x},\mathbf{y}) = 0$ if $(\mathbf{x},\mathbf{y}) \in K_0$. We also denote the state of the continuous component $\mathbf{Y}(t)$ that attains just before this time by $\psi(\mathbf{x},\mathbf{y})$. Let H be the transition kernel H of the embedded process $\{\mathbf{Z}(t_n-)\}$. That is,

$$H(\mathbf{z},\{\mathbf{x}'\} \times B) = P(\mathbf{Z}(t_{n+1}-) \in \{\mathbf{x}'\} \times B | \mathbf{Z}(t_n-) = \mathbf{z}),$$
$$\mathbf{z} \in K_0, \mathbf{x}' \in \mathscr{X}, B \in \mathscr{B}(K_{\mathbf{x}'}).$$

Then, it is easy to see that

$$H(\mathbf{z},\{\mathbf{x}'\} \times B) = \int_{K_{\mathbf{x}'}} Q(\mathbf{z},\{\mathbf{x}'\} \times d\mathbf{y}') 1(\psi(\mathbf{x}',\mathbf{y}') \in B). \tag{45}$$

Example 13. As an example of PDMP, let us consider the workload process $V(t)$ of Example 8 for the $M/G/1$ queue with state dependent processing rate r. Let $X(t) \equiv 0$, $Y_1(t) = t - T_{N(t)}$ and $Y_2(t) = V(t)$, then $Y_1'(t) = -1$ and $Y_2'(t) = r(V(t))1(V(t) > 0)$. Hence, if we let $\mathscr{X} = \{0\}$ and $K = \{0\} \times [0,\infty)^2$ with $K_0 = \{0\}^2 \times [0,\infty)$, then $(X(t),(Y_1(t),Y_2(t)))$ is a PDMP, where the jump transition Q is given by

$$Qf(0,(0,x)) = E(f(0,(T_1, x+S_1))), \qquad x \geq 0,$$

for a nonnegative valued function f on $K_+ \equiv \{0\} \times (0,\infty) \times [0,\infty)$. Note that $Y_2(t)$ has no boundary in this formulation. □

We next to consider the stationary distribution of PDMP (piece-wise deterministic Markov process). We are interested to characterize it using the rate conservation law. We first consider its transition operator of the Markov process $\{\mathbf{Z}(t)\}$. Since its state space K includes continuous components, we consider the transition operator to work on the space of suitable functions on K.

Let $\mathscr{M}_b(K)$ be the set of all bounded functions from $\mathscr{K}$ to $\mathbb{R}$ which are $\mathscr{B}(K)/\mathscr{B}(\mathbb{R})$-measurable. For each $t \geq 0$, define operator T_t on $\mathscr{M}_b(K)$ as

$$T_t f(\mathbf{z}) = E(f(\mathbf{Z}(t)) | \mathbf{Z}(0) = \mathbf{z}), \qquad \mathbf{z} \in K, f \in \mathscr{M}_b(K).$$

Note that T_t is a linear function from $\mathscr{M}_b(K)$ to $\mathscr{M}_b(K)$. Furthermore, it maps a nonnegative function to a nonnegative function. Thus, T_t is nonnegative and linear operator on $\mathscr{M}_b(K)$, which uniquely determines a distribution on $(K, \mathscr{B}(K))$ as is well known.

Define operator $\mathscr{A}_+$ as

$$\mathscr{A}_+ f(\mathbf{z}) = \lim_{t\downarrow 0} \frac{1}{t}(T_t f(\mathbf{z}) - f(\mathbf{z})), \qquad \mathbf{z} \in K_+,$$

as long as it exists. We refer to this $\mathscr{A}_+$ as a weak generator. Let $\mathscr{D}_{\mathscr{A}_+}$ be the set of all $f \in \mathscr{M}_b(K)$ such that $\mathscr{A}_+ f$ exists. Note that $\mathscr{A}_+$ is a generator only for the continuous part of $\mathbf{Z}(t)$, and does not include the information on state changes due to the macro state transitions. Hence, $\mathscr{A}_+$ is not a generator in the sense that it determines the operator T_t. This is the reason why we call it weak.

For each macro state $\mathbf{x} \in \mathscr{X}$, let $\mathscr{Y}_\mathbf{x}$ be the set of all solutions $\{\mathbf{y}(t)\}$ for the differential equation (12a), i.e.,

$$\frac{dy_\ell(t)}{dt} = g_\mathbf{x}(\mathbf{y}(t)), \qquad 0 \le t < \zeta(\mathbf{x},\mathbf{y}).$$

Let $\mathscr{M}^1_b(K)$ be the set of all functions $f \in \mathscr{M}_b(K)$ such that $f(\mathbf{x}, \xi(t))$ has the right-hand derivative in all t in the domain of ξ and is continuous from the left at $t = \zeta(\mathbf{x},\mathbf{y})$ for $\mathbf{x} \in \mathscr{X}$ and $\xi \in \mathscr{Y}_\mathbf{x}$. Let $C^1_b(K)$ be the set of all functions $f \in \mathscr{M}_b(K)$ such that $f(\mathbf{x},\mathbf{y})$ has bounded and continuous partial derivatives $\frac{\partial}{\partial y_\ell} f(\mathbf{x},\mathbf{y})$ ($\ell = 1,2,\ldots,m(\mathbf{x})$) for each $\mathbf{x} \in \mathscr{X}$ and $\mathbf{y} \in K_{\mathbf{x}+}$. Clearly, $C^1_b(K) \subset \mathscr{M}^1_b(K)$.

For $f \in \mathscr{M}^1_b(K)$ and $\mathbf{Z}(t) \in K$, it follows from the definition of the PDMP that

$$f(\mathbf{Z}(t)) - f(\mathbf{Z}(0)) = \int_0^t \frac{d}{du} f(\mathbf{Z}(u))du + \int_0^t (f(\mathbf{Z}(u)) - f(\mathbf{Z}(u-)))N(du).$$

Hence, for $\mathbf{z} \equiv (\mathbf{x},\mathbf{y}) \in K_+$ and $\xi \in \mathscr{M}^1_b(K)$ with $\xi(0) = \mathbf{y}$, we have

$$\begin{aligned} \mathscr{A}_+ f(\mathbf{z}) &= \lim_{t\downarrow 0} \frac{1}{t} E\left(f(\mathbf{Z}(t)) - f(\mathbf{Z}(0)) | \mathbf{Z}(0) = \mathbf{z}\right) \\ &= \lim_{t\downarrow 0} \frac{1}{t} E\left(\int_0^t \frac{d}{du} f(\mathbf{Z}(u))du \middle| \mathbf{Z}(0) = \mathbf{z}\right) \\ &= \left.\frac{d}{du} f(\mathbf{x}, \xi(u))\right|_{u=0}, \end{aligned}$$

where the second equality is obtained since $\mathbf{Z}(u)$ must stay in K_+ for a finite time under the condition that $\mathbf{Z}(0) = \mathbf{z} \in K_+$. In particular, for $f \in C^1_b(K)$,

$$\mathscr{A}_+ f(\mathbf{z}) = \sum_{\ell=1}^{m(\mathbf{x})} g_{\mathbf{x}\ell}(\mathbf{y}) \frac{\partial}{\partial y_\ell} f(\mathbf{x},\mathbf{y}). \tag{46}$$

Hence, $C^1_b(K) \subset \mathscr{M}^1_b(K) \subset \mathscr{D}_{\mathscr{A}_+}$. However, $C^1_b(K) \neq \mathscr{M}^1_b(K)$ in general. For example, $\zeta \in \mathscr{M}^1_b(K)$, but $\zeta \notin C^1_b(K)$ after Definition 11.

Theorem 8. Let $\{\mathbf{Z}(t)\}$ be the PDMP and let N be the point process N generated by hitting times at the boundary. If $\{\mathbf{Z}(t)\}$ has the stationary distribution ν and if N has a finite intensity λ, then there exists a probability distribution ν_0 on $(K_0, \mathscr{B}(K_0))$ satisfying

$$\int_{K_+} \mathscr{A}_+ f(\mathbf{z}) \nu(d\mathbf{z}) = \lambda \int_{K_0} (f(\mathbf{z}) - Qf(\mathbf{z})) \nu_0(d\mathbf{z}), \qquad f \in \mathscr{M}_b^1(K). \tag{47}$$

Conversely, if there exist probability distributions ν on $(K_+, \mathscr{B}(K_+))$ and ν_0 on $(K_0, \mathscr{B}(K_0))$ satisfying (47) with some positive number λ, then

$$\overline{\nu}(B) = \nu(B \cap K_+), \qquad B \in \mathscr{B}(K)$$

is the stationary distribution of $\mathbf{Z}(t)$, and the point process N has the finite intensity λ. Furthermore, let P be a probability measure on $(\Omega, \mathscr{F})$ such that $\{\mathbf{Z}(t)\}$ is the stationary process with the stationary distribution ν, then ν_0 is the distribution of $\mathbf{Z}(0-)$ under the Palm distribution P_0 with respect to N.

Remark 6. Davis [11] computes an extended generator, which characterizes the stationary distribution, for the PDMP supplemented by the attained lifetimes. We can rewrite (47) in a similar form. Namely, let $\lambda(\mathbf{z}) = \lambda \frac{\nu_0(d\mathbf{z})}{\nu(d\mathbf{z})}$, where $\frac{\nu_0(d\mathbf{z})}{\nu(d\mathbf{z})}$ is the Radon Nikodym derivative of ν_0 with respect to ν. Then, we have

$$\int_K (\mathscr{A}_+ f(\mathbf{z}) + \lambda(\mathbf{z})(Qf(\mathbf{z}) - f(\mathbf{z})))\, \nu(d\mathbf{z}) = 0. \tag{48}$$

$\lambda(\mathbf{z})$ can be considered as a stochastic intensity, and the integrant corresponds with the extended generator. (48) is particularly useful when $\lambda(\mathbf{z})$ is available, but this may not be always the case. In this situation, (47) is more flexible. □

Proof. Assume that $\{\mathbf{Z}(t)\}$ is a stationary process under probability measure P. Denote the stationary distribution of $\mathbf{Z}(t)$ by ν. Since the set of the times when $\mathbf{Z}(t)$ is on the boundary is countable, $P(\mathbf{Z}(0) \in K_0) = 0$. Hence, ν can be viewed as a probability distribution on $(K_+, \mathscr{B}(K_+))$. Let P_0 be the Palm distribution of P with respect to N. Since the distribution $\mathbf{Z}(0-)$ under P_0 is determined by ν_0, (47) is immediate from (46) and Corollary 4.

We next prove the converse. Suppose that there exists probability measures ν, ν_0 satisfying (47) and positive constant λ. Let $f \in \mathscr{M}_b(K)$. Since $T_u f$ is continuous in u, we have, from the definition of $\mathscr{A}_+$

$$\begin{aligned}
\mathscr{A}_+ \left(\int_0^t T_u f du \right)(\mathbf{z}) &= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \left(T_\varepsilon \left(\int_0^t T_u f du \right)(\mathbf{z}) - \int_0^t T_u f(\mathbf{z}) du \right) \\
&= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \left(\int_0^t T_{u+\varepsilon} f(\mathbf{z}) du - \int_0^t T_u f(\mathbf{z}) du \right) \\
&= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \left(\int_t^{t+\varepsilon} T_u f(\mathbf{z}) du - \int_0^\varepsilon T_u f(\mathbf{z}) du \right) \\
&= T_t f(\mathbf{z}) - f(\mathbf{z}), \qquad \mathbf{z} \in K_+.
\end{aligned} \tag{49}$$

Define h for $f \in \mathscr{M}_b(K)$ as

$$h(\mathbf{z}) = \int_0^t T_u f(\mathbf{z}) du, \qquad \mathbf{z} \in K.$$

Then, (49) implies that $h \in \mathscr{D}_{\mathscr{A}_+}$. In general, h may not be in $\mathscr{M}_b^1(K)$, but we can prove that (47) holds for this h in the place of f by approximating h by functions in $\mathscr{M}_b^1(K)$. Since this proof is complicated, we omit it, but the reader can find it in [27]. Since, for $\mathbf{Z}(0-) \in K_0$,

$$\begin{aligned} QT_u f(\mathbf{Z}(0-)) &= E(T_u f(\mathbf{Z}(0))|\mathbf{Z}(0-)) \\ &= E(f(\mathbf{Z}(u))|\mathbf{Z}(0-)) = T_u f(\mathbf{Z}(0-)), \qquad u > 0, \end{aligned}$$

implies

$$\int_{K_0} T_u f(\mathbf{z}) \nu_0(d\mathbf{z}) = \int_{K_0} QT_u f(\mathbf{z}) \nu_0(d\mathbf{z}).$$

Integrating both sides for $u \in [0,t]$, we have

$$\int_{K_0} h(\mathbf{z}) \nu_0(d\mathbf{z}) = \int_{K_0} Qh(\mathbf{z}) \nu_0(d\mathbf{z}).$$

Hence substituting h into f of (47), (49) yields

$$\int_{K_+} T_t f(\mathbf{z}) \nu(d\mathbf{z}) = \int_{K_+} f(\mathbf{z}) \nu(d\mathbf{z}), \qquad t > 0.$$

Thus, $\overline{\nu}$ is the stationary distribution of $\mathbf{Z}(t)$.

We next prove that N has the finite intensity λ. To this end, define φ_ε for $\varepsilon > 0$ as

$$\varphi_\varepsilon(u) = \frac{1}{\varepsilon} \min(\varepsilon, u), \qquad u \geq 0.$$

We remind that $\zeta(\mathbf{x},\mathbf{y})$ is the hitting time at the boundary starting from the state $(\mathbf{x},\mathbf{y}) \in K$, where $\zeta(\mathbf{x},\mathbf{y}) = 0$ for $(\mathbf{x},\mathbf{y}) \in K_0$. For the trajectory $\xi \in \mathscr{Y}_{\mathbf{x}}$, $\frac{d}{dt}\zeta(\mathbf{x},\xi(t)) = -1$. Hence,

$$\frac{d}{dt}\varphi_\varepsilon(\zeta(\mathbf{x},\xi(t))) = -\frac{1}{\varepsilon} 1(0 < \zeta(\mathbf{x},\xi(t)) \leq \varepsilon).$$

Let $f(\mathbf{x},\mathbf{y}) = \varphi_\varepsilon(\zeta(\mathbf{x},\mathbf{y}))$. Then, $f \in \mathscr{M}_b^1(K)$. We apply this f in (47), and let $\varepsilon \downarrow 0$. Then

$$\begin{aligned} &\lim_{\varepsilon\downarrow 0} \int_{K_0} \varphi_\varepsilon(\zeta(\mathbf{x},\mathbf{y}))\nu_0(d\mathbf{x},d\mathbf{y}) = \int_{K_0} 1(\zeta(\mathbf{x},\mathbf{y}) > 0)\nu_0(d\mathbf{x},d\mathbf{y}) = 0, \\ &\lim_{\varepsilon\downarrow 0} \int_{K_0} \sum_{\mathbf{x}' \in \mathscr{X}} Q((\mathbf{x}',\mathbf{y}'),(\mathbf{x},\mathbf{y}))\varphi_\varepsilon(\zeta(\mathbf{x},\mathbf{y}))\nu_0(d\mathbf{x}',d\mathbf{y}') = \nu_0(K_0) = 1. \end{aligned}$$

Here, we have used the fact that ν_0 is the distribution of $\mathbf{Z}(t)$ just before hitting the boundary K_0. Since ν is the stationary distribution, the above computations yield

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_{K_+} 1\,(0 < \zeta(\mathbf{z}) \le \varepsilon)\, \nu(d\mathbf{z}) = \lambda.$$

Reminding that N counts the hitting times at the boundary, we have

$$E(N((0,1]) = \lambda.$$

That is, λ is the intensity of N. Since λ is finite, we can define Palm distribution P_0 of P with respect to N. Denote the distribution of $\mathbf{Z}(0)$ under this P_0 by $\tilde{\nu}_0$, then the rate conservation law (23) yields

$$\int_{K_+} \mathscr{A}_+ f(\mathbf{z}) \nu(d\mathbf{z}) = \lambda \int_{K_0} (f(\mathbf{z}) - Qf(\mathbf{z})) \tilde{\nu}_0(d\mathbf{z}), \qquad f \in \mathscr{M}_b^1(K).$$

This together with (47) concludes

$$\int_{K_0} (f(\mathbf{z}) - Qf(\mathbf{z})) \tilde{\nu}_0(d\mathbf{z}) = \int_{K_0} (f(\mathbf{z}) - Qf(\mathbf{z})) \nu_0(d\mathbf{z}), \qquad f \in \mathscr{M}_b^1(K).$$

Here, we choose f_θ for f such that, for each $\mathbf{x} \in \mathscr{X}$ and $\theta_\ell \ge 0$ ($\ell = 1, 2, \ldots, J(\mathbf{x})$),

$$f_\theta(\mathbf{x}', \mathbf{y}') = 1(\mathbf{x}' = \mathbf{x}) \prod_{\ell \in J(\mathbf{x})} e^{-\theta_\ell y'_\ell}, \qquad (\mathbf{x}', \mathbf{y}') \in K_+.$$

For each subset U of $J(\mathbf{x})$, we let $\theta_\ell \to \infty$ for all $\ell \in U$. Then, we have $\nu_0 = \tilde{\nu}_0$ on the boundary $\{\mathbf{y} \in K_{\mathbf{x}}; y_\ell = 0 \text{ for all } \ell \in U\}$ since $Qf_\theta(\mathbf{z})$ goes to zero. Changing U over all subsets of $J(\mathbf{x})$, we obtain $\tilde{\nu}_0 = \nu_0$ on K_0. This completes the proof. □

For applications, Theorem 8 is not convenient since $\mathscr{M}_b^1(K)$ is too large for verifying (47). We can replace it by a smaller class of functions.

Corollary 6. In Theorem 8, the condition (47) can be replaced by

$$\int_{K_{\mathbf{x}+}} \sum_{\ell=1}^{m(\mathbf{x})} g_{\mathbf{x}\ell}(\mathbf{y}) \frac{\partial}{\partial y_\ell} f(\mathbf{x}, \mathbf{y}) \nu(d\mathbf{x} \times d\mathbf{y})$$
$$= \lambda \int_{K_0} (f(\mathbf{z}) - Qf(\mathbf{z})) \nu_0(d\mathbf{z}), \qquad f \in C_b^1(K). \qquad (50)$$

The necessity of (50) is immediate, but its sufficiency needs approximation arguments for functions in $\mathscr{M}_b^1(K)$ by those in $C_b^1(K)$. This argument can be found in [26], and omitted here.

13 Exponentially distributed lifetime

For the PDMP, some of its continuous components $Y_i(t)$ may be exponentially distributed and be decreased with constant rates. For example, this is the case when customers arrive subject to a Poisson process, and $Y_i(t)$ is the remaining time to the next arrival. In such a case, we can remove those continuous components to have a stochastically equivalent Markov process because the exponential distribution has memoryless property, that is, if random variable T has the exponential distribution, then

$$P(T > s+t | T > s) = P(T > t), \qquad s,t \geq 0.$$

Since this case is particularly interested in our applications, we make the following assumption.

(13a) The jump transition kernel Q of the PDMP does not depend on the continuous components which are exponentially distributed and decreased with constant rates.

In this section, we characterize the stationary distribution for this type of piece-wise deterministic Markov processes (PDMP).

Although we do not need to keep track such continuous components, the standard description of PDMP must include them. Thus, after removing them from the PDMP, we have to care about the modified process, which is not exactly PDMP. In this section, we are particularly interested in the stationary distribution of this modified process.

Assume that the PDMP satisfies the assumption (13a). For macro state $\mathbf{x} \in \mathscr{X}$ and continuous component $\mathbf{y} \in K_{\mathbf{x}}$, let $J_e(\mathbf{x})$ be the index set of $\mathbf{y}$'s which have exponential distributions. We denote the decreasing rate of the i-th component for $i \in J_e(\mathbf{x})$ by $c_{\mathbf{x}i} \geq 0$. We replace the i-th entry of $\mathbf{y}$ by 0 for $i \in J_e(\mathbf{x})$, and denote this modified vector by $\tilde{\mathbf{y}}_{\mathbf{x}}$. Let $\tilde{K} = \{(\mathbf{x}, \tilde{\mathbf{y}}_{\mathbf{x}}); \mathbf{x} \in \mathscr{X}, \mathbf{y} \in K_{\mathbf{x}}\}$.

Note that the process $(\mathbf{X}(t), \tilde{\mathbf{Y}}(t))$ is a continuous time Markov process with state space $\tilde{K}$. Its macro state transition kernel $\tilde{Q}$ is unchanged for this process. Let $\tilde{\mathscr{A}_+}$ be the restriction of the weak generator $\mathscr{A}_+$on $\mathscr{M}_b^1(\tilde{K})$ for the PDMP $(\mathbf{X}(t), \mathbf{Y}(t))$. That is, for $\tilde{f} \in \mathscr{M}_b^1(\tilde{K})$ and $\tilde{f}_K(\mathbf{x}, \mathbf{y}) \equiv \tilde{f}(\mathbf{x}, \tilde{\mathbf{y}}_{\mathbf{x}})$,

$$\tilde{\mathscr{A}_+} \tilde{f}(\mathbf{x}, \tilde{\mathbf{y}}_{\mathbf{x}}) = \mathscr{A}_+ \tilde{f}_K(\mathbf{x}, \mathbf{y}), \qquad (\mathbf{x}, \mathbf{y}) \in K. \tag{51}$$

The following fact is intuitively clear, but its proof clarifies the role of the stationary equation (47) of Theorem 8.

Theorem 9. Let the PDMP $(\mathbf{X}(t), \mathbf{Y}(t))$ have weak kernel $\mathscr{A}_+$ and jump transition kernel Q. Assume that this PDMP satisfies the assumption (13a) and the mean lifetime of the i-th continuous component is $1/\mu_i(\mathbf{x})$ for $i \in J_e(\mathbf{x})$. Then, $\tilde{\nu}$ is the stationary distribution of $(\mathbf{X}(t), \tilde{\mathbf{Y}}(t))$ that has a finite intensity for the embedded point process generated by macro state transitions if and only if there exists a finite measure $\tilde{\nu}_{\mathbf{x}}$ on $(\tilde{K}_{\mathbf{x}}, \mathscr{B}(\tilde{K}_{\mathbf{x}}))$ for each $\mathbf{x} \in \mathscr{X}$ such that

$$\lambda \tilde{\nu}_0(\{\mathbf{x}\} \times d\tilde{\mathbf{y}}_{\mathbf{x}}) = \tilde{\nu}_{\mathbf{x}}(d\tilde{\mathbf{y}}_{\mathbf{x}}) + \sum_{i \in J_e(\mathbf{x})} c_{\mathbf{x}i} \mu_i(\mathbf{x}) \tilde{\nu}(\{\mathbf{x}\} \times d\tilde{\mathbf{y}}_{\mathbf{x}}), \qquad \mathbf{x} \in \mathscr{X}, \tag{52}$$

$$\int_{\tilde{K}} \tilde{\mathscr{A}_+} \tilde{f}(\tilde{\mathbf{z}}) \tilde{\nu}(d\tilde{\mathbf{z}}) = \lambda \int_{\partial \tilde{K}} \left(\tilde{f}(\tilde{\mathbf{z}}) - Q\tilde{f}(\tilde{\mathbf{z}})\right) \tilde{\nu}_0(d\tilde{\mathbf{z}}), \qquad \tilde{f} \in \mathscr{M}_b^1(\tilde{K}), \tag{53}$$

where $\tilde{\mathscr{A}}_+$ is the weak generator of $(\mathbf{X}(t), \tilde{\mathbf{Y}}(t))$, that is given by (51).

Proof. For necessity, (52) is immediate from the decomposition formula of Palm distributions while (53) is obtained from Theorem 8 and (51). To prove sufficiency, we let $\tilde{f}(\tilde{\mathbf{x}}, \mathbf{y}) = 1(\tilde{\mathbf{x}} = \mathbf{x})\tilde{g}(\tilde{\mathbf{y}}_{\mathbf{x}})$ in (53). Then, with help of (52), we have

$$\begin{aligned}\int_{\tilde{K}_{\mathbf{x}}} \tilde{\mathscr{A}}_+ \tilde{g}(\mathbf{x}, \tilde{\mathbf{y}}_{\mathbf{x}}) \tilde{v}(\{\mathbf{x}\} \times d\tilde{\mathbf{y}}_{\mathbf{x}}) = &\int_{\partial \tilde{K}_{\mathbf{x}}} \tilde{g}(\tilde{\mathbf{y}}_{\mathbf{x}}) \tilde{v}_{\mathbf{x}}(d\tilde{\mathbf{y}}_{\mathbf{x}}) \\ &+ \sum_{i \in J_e(\mathbf{x})} c_{\mathbf{x}i} \mu_i(\mathbf{x}) \int_{\partial \tilde{K}_{\mathbf{x}}} \tilde{g}(\tilde{\mathbf{y}}_{\mathbf{x}}) \tilde{v}(\{\mathbf{x}\} \times d\tilde{\mathbf{y}}_{\mathbf{x}}) \\ &- \lambda \int_{\partial \tilde{K}} Q(\tilde{\mathbf{z}}', \{\mathbf{x}\} \times d\tilde{\mathbf{y}}_{\mathbf{x}}) \tilde{g}(\tilde{\mathbf{y}}_{\mathbf{x}}) \tilde{v}_0(d\tilde{\mathbf{z}}'). \qquad (54)\end{aligned}$$

Multiply both sides of this equation by $\prod_{j \in J_e(\mathbf{x})} \frac{\mu_j(\mathbf{x})}{\mu_j(\mathbf{x}) + \theta_j}$, which is the joint Laplace transform of independent and exponentially distributed random variables with means $1/\mu_j(\mathbf{x})$, where θ_j is a nonnegative number, and should not be confused with the shift operator θ_t. Then, the second term in the right-hand side can be computed as

$$\begin{aligned}&\sum_{i \in J_e(\mathbf{x})} c_{\mathbf{x}i} \mu_i(\mathbf{x}) \int_{\partial \tilde{K}_{\mathbf{x}}} \tilde{g}(\tilde{\mathbf{y}}_{\mathbf{x}}) \tilde{v}(\{\mathbf{x}\} \times d\tilde{\mathbf{y}}_{\mathbf{x}}) \prod_{j \in J_e(\mathbf{x})} \frac{\mu_j(\mathbf{x})}{\mu_j(\mathbf{x}) + \theta_j} \\ &= \sum_{i \in J_e(\mathbf{x})} c_{\mathbf{x}i} \mu_i(\mathbf{x}) \left(1 - \frac{\theta_i}{\mu_i(\mathbf{x}) + \theta_i}\right) \int_{\partial \tilde{K}_{\mathbf{x}}} \tilde{g}(\tilde{\mathbf{y}}_{\mathbf{x}}) \tilde{v}(\{\mathbf{x}\} \times d\tilde{\mathbf{y}}_{\mathbf{x}}) \prod_{j \in J_e(\mathbf{x}) \setminus \{i\}} \frac{\mu_j(\mathbf{x})}{\mu_j(\mathbf{x}) + \theta_j} \\ &= \sum_{i \in J_e(\mathbf{x})} c_{\mathbf{x}i} \mu_i(\mathbf{x}) \int_{\partial \tilde{K}_{\mathbf{x}}} \tilde{g}(\tilde{\mathbf{y}}_{\mathbf{x}}) \tilde{v}(\{\mathbf{x}\} \times d\tilde{\mathbf{y}}_{\mathbf{x}}) \prod_{j \in J_e(\mathbf{x}) \setminus \{i\}} \frac{\mu_j(\mathbf{x})}{\mu_j(\mathbf{x}) + \theta_j} \\ &\quad - \sum_{i \in J_e(\mathbf{x})} c_{\mathbf{x}i} \theta_i \int_{\partial \tilde{K}_{\mathbf{x}}} \tilde{g}(\tilde{\mathbf{y}}_{\mathbf{x}}) \tilde{v}(\{\mathbf{x}\} \times d\tilde{\mathbf{y}}_{\mathbf{x}}) \prod_{j \in J_e(\mathbf{x})} \frac{\mu_j(\mathbf{x})}{\mu_j(\mathbf{x}) + \theta_j}.\end{aligned}$$

Thus, we have

$$\begin{aligned}&\int_{\tilde{K}_{\mathbf{x}}} \tilde{\mathscr{A}}_+ \tilde{g}(\mathbf{y}) \tilde{v}(\{\mathbf{x}\} \times d\tilde{\mathbf{y}}_{\mathbf{x}}) \prod_{j \in J_e(\mathbf{x})} \frac{\mu_j(\mathbf{x})}{\mu_j(\mathbf{x}) + \theta_j} \\ &\qquad + \sum_{i \in J_e(\mathbf{x})} c_{\mathbf{x}i} \theta_i \int_{\partial \tilde{K}_{\mathbf{x}}} \tilde{g}(\tilde{\mathbf{y}}_{\mathbf{x}}) \tilde{v}(\{\mathbf{x}\} \times d\tilde{\mathbf{y}}_{\mathbf{x}}) \prod_{j \in J_e(\mathbf{x})} \frac{\mu_j(\mathbf{x})}{\mu_j(\mathbf{x}) + \theta_j} \\ &\quad = \int_{\partial \tilde{K}_{\mathbf{x}}} \tilde{g}(\tilde{\mathbf{y}}_{\mathbf{x}}) \tilde{v}_{\mathbf{x}}(d\tilde{\mathbf{y}}_{\mathbf{x}}) \prod_{j \in J_e(\mathbf{x})} \frac{\mu_j(\mathbf{x})}{\mu_j(\mathbf{x}) + \theta_j} \\ &\qquad + \sum_{i \in J_e(\mathbf{x})} c_{\mathbf{x}i} \mu_i(\mathbf{x}) \int_{\partial \tilde{K}_{\mathbf{x}}} \tilde{g}(\tilde{\mathbf{y}}_{\mathbf{x}}) \tilde{v}(\{\mathbf{x}\} \times d\tilde{\mathbf{y}}_{\mathbf{x}}) \prod_{j \in J_e(\mathbf{x}) \setminus \{i\}} \frac{\mu_j(\mathbf{x})}{\mu_j(\mathbf{x}) + \theta_j} \\ &\qquad - \lambda \int_{\partial \tilde{K}} Q(\tilde{\mathbf{z}}', \{\mathbf{x}\} \times d\tilde{\mathbf{y}}_{\mathbf{x}}) \tilde{g}(\tilde{\mathbf{y}}_{\mathbf{x}}) \tilde{v}_0(d\tilde{\mathbf{z}}') \prod_{j \in J_e(\mathbf{x})} \frac{\mu_j(\mathbf{x})}{\mu_j(\mathbf{x}) + \theta_j}.\end{aligned}$$

This is identical with (47) with f given by

$$f(\mathbf{x}',\mathbf{y}) = 1(\mathbf{x}'=\mathbf{x})\tilde{g}(\tilde{\mathbf{y}}_{\mathbf{x}}) \prod_{j\in J_e(\mathbf{x})} e^{-\theta_i y_i}.$$

Since θ_i can be any positive number, this class of function f is sufficiently large to determine a distribution on K. Hence,

$$\nu(\{\mathbf{x}\}\times d\mathbf{y}) = \tilde{\nu}(\{\mathbf{x}\}\times d\tilde{\mathbf{y}}_{\mathbf{x}}) \prod_{j\in J_e(\mathbf{x})} \mu_j(\mathbf{x}) e^{-\mu_j(\mathbf{x})y_j} dy_j$$

is the stationary distribution of the PDMP $(\mathbf{X}(t),\mathbf{Y}(t))$, so $\tilde{\nu}$ is that of $(\mathbf{X}(t),\tilde{\mathbf{Y}}(t))$. □

It is notable that (52) is necessary to get the stationary distribution. Of course, we can combine (52) and (53) substituting the former into the latter.

Example 14 (State dependent workload, revisited). In Example 13, we formulate the workload process $V(t)$ of Example 8 for the $M/G/1$ queue with state dependent processing rate r as the PDMP $(X(t),(Y_1(t),Y_2(t)))$. Since $T_n - T_{n-1}$ is exponentially distributed and independent of everything else, we can drop $Y_1(t)$, and Theorem 9 is applicable. Although this is not so much helpful to find the stationary distribution since (52) and (53) are equivalent to (32), we can see how Theorem 9 is applied. □

14 GSMP and RGSMP

In many queueing applications of the PDMP, all the continuous components $Y_\ell(t)$ count the remaining lifetimes, and the macro state transitions due to Q is independent of non-zero remaining lifetimes. Assume that the remaining lifetimes decrease with constant rates. In this case, the weak generator $\mathscr{A}_+$ has a simpler form:

$$\mathscr{A}_+ f(\mathbf{x},\mathbf{y}) = -\sum_{i=1}^{m(\mathbf{x})} c_{\mathbf{x}i} \frac{\partial}{\partial y_i} f(\mathbf{x},\mathbf{y}), \tag{55}$$

where $c_{\mathbf{x}i}$ are nonnegative constants for each $\mathbf{x}$ and i. Furthermore, Q is also simpler. We introduce this class of models adding more structure to the macro states.

Definition 11 (GSMP). Let $\mathscr{X}$ and $\mathscr{S}$ be countable or finite sets. Their elements are called a macro state and a site, respectively. For each $\mathbf{x}\in\mathscr{X}$, a finite and non-empty subset of $\mathscr{S}$ is associated, and denoted by $A(\mathbf{x})$, whose element is called an active site under macro state $\mathbf{x}$. For each $s\in A(\mathbf{x})$, a clock is attached, and counts its remaining life time r_s. Let $r(\mathbf{x}) = \{(s,r_s); s\in A(\mathbf{x})\}$.

Assume the following dynamics of macro states and clocks.

(14a) Under macro state $\mathbf{x}$, the clock at site $s\in A(\mathbf{x})$ advances with speed $\overline{c}_{\mathbf{x}s}$.
(14b) If the remaining lifetime of clocks at sites in $U\subset A(\mathbf{x})$ simultaneously expire under macro state $\mathbf{x}$, then the macro state changes to $\mathbf{x}'$ with probability $p_U(\mathbf{x},\mathbf{x}')$.

(14c) Under the above transition, the remaining lifetimes of clocks at sites $A(\mathbf{x}) \setminus U$ are retained, and new clocks are activates on sites $A(\mathbf{x}') \setminus A(\mathbf{x})$ with lifetimes independently sampled from the distribution determined by their sites and new macro state $\mathbf{x}'$.

Thus, $A(\mathbf{x}) \setminus U$ must be a subset of $A(\mathbf{x}')$. Let $\mathbf{X}(t)$ be a macro state at time t, and let $R_s(t)$ be remaining lifetime of the clock at site $s \in A(\mathbf{x})$ at time t. Then, $(\mathbf{X}(t), \{R_s(t); s \in A(\mathbf{x})\})$ is a Markov process. We refer to this Markov process as a generalized semi-Markov process, GSMP for short, with macro state space $\mathscr{X}$ and site space $\mathscr{S}$. □

This GSMP is not exactly the PDMP (piece-wise deterministic Markov process), but can be reduced to it. To see this, let $m(\mathbf{x})$ be the number of elements of $A(\mathbf{x})$, and let $J(\mathbf{x}) = \{1, 2, \ldots, m(\mathbf{x})\}$, where $m(\mathbf{x})$ is a finite positive integer by the assumption on $A(\mathbf{x})$. For each $\mathbf{x} \in \mathscr{X}$, define one to one mapping $\xi_{\mathbf{x}}$ from $A(\mathbf{x})$ to $J(\mathbf{x})$. For each $\ell \in J(\mathbf{x})$, let $y_\ell = v_{\xi_{\mathbf{x}}^{-1}(\ell)}$. Thus, site $s \in A(\mathbf{x})$ is mapped to $\ell \in J(\mathbf{x})$ with the remaining lifetime of the clock attached to s. Let $K_{\mathbf{x}} = [0, \infty)^{m(\mathbf{x})}$ and $K = \cup_{\mathbf{x} \in \mathscr{X}} \{\mathbf{x}\} \times K_{\mathbf{x}}$. Let $R_s(t)$ be the remaining lifetime of the clock at site s, and with $Y_\ell(t) = R_{\xi^{-1}_{\mathbf{X}(t)}(\ell)}(t)$ let

$$\mathbf{Y}(t) = (Y_1(t), Y_2(t), \ldots, Y_{m(\mathbf{X}(t))}(t)).$$

We also define the jump transition kernel Q as, for $\mathbf{x}, \mathbf{x}' \in \mathscr{X}$, $\mathbf{y} \in K_{\mathbf{x}}$ and $B_\ell \in \mathscr{B}(\mathbb{R})$,

$$\begin{aligned} &Q((\mathbf{x}, \mathbf{y}), \{\mathbf{x}'\} \times B_1 \times \cdots \times B_{m(\mathbf{x}')}) \\ &\qquad = p_U(\mathbf{x}, \mathbf{x}') \prod_{\ell \in \xi(A(\mathbf{x}) \setminus U)} 1(y_\ell \in B_\ell) \prod_{\ell \in \xi(A(\mathbf{x}') \setminus A(\mathbf{x}))} F_{\mathbf{x}' \xi^{-1}_{\mathbf{x}'}(\ell)}(B_\ell), \end{aligned}$$

where U is the set of all expiring sites under $\mathbf{x}$ and the remaining lifetimes $y_{\xi_{\mathbf{x}}^{-1}(s)}$ of the clock at site $s \in A(\mathbf{x})$, and $F_{\mathbf{x}s}$ is the new lifetime distribution of the clock at site s under macro state $\mathbf{x}$. Note that $F(B)$ is defined for distribution F as

$$F(B) = \int_B F(du), \qquad B \in \mathscr{B}(\mathbb{R}).$$

Then, we have PDMP $\{(\mathbf{X}(t), \mathbf{Y}(t))\}$ with state space K and jump transition kernel Q. We refer to $\{(\mathbf{X}(t), \mathbf{Y}(t))\}$ as a canonical form of GSMP.

From the assumption on the speed of clocks, we have

$$\frac{dY_\ell(t)}{dt} = -c_{\mathbf{X}(t)\ell}, \qquad \ell \in J(\mathbf{X}(t)),$$

where $c_{\mathbf{x}\ell} = \overline{c}_{\mathbf{x}\xi^{-1}(\ell)}$. Hence, let $\mathbf{y} = (y_1, \ldots, y_{m(\mathbf{x})})$, then

$$\zeta(\mathbf{x},\mathbf{y}) = \min\left\{\frac{y_1}{c_{\mathbf{x}1}},\ldots,\frac{y_{m(\mathbf{x})}}{c_{\mathbf{x}m(\mathbf{x})}}\right\},$$
$$\psi(\mathbf{x},\mathbf{y}) = \left(\frac{y_1}{c_{\mathbf{x}1}} - \zeta(\mathbf{x},\mathbf{y}),\ldots,\frac{y_{m(\mathbf{x})}}{c_{\mathbf{x}m(\mathbf{x})}} - \zeta(\mathbf{x},\mathbf{y})\right).$$

Many queueing models and their networks can be described by GSMP. For those models, sites correspond with arrivals and services, and the remaining lifetimes are the remaining arrival times and the remaining workloads. Particularly, GSMP is useful for those queues with the first-come and first-served discipline since sites for service are unchanged for them.

However, GSMP is not so convenient when services are interrupted. In this case, we have to keep track of the remaining workloads of all customers who once started service. Then, the macro state has to accommodate all the sites as they are since clocks are fixed at sites in GSMP. This often unnecessarily complicates analysis, particularly when the number of active sites is unbounded.

To reduce this complication, one may think of reallocating clocks on sites at each transition instants. This is basically equivalent to only work on the canonical form with the reallocation. Let us define this model.

Definition 12 (RGSMP). Let $\mathscr{X}$ be a finite or countable set for a macro state space, and let $J(\mathbf{x}) \equiv \{1,2,\ldots,m(\mathbf{x})\}$ of the set of all active sites under macro state $\mathbf{x}$. Let D be the index set of lifetime distributions for clocks at their activation, where the same distribution may have different indexes. An active clock is allocated to each element of $J(\mathbf{x})$ and has the remaining lifetime, but this allocation may change at the macro state transitions in the following way.

(14d) Under macro state $\mathbf{x}$, the remaining lifetime of the clock at site $\ell \in J(\mathbf{x})$ decreases with rate $c_{\mathbf{x}\ell}$, where there is at least one positive rate.
(14e) Each clock at site $\ell \in J(\mathbf{x})$ has an index in D. Denote this index by $\gamma_{\mathbf{x}}(\ell)$. This $\gamma_{\mathbf{x}}$ is a mapping from $J(\mathbf{x})$ to D, which is not necessarily one-to-one.
(14f) When all clocks of sites in set U simultaneously expire, macro state $\mathbf{x}$ changes to $\mathbf{x}'$ activating clocks on sites in set U' with probability $p((\mathbf{x},U),(\mathbf{x}',U'))$.
(14g) At this macro state transition, clocks on $J(\mathbf{x})\setminus U$ are reallocated on $J(\mathbf{x}')\setminus U'$ by one-to-one mapping $\Gamma_{\mathbf{x}U,\mathbf{x}'U'}$ onto $J(\mathbf{x}')\setminus U'$, whose domain $\Gamma^{-1}_{\mathbf{x}U,\mathbf{x}'U'}(J(\mathbf{x}')\setminus U')$ is a subset of $J(\mathbf{x})\setminus U$. The clocks at sites in the set:
$$(J(\mathbf{x})\setminus U)\setminus \Gamma^{-1}_{\mathbf{x}U,\mathbf{x}'U'}(J(\mathbf{x}')\setminus U')$$
are said to be interrupted. Under this reallocation, the remaining lifetimes of the reallocated clocks and their indexes are unchanged while newly activated clocks with indexes $d \in D$ have the lifetimes independently sampled subject to distribution F_d's.

Let $\mathbf{X}(t)$ and $\mathbf{Y}(t)$ be the macro state and the remaining lifetime vector at time t. Then, $\{(\mathbf{X}(t),\mathbf{Y}(t))\}$ is the PDMC, and we refer to it as a reallocatable generalized semi-Markov process, RGSMP for short. □

Note that the transition kernel Q at macro state transitions is given by

$$\begin{aligned} &Q((\mathbf{x},\mathbf{y}),(\mathbf{x}',B_1\times\cdots\times B_{m(\mathbf{x}')}))\\ &\quad = p((\mathbf{x},U),(\mathbf{x}',U'))\prod_{j\in\Gamma_{\mathbf{x}U\mathbf{x}'}(J(\mathbf{x})\setminus U)} 1(y_j\in B_j)\prod_{j\in U'} F_{\gamma_{\mathbf{x}'}(j)}(B_j). \end{aligned}$$

We assume that $\Gamma_{\mathbf{x}U,\mathbf{x}'U'}$ is deterministic for simplicity, but it could be random without any difficulty. In applications, U is usually a singleton, that is, $U=\{\ell\}$ for some ℓ. In this case, $p((\mathbf{x},U),(\mathbf{x}',U'))$ and $\Gamma_{\mathbf{x}U,\mathbf{x}'U'}$ are simply written as $p((\mathbf{x},\ell),(\mathbf{x}',U'))$ and $\Gamma_{\mathbf{x}\ell,\mathbf{x}'U'}$, respectively.

Although the canonical form of the remaining lifetimes is sufficient for RGSMP, it may not be always convenient. For example, if there are different groups of sites and reallocations are only taken within each group, then a finite set of multidimensional vectors is more convenient than one multidimensional vector $\mathbf{y}$. In this case, $J(\mathbf{x})$ is divided into subsets $J_{s_1}(\mathbf{x}),\ldots,J_{s_k}(\mathbf{x})$ for the number k of such groups and their indexes $s_1,\ldots,s_k$. Similarly, $\gamma_{\mathbf{x}}(\ell)$ is divided into $\gamma_{s_1\mathbf{x}}(\ell_1),\ldots,\gamma_{s_k\mathbf{x}}(\ell_k)$.

In our definition of RGSMP, some of active clocks may expire at the transition. In queueing applications, this may be the case that a customer being serviced is forced to leave. For example, so called a negative customer causes such an event.

We now specialize Corollary 6 to RGSMP (reallocatable generalized semi-Markov process) of Definition 12. In this case, Laplace transform is convenient.

Corollary 7. Assume RGSMP satisfies the assumptions in Theorem 8. For each $\mathbf{x}\in\mathscr{X}$, let $\theta_{\mathbf{x}}=(\theta_1,\ldots,\theta_{m(\mathbf{x})})$, $\theta_\ell\ge 0$, $\langle\theta_{\mathbf{x}},\mathbf{y}\rangle=\sum_{\ell=1}^{m(\mathbf{x})}\theta_\ell y_\ell$, then (50) can be replaced by

$$\sum_{\ell=1}^{m(\mathbf{x})} c_{\mathbf{x}\ell}\theta_\ell\hat{\nu}(\mathbf{x},\theta_{\mathbf{x}})=\lambda\left(\hat{\nu}_0(\mathbf{x},\theta_{\mathbf{x}})-\hat{\nu}_0^{+}(\mathbf{x},\theta_{\mathbf{x}})\right),\qquad \theta\ge 0,\mathbf{x}\in\mathscr{X}, \tag{56}$$

where

$$\begin{aligned} \hat{\nu}(\mathbf{x},\theta_{\mathbf{x}}) &= \int_{K_{\mathbf{x}+}} e^{-\langle\theta_{\mathbf{x}},\mathbf{y}\rangle}\nu(\mathbf{x},d\mathbf{y}),\\ \hat{\nu}_0(\mathbf{x},\theta_{\mathbf{x}}) &= \int_{K_{\mathbf{x}0}} e^{-\langle\theta_{\mathbf{x}},\mathbf{y}\rangle}\nu_0(\mathbf{x},d\mathbf{y}),\\ \hat{\nu}_0^{+}(\mathbf{x},\theta_{\mathbf{x}}) &= \int_{K_0}\int_{K_{\mathbf{x}+}} e^{-\langle\theta_{\mathbf{x}},\mathbf{y}\rangle}Q(\mathbf{z}',(\mathbf{x},d\mathbf{y}))\nu_0(\mathbf{z}'). \end{aligned}$$

Since function $e^{-\langle\theta_{\mathbf{x}},\mathbf{y}\rangle}$ is in $C_b^1(K)$, the necessity is immediate. For the necessity, we need to approximate functions in C_b^1 with compact supports by Fourier series. This can be found in [26] again.

Up to now, we are mainly concerned with a single point process N for the macro state transitions. We can decompose this N into point processes observed at sites. For each subset U of $\{1,2,\ldots\}$, define point process N_U as

$$N_U(B) = \sum_{t \in B} \sum_{\mathbf{x} \in \mathscr{X}} 1(\mathbf{X}(t) = \mathbf{x}, Y_\ell(t) = 0, \ell \in U \cap J(\mathbf{x})), \qquad B \in \mathscr{B}(\mathbb{R}),$$

and let $\lambda_U = E(N_U((0,1]))$. Since $\lambda_U \le \lambda < \infty$, we can define Palm distribution P_U of P with respect to N_U. Denote the distribution of $\mathbf{Z}(t)$ under P_U by ν_U. Let $\hat{\nu}_U(\mathbf{x}, \theta_\mathbf{x})$ be the Laplace transform with respect to the remaining lifetimes under macro state $\mathbf{x} \in \mathscr{X}$, and let

$$\hat{\nu}_U^+(\mathbf{x}, \theta_\mathbf{x}) = \int_{K_0} \int_{K_{\mathbf{x}+}} e^{-\langle \theta_\mathbf{x}, \mathbf{y} \rangle} Q(\mathbf{z}', (\mathbf{x}, d\mathbf{y})) \nu_U(\mathbf{z}').$$

Then, (56) can be replaced by

$$\sum_{\ell=1}^{m(\mathbf{x})} c_{\mathbf{x}\ell} \theta_\ell \hat{\nu}(\mathbf{x}, \theta_\mathbf{x}) = \sum_U \lambda_U \left(\hat{\nu}_U(\mathbf{x}, \theta_\mathbf{x}) - \hat{\nu}_U^+(\mathbf{x}, \theta_\mathbf{x}) \right), \qquad \theta \ge 0. \tag{57}$$

In many cases, U is a singleton for $\lambda_U > 0$. In this case, we simply write $\lambda_{\{\ell\}}$ and $\nu_{\{\ell\}}$ as λ_ℓ and ν_ℓ, respectively, for $U = \{\ell\}$.

15 Exponential and non-exponential clocks in RGSMP

In this section, we consider the stationary distribution of RGSMP (reallocatable generalize semi-Markov process), provided it exists. In what follows, we use the notations in Definition 12, and assume that $\{(\mathbf{X}(t), \mathbf{Y}(t))\}$ is stationary under P.

As we have considered in Section 13, it is interesting to see the case where some of lifetime distributions are exponential. We here consider such a case for RGSMP. Denote the set of the indexes in D which specify the exponential distributions by D_e. Similarly, let $J_\mathrm{e}(\mathbf{x})$ be the set of the sites whose clocks have indexes in D_e. Those clocks are activated with lifetimes subject to the exponential distributions. For the other distributions, we let

$$D_\mathrm{g} = D \setminus D_\mathrm{e}, \qquad J_\mathrm{g}(\mathbf{x}) = J(\mathbf{x}) \setminus J_\mathrm{e}(\mathbf{x}).$$

In this section, we shall use Theorem 9 and Corollary 7 to characterize the stationary distribution. We first prepare some notations. For each $d \in D$, denote the mean of distribution F_d by m_d, and its reciprocal by μ_d. Denote the Laplace transform of F_d by $\hat{F}_d(\theta)$. Since F_d is exponential for $d \in D_\mathrm{e}$,

$$F_d(x) = 1 - e^{-\mu_d x}, \quad x \ge 0, \qquad\qquad \hat{F}_d(\theta) = \frac{\mu_d}{\mu_d + \theta}, \quad \theta \ge 0.$$

Let $\theta_\mathbf{x} = (\theta_1, \ldots, \theta_{m(\mathbf{x})})$. For $U \subset J(\mathbf{x})$, let $\theta_\mathbf{x}(U)$ denote the $\theta_\mathbf{x}$ in which the components with indexes in U is replaced by 0. In particular, if $U = \{\ell\}$, then $\theta_\mathbf{x}(U)$ is denoted by $\theta_\mathbf{x}(\ell)$. Let N_ℓ be the point process generated by expiring instants of clocks at site ℓ. This point process is obviously stationary under P. In what follows, we also assume

(15a) The mean m_d of F_d is finite for all $d \in D$.
(15b) Not more than one clock simultaneously expires.
(15c) $\sum_{\ell=1}^{\infty} \lambda_\ell < \infty$, where λ_ℓ is the intensity of N_ℓ.

Let P_ℓ be the Palm distribution concerning N_ℓ. We denote the distribution of $(\mathbf{X}(t), \mathbf{Y}(t))$ under P by ν, and its Laplace transform concerning $\mathbf{Y}(t)$ under $\mathbf{X}(t) = \mathbf{x}$ by $\hat{\nu}(\mathbf{x}, \theta_\mathbf{x})$ for each $\mathbf{x} \in \mathscr{X}$. Similarly, the distribution of $(\mathbf{X}(0-), \mathbf{Y}(0-))$ under the Palm distribution P_ℓ and its Laplace transform under $\mathbf{X}(0-) = \mathbf{x}$ are denoted by ν_ℓ and $\hat{\nu}_\ell(\mathbf{x}, \theta_\mathbf{x})$, respectively.

Lemma 12. For $\mathbf{x} \in \mathscr{X}$ and $\theta_\mathbf{x} \geq 0$, we have

$$\hat{\nu}(\mathbf{x}, \theta_\mathbf{x}) = \hat{\nu}(\mathbf{x}, \theta_\mathbf{x}(J_\mathrm{e}(\mathbf{x}))) \prod_{i \in J_\mathrm{e}(\mathbf{x})} \frac{\mu_{\gamma_\mathbf{x}(i)}}{\mu_{\gamma_\mathbf{x}(i)} + \theta_i}, \tag{58}$$

$$\hat{\nu}_\ell(\mathbf{x}, \theta_\mathbf{x}) = \hat{\nu}_\ell(\mathbf{x}, \theta_\mathbf{x}(J_\mathrm{e}(\mathbf{x}))) \prod_{i \in J_\mathrm{e}(\mathbf{x}) \setminus \{\ell\}} \frac{\mu_{\gamma_\mathbf{x}(i)}}{\mu_{\gamma_\mathbf{x}(i)} + \theta_i}, \tag{59}$$

$$c_{\mathbf{x}\ell} \mu_{\gamma_\mathbf{x}(\ell)} \hat{\nu}(\mathbf{x}, \theta_\mathbf{x}(J_\mathrm{e}(\mathbf{x}))) = \lambda_\ell \hat{\nu}_\ell(\mathbf{x}, \theta_\mathbf{x}(J_\mathrm{e}(\mathbf{x}))), \qquad \ell \in J_\mathrm{e}(\mathbf{x}). \tag{60}$$

Proof. (58) and (59) are immediate from the memoryless property of the exponential distribution. Substituting them into (57) and letting $\theta_\ell \to \infty$ yield (60). □

The next result is a specialization of Corollary 7 to the case that some of lifetime distributions are exponential, but can be viewed as a special case of Theorem 9.

Theorem 10. Under the assumptions (15a), (15b) and (15c), RGSMP has the stationary distribution if and only if there exist Laplace transforms $\hat{\nu}, \hat{\nu}_\ell$ and λ_ℓ $(\ell = 1, 2, \ldots)$ such that (60) holds and, for each $\mathbf{x} \in \mathscr{X}$ and $\theta_\mathbf{x}(J_\mathrm{e}(\mathbf{x})) \geq 0$,

$$\begin{aligned}
&\sum_{i \in J_\mathrm{g}(\mathbf{x})} c_{\mathbf{x}i} \theta_i \hat{\nu}(\mathbf{x}, \theta_\mathbf{x}(J_\mathrm{e}(\mathbf{x}))) \\
&\quad = \sum_{i \in J(\mathbf{x})} \lambda_i \hat{\nu}_i(\mathbf{x}, \theta_\mathbf{x}(J_\mathrm{e}(\mathbf{x}))) - \sum_{\mathbf{x}' \in \mathscr{X}} \sum_{i \in J(\mathbf{x}')} \sum_{U \subset J(\mathbf{x})} \lambda_i \hat{\nu}_i(\mathbf{x}', \hat{\Gamma}^{-1}_{\mathbf{x}'i,\mathbf{x}U}(\theta_\mathbf{x}(J_\mathrm{e}(\mathbf{x})))) \\
&\qquad \times p((\mathbf{x}', i), (\mathbf{x}, U)) \prod_{j \in U \cap J_\mathrm{g}(\mathbf{x})} \hat{F}_{\gamma_\mathbf{x}(j)}(\theta_j),
\end{aligned} \tag{61}$$

where $\hat{\Gamma}^{-1}_{\mathbf{x}'i,\mathbf{x}U}(\theta_\mathbf{x})$ is the $m(\mathbf{x}')$-dimensional vector whose j-th entry is $\theta_{\Gamma_{\mathbf{x}'i,\mathbf{x}U}(j)}$ if $j \neq i$ and $\Gamma_{\mathbf{x}'i,\mathbf{x}U}(j) \in J(\mathbf{x})$ and equals 0 otherwise. In this case, $\hat{\nu}$ is the Laplace transform of the stationary distribution ν.

Remark 7. It is not hard to see that (61) is a special case of (54).

Proof. We apply Corollary 7. From the assumption (15a),

$$\lambda \hat{\nu}_0(\mathbf{x}, \theta_\mathbf{x}) = \sum_{\ell=1}^{\infty} \lambda_\ell \hat{\nu}_\ell(\mathbf{x}, \theta_\mathbf{x}).$$

Similarly, from the definition of $\hat{\Gamma}^{-1}_{\mathbf{x}' i \mathbf{x} U}$,

$$\lambda \hat{v}_0^+(\mathbf{x}, \theta_{\mathbf{x}}) = \sum_{i=1}^{\infty} \lambda_i \hat{v}_i^+(\mathbf{x}, \theta_{\mathbf{x}})$$
$$= \sum_{i=1}^{\infty} \sum_{\mathbf{x}' \in \mathscr{X}} \sum_{U \subset J(\mathbf{x})} \lambda_i \hat{v}_i^-(\mathbf{x}', \hat{\Gamma}^{-1}_{\mathbf{x}' i, \mathbf{x} U}(\theta_{\mathbf{x}'})) p((\mathbf{x}', i), (\mathbf{x}, U)) \prod_{j \in U \cap J_{\mathrm{g}}(\mathbf{x})} \hat{F}_{\gamma_{\mathbf{x}}(j)}(\theta_j).$$

Substituting these formulas together with (58) and (59) into (56) and dividing both sides by $\prod_{j \in J_{\mathrm{e}}(\mathbf{x})} \frac{\mu_{\gamma_{\mathbf{x}}(j)}}{\mu_{\gamma_{\mathbf{x}}(j)} + \theta_j}$, we have

$$\sum_{i=1}^{m(\mathbf{x})} c_{\mathbf{x} i} \theta_i \hat{v}(\mathbf{x}, \theta_{\mathbf{x}}(J_{\mathrm{e}}(\mathbf{x})))$$
$$= \sum_{i \in J_{\mathrm{g}}(\mathbf{x})} \lambda_i \hat{v}_i(\mathbf{x}, \theta_{\mathbf{x}}(J_{\mathrm{e}}(\mathbf{x})) + \sum_{i \in J_{\mathrm{e}}(\mathbf{x})} \frac{\lambda_i (\mu_{\gamma_{\mathbf{x}}(i)} + \theta_i)}{\mu_{\gamma_{\mathbf{x}}(i)}} \hat{v}_i(\mathbf{x}, \theta_{\mathbf{x}}(J_{\mathrm{e}}(\mathbf{x}))$$
$$- \sum_{\mathbf{x}' \in \mathscr{X}} \sum_{i=1}^{m(\mathbf{x}')} \sum_{U \subset J(\mathbf{x})} \lambda_i \hat{v}_i^-(\mathbf{x}', \hat{\Gamma}_{\mathbf{x}' i, \mathbf{x} U}(\theta_{\mathbf{x}'}(J_{\mathrm{e}}(\mathbf{x}')))) p((\mathbf{x}', i), (\mathbf{x}, U)) \prod_{j \in U \cap J_{\mathrm{g}}(\mathbf{x})} \hat{F}_{\gamma_{\mathbf{x}}(j)}(\theta_j).$$

By Lemma 12, (60) is necessary. We apply it to the second term in the right-hand side of this equation, then we can see that the terms of $i \in J_{\mathrm{e}}(\mathbf{x})$ in the left-hand side are cancelled, which yields (61). Thus, (60) and (61) are necessary. These arguments can be traced back, so the converse is proved (see also the proof of Theorem 9). □

Example 15. Let us formulate the $M/G/1$ queue of Example 7 by the RGSMP. We here assume the first-come first-served discipline. Let $X(t)$ be the number of customers in the system, and $R(t)$ be the remaining service time of a customer in service, where $R(t) = 0$ if the system is empty. Obviously, $(X(t), R(t))$ is a continuous-time Markov chain, and it is easy to see that this process is a RGSMP.

We show how the notations of the RGSMP are specified in this case. Let $\mathscr{X} = \{0, 1, 2, \ldots\}$. $D = \{0, 1\}$, where 0 represents the exponential distribution with mean λ^{-1}, and 1 represents a generic distribution with mean μ^{-1} and distribution F. Define the jump transition function by

$$p((n,0),(n+1,U)) = 1 \text{ if } n = 0 \text{ and } U = \{0,1\} \text{ or if } n \geq 1 \text{ and } U = \{0\},$$
$$p((n,1),(n-1,U)) = 1 \text{ if } n = 1 \text{ and } U = \emptyset \text{ or if } n \geq 2 \text{ and } U = \{1\},$$

and let $c_{n,0} = c_{n+1,1} = 1$ for $n \geq 0$. Let

$$J_e(0) = \{0,1\}, \quad J_e(n) = \{0\}, \quad J_g(0) = \emptyset, \quad J_g(n) = \{1\}, \qquad n \geq 1.$$

Thus, we indeed have the RGSMP. Assume the stability condition $\rho \equiv \lambda/\mu < 1$. In what follows we solve the stationary equation (61), which becomes

$$\begin{aligned}
&0 = \lambda \hat{v}(0,0) - \lambda \hat{v}(1,0),\\
&\theta \hat{v}(1,\theta) = \lambda(\hat{v}(1,\theta) + \hat{v}(1,0)) - \lambda(\hat{v}(0,0) + \hat{v}(2,0))\hat{F}(\theta),\\
&\theta \hat{v}(n,\theta) = \lambda(\hat{v}(n,\theta) + \hat{v}(n,0)) - \lambda(\hat{v}(n-1,\theta) + \hat{v}(n+1,0)\hat{F}(\theta)),
\end{aligned}$$

for $n \geq 2$, where we have used the fact that $\lambda_1 = \lambda$. By letting $\theta = 0$ in these formulas. it is easy to see that $\hat{v}_1(n,0) = \hat{v}(n-1,0)$ for $n \geq 1$. Then, it is routine to solve these stationary equations by taking the generating function:

$$\hat{v}_*(z,\theta) = 1 - \rho + \sum_{n=1}^{\infty} z^n \hat{v}(n,\theta).$$

Let $\pi(0) = \hat{v}(0,0)$. This yields

$$(\theta - \lambda(1-z))(\hat{v}_*(z,\theta) - \pi(0)) = \lambda(1-z)\hat{F}(\theta)\pi(0) + \lambda(z - \hat{F}(\theta))\hat{v}_*(z,0). \quad (62)$$

Let $\theta = \lambda(1-z)$ in this equation, then we have

$$\hat{v}_*(z,0) = \frac{(1-\rho)(1-z)\hat{F}(\lambda(1-z))}{\hat{F}(\lambda(1-z)) - z}. \quad (63)$$

We can compute $\hat{v}_*(z,\theta)$ by substituting this into (62). These results are well known. The advantage of the present derivation is that the existence of the density of F is not needed, which is often assumed in the literature. □

If $D_g = \emptyset$ in Theorem 10, i.e., all the lifetimes are exponentially distributed, then the set of equations (60) and (61) with $\theta_i = 0$ uniquely determines the stationary distribution of the macro states. Hence, we have the following corollary.

Corollary 8. For the RGSMP satisfying (15a), (15b) and (15c), if all the lifetime distributions are exponential, then a probability distribution π on $\mathscr{X}$ is the stationary distribution of $\mathbf{X}(t)$ if and only if, for all $\mathbf{x} \in \mathscr{X}$,

$$\sum_{\ell \in J(\mathbf{x})} c_{\mathbf{x}\ell}\mu_{\gamma_{\mathbf{x}}(\ell)}\pi(\mathbf{x}) = \sum_{\mathbf{x}' \in \mathscr{X}} \sum_{\ell \in J(\mathbf{x}')} \sum_{U \subset J(\mathbf{x})} c_{\mathbf{x}'\ell}\mu_{\gamma_{\mathbf{x}'}(\ell)}\pi(\mathbf{x}')p((\mathbf{x}',\ell),(\mathbf{x},U)). \quad (64)$$

Equation (64) can be interpreted as the stationary equation for the macro state. To see this, define the transition rate function $q(\mathbf{x},\mathbf{x}')$ as

$$q(\mathbf{x},\mathbf{x}') = \sum_{\ell \in J(\mathbf{x}')} \sum_{U \subset J(\mathbf{x})} c_{\mathbf{x}\ell}\mu_{\gamma_{\mathbf{x}}(\ell)}p((\mathbf{x},\ell),(\mathbf{x}',U)).$$

Then, it is not hard to see that (64) is equivalent to

$$\pi(\mathbf{x}) \sum_{\mathbf{x}' \in \mathscr{X}} q(\mathbf{x},\mathbf{x}') = \sum_{\mathbf{x}' \in \mathscr{X}} \pi(\mathbf{x}')q(\mathbf{x}',\mathbf{x}), \qquad \mathbf{x} \in \mathscr{X}.$$

Thus, we can find the time-reversed process of $\{\mathbf{X}(t)\}$, which has the transition rate function:

$$\tilde{q}(\mathbf{x},\mathbf{x}') = \frac{\pi(\mathbf{x}')}{\pi(\mathbf{x})} q(\mathbf{x}',\mathbf{x}).$$

Then, it is not hard to see the following result.

Corollary 9. Under all the conditions of Corollary 8, the time-reversed macro process can be considered as that of the RGSMP with the speeds $\tilde{c}_{\mathbf{x}U}$ and the rates of the exponential distributions $\tilde{\mu}_{\gamma_{\mathbf{x}}(U)}$ and jump transition $\tilde{p}((\mathbf{x},U),(\mathbf{x}',\ell))$ as long as they satisfy

$$\tilde{c}_{\mathbf{x}U}\tilde{\mu}_{\gamma_{\mathbf{x}}(U)} = \frac{1}{\pi(\mathbf{x})} \sum_{\mathbf{x}' \in \mathscr{X}} \sum_{\ell \in J(\mathbf{x}')} c_{\mathbf{x}'\ell}\mu_{\gamma_{\mathbf{x}'}(\ell)}\pi(\mathbf{x}')p((\mathbf{x}',\ell),(\mathbf{x},U)), \tag{65}$$

$$\tilde{p}((\mathbf{x},U),(\mathbf{x}',\ell)) = \frac{1}{\pi(\mathbf{x})\tilde{c}_{\mathbf{x}U}\tilde{\mu}_{\gamma_{\mathbf{x}}(U)}} c_{\mathbf{x}'\ell}\mu_{\gamma_{\mathbf{x}'}(\ell)}\pi(\mathbf{x}')p((\mathbf{x}',\ell),(\mathbf{x},U)). \tag{66}$$

Remark 8. If U is not a singleton in this corollary, then clocks in U are forced to expire except for one in the constructed RGSMP for reversed time. However, active clocks are singly created. This is contrasted with the forward process.

Example 16 (Reversibility of the $M/M/1$ queue). We show how Corollary 9 can be used for applications. Consider the $M/M/1$ queue with arrival rate λ and service rate μ. We assume the stability condition $\rho \equiv \frac{\lambda}{\mu} < 1$. This model is a special case of the $M/G/1$ queue, which is formulated by the RGSMP in Example 15, and we can apply Corollary 9 because all lifetime distributions are exponential. Since $\hat{F}(\theta) = \mu/(\mu+\theta)$, (63) becomes

$$\hat{v}_*(z,0) = \frac{1-\rho}{1-\rho z}.$$

Hence, the stationary distribution $\{\pi(n)\}$ is given by $\pi(n) = (1-\rho)\rho^n$, as is well known. Remind that $D = D_e = \{0,1\}$, $J(0) = \{0\}$, and $J(n) = \{0,1\}$ for $n \geq 1$. From (65), we have, for $n \geq 0$,

$$\tilde{c}_{(n+1)0}\tilde{\mu}_0 = \frac{1}{\pi(n+1)}\lambda\pi(n) = \lambda\rho^{-1} = \mu,$$
$$\tilde{c}_{n1}\tilde{\mu}_1 = \frac{1}{\pi(n)}\mu\pi(n+1) = \mu\rho = \lambda,$$

Similarly we have

$$\tilde{p}((n,U),(n+1,1)) = p((n+1,1),(n,U)) = 1,$$
$$\tilde{p}((n+1,U),(n,0)) = p((n,0),(n+1,U)) = 1.$$

Thus, letting $\tilde{c}_{(n+1)0} = \tilde{c}_{n1} = 1$ for $n \geq 0$, the time reversed RGSMP is identical with the same $M/M/1$ queue except for the indexes, which are exchanged. Thus, the departure process of the original $M/G/1$ queue is the Poisson process with rate

λ and independent of the past history of the system. This is known as the Burke's theorem [7], which is obtained for the $M/M/s$ queue. □

16 Product form decomposability

Under the assumptions of Corollary 8, $\{\mathbf{X}(t)\}$ is a continuous time Markov chain. However, this is not the case if there are non exponential lifetime distributions, so π determined by (64) may not be the stationary distribution of the macro state. We are interested in the case that this π is either still the stationary distribution of $\mathbf{X}(t)$ or can be modified to be the stationary distribution. We guess this could occurs when the remaining lifetimes are independent, and give the following definition.

Definition 13. If RGSMP $\{(\mathbf{X}(t),\mathbf{Y}(t))\}$ is stationary and if there exist distribution function H_d for each $d \in D$ such that $H_d(0) = 0$ and, under the stationary probability measure P,

$$P\Big(\mathbf{X}(0) = \mathbf{x}, \mathbf{Y}(0) \in \prod_{\ell=1}^{m(\mathbf{x})} [0,u_\ell]\Big) = P(\mathbf{X}(0) = \mathbf{x}) \prod_{\ell=1}^{m(\mathbf{x})} H_{\gamma_{\mathbf{x}}(\ell)}(u_\ell), \quad \mathbf{x} \in \mathscr{X}, u_\ell \geq 0,$$

then the RGSMP or its stationary distribution is said to have product form decomposition with respect to remaining lifetimes

Remark 9. The product form decomposability is slightly different from the conditionally independence of $Y_\ell(t)$ $(\ell = 1,2,\ldots,m(\mathbf{X}(t))$ given $\mathbf{X}(t) = \mathbf{x}$, that is,

$$P\Big(\mathbf{X}(t) = \mathbf{x}, \mathbf{Y}(t) \in \prod_{\ell=1}^{m(\mathbf{x})} [0,u_\ell]\Big) = P(\mathbf{X}(t) = \mathbf{x}) \prod_{\ell=1}^{m(\mathbf{x})} P(Y_\ell(t) \leq u_\ell).$$

Clearly, they are equivalent if no reallocation occurs.

Lemma 13. Assume that the RGSMP satisfies the assumptions (15a) and (15c). If the RGSMP has a product form decomposable stationary distribution ν, then (15b) is satisfied, and there exists $\alpha_d > 0$ for each $d \in D$ such that

$$H_d(x) = 1 - \beta_d \int_x^\infty (1 - F_d(u))e^{-\alpha_d(u-x)}du, \qquad x \geq 0, d \in D, \tag{67}$$

$$c_{\mathbf{x}\ell}\mu^*_{\gamma_{\mathbf{x}}(\ell)} \hat{\nu}(\mathbf{x}, \theta_{\mathbf{x}}(\ell)) = \lambda_\ell \hat{\nu}_\ell(\mathbf{x}, \theta_{\mathbf{x}}(\ell)), \qquad \mathbf{x} \in \mathscr{X}, \ell \in J(\mathbf{x}), \tag{68}$$

where β_d and μ^*_d are given by

$$\beta_d = \begin{cases} \frac{\alpha_d}{1-\hat{F}_d(\alpha_d)}, & \alpha_d \neq 0, \\ \mu_d & \alpha_d = 0\,, \end{cases} \qquad \mu^*_d = \begin{cases} \frac{\alpha_d \hat{F}_d(\alpha_d)}{1-\hat{F}_d(\alpha_d)} & \alpha_d \neq 0, \\ \mu_d & \alpha_d = 0\,. \end{cases} \tag{69}$$

Proof. Assume that $(X(t),Y(t))$ is a stationary process with the stationary distribution ν. When F_d is exponential, we obviously have (67), and (68) is easily obtained similarly to (60). Hence, it is sufficient to prove (67) and (68) for $d \in D_{\rm g}$ and for $\ell \in J_{\rm g}(\mathbf{x})$. If $c_{\mathbf{x}\ell} = 0$, then (68) obviously holds, so we assume that $c_{\mathbf{x}\ell} > 0$. Let $T_{\ell 1} = \inf\{t > 0; N_\ell((0,t]) = 1\}$. Since $Y_\ell(0) = c_{\mathbf{x}\ell} T_{\ell 1}$, we have, from Corollary 1,

$$\begin{aligned}
&\lambda_\ell P_\ell \left(\mathbf{X}(0-) = \mathbf{x}, Y_i(0-) \le u_i, i \in J(\mathbf{x}) \setminus \{\ell\}\right) \\
&\quad = \lim_{t \downarrow 0} \frac{1}{t} P\left(\mathbf{X}(T_{\ell 1}-) = \mathbf{x}, Y_i(T_{\ell 1}-) \le u_i, i \in J(\mathbf{x}) \setminus \{\ell\}, T_{\ell 1} \le t\right) \\
&\quad = \lim_{t \downarrow 0} \frac{c_{\mathbf{x}\ell}}{c_{\mathbf{x}\ell} t} P\left(\mathbf{X}(0) = \mathbf{x}, Y_i(0) - c_{\mathbf{x}i} T_{\ell 1} \le u_i, i \in J(\mathbf{x}) \setminus \{\ell\}, Y_\ell(0) \le c_{\mathbf{x}\ell} t\right).
\end{aligned}$$

Hence, from the product form decomposability and the definition of Palm distribution, we have

$$\begin{aligned}
\lambda_\ell \nu_\ell \left(\mathbf{x}, \prod_{i=1}^{m(\mathbf{x})} [0, u_i]\right) &= c_{\mathbf{x}\ell} \frac{\partial}{\partial u_\ell} \nu \left(\mathbf{x}, \prod_{i=1}^{m(\mathbf{x})} [0, u_i]\right)\Bigg|_{u_\ell = 0} \\
&= c_{\mathbf{x}\ell} H'_{\gamma_{\mathbf{x}}(\ell)}(0) \pi(\mathbf{x}) \prod_{i \in J(\mathbf{x}) \setminus \{\ell\}} H_{\gamma_{\mathbf{x}}(i)}(u_i), \qquad (70)
\end{aligned}$$

where $H'_{\gamma_{\mathbf{x}}(\ell)}(0)$ must exist and be finite because the left-hand side is finite. Note that this formula also holds for $\ell \in J_{\rm e}(\mathbf{x})$. Furthermore, if more than one clocks simultaneously expire, then we can put $u_i = u_\ell$ for some $i \neq \ell$ in the right-hand side of (70), which implies that the corresponding Palm distribution vanishes. Thus, (15b) is satisfied.

Letting $u_i = \infty$ in (70) and summing both sides of it for all $\mathbf{x} \in \mathscr{X}$ and $\ell \in \gamma_{\mathbf{x}}^{-1}(d)$ for each $d \in D$, we have

$$\sum_{\ell \in \gamma_{\mathbf{x}}^{-1}(d)} \lambda_\ell = H'_d(0) \sum_{\mathbf{x} \in \mathscr{X}} \sum_{\ell \in \gamma_{\mathbf{x}}^{-1}(d)} c_{\mathbf{x}\ell} \pi(\mathbf{x}). \qquad (71)$$

Thus, the right-hand side is finite by the assumption (15c).

Let $d = \gamma_{\mathbf{x}}(\ell)$, and letting $\theta_i = 0$ for all $i \in J(\mathbf{x}) \setminus \{\ell\}$ and $\theta_\ell = \theta$ in (61) of Theorem 10 and substituting (70) yield, for $\ell \in J_g(\mathbf{x})$,

$$\begin{aligned}
c_{\mathbf{x}\ell} \theta \pi(\mathbf{x}) \hat{H}_d(\theta) &= c_{\mathbf{x}\ell} H'_d(0) \pi(\mathbf{x}) + \sum_{i \in J(\mathbf{x}) \setminus \{\ell\}} c_{\mathbf{x}i} H'_{\gamma_{\mathbf{x}}(i)}(0) \pi(\mathbf{x}) \hat{H}_d(\theta) \\
&\quad - \sum_{\mathbf{x}' \in \mathscr{X}} \sum_{i \in J(\mathbf{x}')} \sum_{\ell \notin U \subset J(\mathbf{x})} c_{\mathbf{x}'i} H'_{\gamma_{\mathbf{x}'}(i)}(0) \pi(\mathbf{x}') p((\mathbf{x}', i), (\mathbf{x}, U)) \hat{H}_d(\theta) \\
&\quad - \sum_{\mathbf{x}' \in \mathscr{X}} \sum_{i \in J(\mathbf{x}')} \sum_{\ell \in U \subset J(\mathbf{x})} c_{\mathbf{x}'i} H'_{\gamma_{\mathbf{x}'}(i)}(0) \pi(\mathbf{x}') p((\mathbf{x}', i), (\mathbf{x}, U)) \hat{F}_d(\theta). \qquad (72)
\end{aligned}$$

Summing this formula for all $\mathbf{x} \in \mathscr{X}$ and $\ell \in \gamma_{\mathbf{x}}^{-1}(d)$ for each fixed $d \in D_g$, we can see that the sum of the left-hand side is finite by (71) and the second sum is not less than the third sum because of the interruption (they must be identical if there is no interruption). So, there exist a nonnegative constant a and positive constants b, c such that

$$\theta \hat{H}_d(\theta) = c + a\hat{H}_d(\theta) - b\hat{F}_d(\theta).$$

Letting $\theta = 0$ in this equation, we have $c = b - a$. Thus, we have, rewriting θ as θ,

$$\theta \hat{H}_d(\theta) - a\hat{H}_d(\theta) = -a + b(1 - \hat{F}_d(\theta)). \tag{73}$$

This is equivalent to the following differential equation.

$$\frac{d}{dx} H_d(x) - aH_d(x) = -a + b(1 - F_d(x)) \tag{74}$$

In fact, using the fact that $H_d(0) = 0$, one can check this equivalence by integrating both sides of the above equation multiplying $e^{-\theta x}$ concerning x over $[0, \infty)$ for each $\theta > 0$. We can easily solve the linear differential equation (74) using the boundary conditions $H_d(0) = 0$ and $\lim_{x\to\infty} H_d(x) = 1$. Thus, we get

$$H_d(x) = 1 - b\int_x^\infty (1 - F_d(x))e^{-a(u-x)}du, \qquad x \geq 0.$$

Denote a by α_d. Then, $b = \beta_b$, and we have (67). From (74), we have

$$H'_d(0) = \beta_d - \alpha_d = \mu^*_d.$$

Hence, (70) implies (68). □

Remark 10. From (73) and the expression of β_d, we have

$$\hat{H}_d(\theta) = \beta_d \frac{\hat{F}_d(\alpha_d) - \hat{F}_d(\theta)}{\theta - \alpha_d}, \qquad \theta \geq 0, d \in D_g. \tag{75}$$

From (68), we can interpret α_d as the rate for the interruption of a clock with index d. This rate does not depend on the macro state $\mathbf{x}$ and site $\ell \in J(\mathbf{x})$ as long as $d = \gamma_{\mathbf{x}}(\ell)$.

Lemma 14. Under the assumptions of Lemma 13, we have, for each $\mathbf{x} \in \mathscr{X}$,

$$\sum_{i \in J(\mathbf{x})} c_{\mathbf{x}i}\mu^*_{\gamma_{\mathbf{x}}(i)}\pi(\mathbf{x}) = \sum_{\mathbf{x}' \in \mathscr{X}} \sum_{i \in J(\mathbf{x}')} \sum_{U \subset J(\mathbf{x})} c_{\mathbf{x}'i}\mu^*_{\gamma_{\mathbf{x}'}(i)}\pi(\mathbf{x}')p((\mathbf{x}', i), (\mathbf{x}, U)), \tag{76}$$

$$c_{\mathbf{x}\ell}(\alpha_{\gamma_{\mathbf{x}}(\ell)} + \mu^*_{\gamma_{\mathbf{x}}(\ell)})\pi(\mathbf{x}) = \sum_{\mathbf{x}' \in \mathscr{X}} \sum_{i \in J(\mathbf{x}')} \sum_{\ell \in U \subset J(\mathbf{x})} c_{\mathbf{x}'i}\mu^*_{\gamma_{\mathbf{x}'}(i)}\pi(\mathbf{x}')p((\mathbf{x}', i), (\mathbf{x}, U)), \quad \ell \in J_g(\mathbf{x}). \tag{77}$$

Remark 11. The right-hand side of (77) is the rate for the event that a new clock is activated at site ℓ. On the other hand, the left-hand side is the expiring rate of a clock at site ℓ. Hence, (77) represents the balance of the rates for expiring and activating clocks at the same site ℓ, so it is referred to as a local balance at site ℓ.

Proof. Substituting $H'_{\gamma_{\mathbf{x}'}(i)}(0) = \mu^*_{\gamma_{\mathbf{x}'}(i)}$ into (72) with $\theta = 0$ in the proof of Lemma 13 and noting the fact that (72) also holds for $\ell \in J_e(\mathbf{x})$, we have (76) for their summation. We next consider (72). For this let

$$K_1(\mathbf{x},\ell) = \sum_{i \in J(\mathbf{x})\setminus\{\ell\}} c_{\mathbf{x}i}\mu^*_{\gamma_{\mathbf{x}}(i)}\pi(\mathbf{x}) - \sum_{\mathbf{x}'\in\mathscr{X}} \sum_{i\in J(\mathbf{x}')} \sum_{\ell\notin U\subset J(\mathbf{x})} c_{\mathbf{x}'i}\mu^*_{\gamma_{\mathbf{x}'}(i)}\pi(\mathbf{x}')p((\mathbf{x}',i),(\mathbf{x},U)), \tag{78}$$

$$K_2(\mathbf{x},\ell) = \sum_{\mathbf{x}'\in\mathscr{X}} \sum_{i\in J(\mathbf{x}')} \sum_{\ell\in U\subset J(\mathbf{x})} c_{\mathbf{x}'i}\mu^*_{\gamma_{\mathbf{x}'}(i)}\pi(\mathbf{x}')p((\mathbf{x}',i),(\mathbf{x},U)). \tag{79}$$

Then, (72) can be written as

$$c_{\mathbf{x}\ell}\theta_\ell\pi(\mathbf{x})\hat{H}_d(\theta) = c_{\mathbf{x}\ell}H'_d(0)\pi(\mathbf{x}) + K_1(\mathbf{x},\ell)\hat{H}_d(\theta) - K_2(\mathbf{x},\ell)\hat{F}_d(\theta)$$

Subtracting this from (73) multiplied by $c_{\mathbf{x}\ell}\theta_\ell\pi(\mathbf{x})$, we have

$$(\alpha_d c_{\mathbf{x}\ell}\pi(\mathbf{x}) - K_1(\mathbf{x},\ell))(1-\hat{H}_d(\theta)) = (\beta_d c_{\mathbf{x}\ell}\pi(\mathbf{x}) - K_2(\mathbf{x},\ell))(1-\hat{F}_d(\theta)).$$

Because $1-\hat{H}_d(\theta)$ can not be a constant multiplication of $(1-\hat{F}_d(\theta))$ for $d \in D_g$ by Lemma 13, their coefficients must vanish. Thus, we have

$$\alpha_d c_{\mathbf{x}\ell}\pi(\mathbf{x}) = K_1(\mathbf{x},\ell), \qquad \beta_d c_{\mathbf{x}\ell}\pi(\mathbf{x}) = K_2(\mathbf{x},\ell). \tag{80}$$

This is nothing but (77) because $\beta_d = \alpha_d + \mu^*_d$. □

It is notable that (76) represents the global balance under macro state $\mathbf{x}$ while (77) is the local balance at site ℓ under macro state $\mathbf{x}$. We are now ready to prove the following theorem.

Theorem 11. The RGSMP satisfying the assumptions (15a) is product form decomposable and satisfies and (15c) if and only if there exist the distribution π on $\mathscr{X}$ and nonnegative numbers $\{\alpha_d; d \in D_g\}$ satisfying the global balance (76), the local balance (77) and the finite intensity condition:

$$\sum_{\mathbf{x}\in\mathscr{X}} \sum_{\ell\in J(\mathbf{x})} c_{\mathbf{x}\ell}\mu^*_{\gamma_{\mathbf{x}}(\ell)}\pi(\mathbf{x}) < \infty. \tag{81}$$

In this case, the stationary distribution ν is given by

$$\nu\Big(\mathbf{x}, \prod_{i\in J(\mathbf{x})}[0,u_i]\Big) = \pi(\mathbf{x}) \prod_{i\in J(\mathbf{x})} H_{\gamma_{\mathbf{x}}(i)}(u_i), \quad \mathbf{x}\in\mathscr{X}, u_i \ge 0, \tag{82}$$

where $\alpha_d = 0$ for $d \in D_e$ and μ^*_d and H_d are defined in Lemma 13. Furthermore, under this stationary distribution, (15b) is satisfied, and not more than one clock is activated at once, that is, we have, for each $\mathbf{x} \in \mathscr{X}$ and any $\ell_1, \ell_2 \in J_g(\mathbf{x})$ such that $\ell_1 \neq \ell_2$,

$$\sum_{\mathbf{x}'\in\mathscr{X}} \sum_{i\in J(\mathbf{x}')} \sum_{\{\ell_1,\ell_2\}\subset U\subset J_g(\mathbf{x})} c_{\mathbf{x}'i}\mu^*_{\gamma_{\mathbf{x}'}(i)}\pi(\mathbf{x}')p((\mathbf{x}',i),(\mathbf{x},U)) = 0. \tag{83}$$

Proof. We have already shown that the product decomposability with conditions (15a), (15b) and (15c) implies (76), (77), (81) and (82) with the nonnegative number α_d for $d \in D$. Thus, for the necessity, we only need to prove the last statement. Suppose that $\ell_1, \ell_2 \in J_g(\mathbf{x})$ satisfying $\ell_1 \neq \ell_2$ are simultaneously activated. Let $d_i = \gamma_{\mathbf{x}}(\ell_i)$ for $i = 1, 2$. Then, similar to (72), it follows from (61) that

$$\begin{aligned}
&(c_{\mathbf{x}\ell_1}\theta_1 + c_{\mathbf{x}\ell_2}\theta_2)\pi(\mathbf{x})\hat{H}_{d_1}(\theta_1)\hat{H}_{d_2}(\theta_2) \\
&\quad = (c_{\mathbf{x}\ell_1}\mu^*_{d_1}\hat{H}_{d_1}(\theta_1) + c_{\mathbf{x}\ell_2}\mu^*_{d_2}\hat{H}_{d_2}(\theta_2))\pi(\mathbf{x}) \\
&\qquad + \sum_{i \in J(\mathbf{x})\setminus\{\ell_1,\ell_2\}} c_{\mathbf{x}i}\mu^*_{\gamma_{\mathbf{x}}(i)}\pi(\mathbf{x})\hat{H}_{d_1}(\theta_1)\hat{H}_{d_2}(\theta_2) \\
&\qquad - \sum_{\mathbf{x}' \in \mathscr{X}} \sum_{i \in J(\mathbf{x}')} \sum_{\ell_1,\ell_2 \notin U \subset J(\mathbf{x})} c_{\mathbf{x}'i}\mu^*_{\gamma_{\mathbf{x}'}(i)}\pi(\mathbf{x}')p((\mathbf{x}',i),(\mathbf{x},U))\hat{H}_{d_1}(\theta_1)\hat{H}_{d_2}(\theta_2) \\
&\qquad - \sum_{\mathbf{x}' \in \mathscr{X}} \sum_{i \in J(\mathbf{x}')} \sum_{\ell_1 \notin U, \ell_2 \in U \subset J(\mathbf{x})} c_{\mathbf{x}'i}\mu^*_{\gamma_{\mathbf{x}'}(i)}\pi(\mathbf{x}')p((\mathbf{x}',i),(\mathbf{x},U))\hat{H}_{d_1}(\theta_1)\hat{F}_{d_2}(\theta_2) \\
&\qquad - \sum_{\mathbf{x}' \in \mathscr{X}} \sum_{i \in J(\mathbf{x}')} \sum_{\ell_1 \in U, \ell_2 \notin U \subset J(\mathbf{x})} c_{\mathbf{x}'i}\mu^*_{\gamma_{\mathbf{x}'}(i)}\pi(\mathbf{x}')p((\mathbf{x}',i),(\mathbf{x},U))\hat{F}_{d_1}(\theta_1)\hat{H}_{d_2}(\theta_2) \\
&\qquad - \sum_{\mathbf{x}' \in \mathscr{X}} \sum_{i \in J(\mathbf{x}')} \sum_{\ell_1,\ell_2 \in U \subset J(\mathbf{x})} c_{\mathbf{x}'i}\mu^*_{\gamma_{\mathbf{x}'}(i)}\pi(\mathbf{x}')p((\mathbf{x}',i),(\mathbf{x},U))\hat{F}_{d_1}(\theta_1)\hat{F}_{d_2}(\theta_2). \quad (84)
\end{aligned}$$

On the other hand, multiplying (72) for $\ell = \ell_1$ and $\theta = \theta_1$ by $\hat{H}_{d_2}(\theta_2)$, we have

$$\begin{aligned}
&c_{\mathbf{x}\ell_1}\theta_1\pi(\mathbf{x})\hat{H}_{d_1}(\theta_1)\hat{H}_{d_2}(\theta_2) \\
&\quad = (c_{\mathbf{x}\ell_1}\mu^*_{d_1}\hat{H}_{d_1}(\theta_1)\hat{H}_{d_2}(\theta_2) + c_{\mathbf{x}\ell_2}\mu^*_{d_2}\hat{H}_{d_2}(\theta_2))\pi(\mathbf{x}) \\
&\qquad + \sum_{i \in J(\mathbf{x})\setminus\{\ell_1,\ell_2\}} c_{\mathbf{x}i}\mu^*_{\gamma_{\mathbf{x}}(i)}\pi(\mathbf{x})\hat{H}_{d_1}(\theta_1)\hat{H}_{d_2}(\theta_2) \\
&\qquad - \sum_{\mathbf{x}' \in \mathscr{X}} \sum_{i \in J(\mathbf{x}')} \sum_{\ell_1,\ell_2 \notin U \subset J(\mathbf{x})} c_{\mathbf{x}'i}\mu^*_{\gamma_{\mathbf{x}'}(i)}\pi(\mathbf{x}')p((\mathbf{x}',i),(\mathbf{x},U))\hat{H}_{d_1}(\theta_1)\hat{H}_{d_2}(\theta_2) \\
&\qquad - \sum_{\mathbf{x}' \in \mathscr{X}} \sum_{i \in J(\mathbf{x}')} \sum_{\ell_1 \notin U, \ell_2 \in U \subset J(\mathbf{x})} c_{\mathbf{x}'i}\mu^*_{\gamma_{\mathbf{x}'}(i)}\pi(\mathbf{x}')p((\mathbf{x}',i),(\mathbf{x},U))\hat{H}_{d_1}(\theta_1)\hat{H}_{d_2}(\theta_2) \\
&\qquad - \sum_{\mathbf{x}' \in \mathscr{X}} \sum_{i \in J(\mathbf{x}')} \sum_{\ell_1 \in U, \ell_2 \notin U \subset J(\mathbf{x})} c_{\mathbf{x}'i}\mu^*_{\gamma_{\mathbf{x}'}(i)}\pi(\mathbf{x}')p((\mathbf{x}',i),(\mathbf{x},U))\hat{F}_{d_1}(\theta_1)\hat{H}_{d_2}(\theta_2) \\
&\qquad - \sum_{\mathbf{x}' \in \mathscr{X}} \sum_{i \in J(\mathbf{x}')} \sum_{\ell_1,\ell_2 \in U \subset J(\mathbf{x})} c_{\mathbf{x}'i}\mu^*_{\gamma_{\mathbf{x}'}(i)}\pi(\mathbf{x}')p((\mathbf{x}',i),(\mathbf{x},U))\hat{F}_{d_1}(\theta_1)\hat{H}_{d_2}(\theta_2). \quad (85)
\end{aligned}$$

Subtracting both sides (85) from (84), we have

$$\begin{aligned}
&c_{\mathbf{x}\ell_2}\theta_2\pi(\mathbf{x})\hat{H}_{d_1}(\theta_1)\hat{H}_{d_2}(\theta_2) = c_{\mathbf{x}\ell_2}\mu^*_{d_2}\pi(\mathbf{x})(1 - \hat{H}_{d_2}(\theta_2))\hat{H}_{d_1}(\theta_1) \\
&- \sum_{\mathbf{x}' \in \mathscr{X}} \sum_{i \in J(\mathbf{x}')} \sum_{\ell_1 \notin U, \ell_2 \in U \subset J(\mathbf{x})} c_{\mathbf{x}'i}\mu^*_{\gamma_{\mathbf{x}'}(i)}\pi(\mathbf{x}')p((\mathbf{x}',i),(\mathbf{x},U))\hat{H}_{d_1}(\theta_1))(\hat{F}_{d_2}(\theta_2) - \hat{H}_{d_2}(\theta_2)) \\
&- \sum_{\mathbf{x}' \in \mathscr{X}} \sum_{i \in J(\mathbf{x}')} \sum_{\ell_1,\ell_2 \in U \subset J(\mathbf{x})} c_{\mathbf{x}'i}\mu^*_{\gamma_{\mathbf{x}'}(i)}\pi(\mathbf{x}')p((\mathbf{x}',i),(\mathbf{x},U))\hat{F}_{d_1}(\theta_1)(\hat{F}_{d_2}(\theta_2) - \hat{H}_{d_2}(\theta_2)).
\end{aligned}$$

Dividing both sides of the above formula by θ_2 and letting $\theta_2 \downarrow 0$ yield

$$
\begin{aligned}
c_{\mathbf{x}\ell_2}\pi(\mathbf{x})\hat{H}_{d_1}(\theta_1) &= c_{\mathbf{x}\ell_2}\mu^*_{d_2}\pi(\mathbf{x})(-\hat{H}'_{d_2}(0))\hat{H}_{d_1}(\theta_1) \\
&- \sum_{\mathbf{x}'\in\mathscr{X}}\sum_{i\in J(\mathbf{x}')}\sum_{\ell_1\notin U,\ell_2\in U\subset J(\mathbf{x})} c_{\mathbf{x}'i}\mu^*_{\gamma_{\mathbf{x}'}(i)}\pi(\mathbf{x}')p((\mathbf{x}',i),(\mathbf{x},U))\hat{H}_{d_1}(\theta_1))(\hat{F}'_{d_2}(0)-H'_{d_2}(0)) \\
&- \sum_{\mathbf{x}'\in\mathscr{X}}\sum_{i\in J(\mathbf{x}')}\sum_{\ell_1,\ell_2\in U\subset J(\mathbf{x})} c_{\mathbf{x}'i}\mu^*_{\gamma_{\mathbf{x}'}(i)}\pi(\mathbf{x}')p((\mathbf{x}',i),(\mathbf{x},U))\hat{F}_{d_1}(\theta_1)(\hat{F}'_{d_2}(0)-H'_{d_2}(0)).
\end{aligned}
$$

Consequently, $\hat{F}_{d_1}(\theta_1)$ must be proportional to $\hat{H}_{d_1}(\theta_1)$ and therefore identical with $\hat{H}_{d_1}(\theta_1)$. This is impossible. Thus, not more than one clock can not be activated at once.

We next show the converse. Summing (77) over all $\ell \in J_g(\mathbf{x})$ and subtracting this sum from (76), we have

$$
\begin{aligned}
\sum_{i\in J_e(\mathbf{x})} c_{\mathbf{x}i}\mu^*_{\gamma_{\mathbf{x}}(i)}\pi(\mathbf{x}) &= \sum_{i\in J_g(\mathbf{x})} c_{\mathbf{x}i}\alpha_{\gamma_{\mathbf{x}}(i)}\pi(\mathbf{x}) \\
&+ \sum_{\mathbf{x}'\in\mathscr{X}}\sum_{i\in J(\mathbf{x}')}\sum_{U\subset J_e(\mathbf{x})} c_{\mathbf{x}'i}\mu^*_{\gamma_{\mathbf{x}'}(i)}\pi(\mathbf{x}')p((\mathbf{x}',i),(\mathbf{x},U)). \qquad (86)
\end{aligned}
$$

From the definition of $\hat{H}_d(\theta)$, it follows that

$$
\beta_d\hat{F}_d(\theta) = (\alpha_d-\theta)\hat{H}_d(\theta)+\mu^*_d.
$$

Multiplying both sides of (77) by $\hat{F}_d(\theta)$ and substituting the above $\hat{F}_d(\theta)$ to its left side, we have

$$
c_{\mathbf{x}\ell}\theta\pi(\mathbf{x})\hat{H}_d(\theta) = c_{\mathbf{x}\ell}\mu^*_d\pi(\mathbf{x}) + c_{\mathbf{x}\ell}\alpha_d\pi(\mathbf{x})\hat{H}_d(\theta) - K_2(\mathbf{x},\ell)\hat{F}_d(\theta). \qquad (87)
$$

From (76) and (77), we have

$$
\begin{aligned}
\sum_{i\in J(\mathbf{x})} c_{\mathbf{x}i}\mu^*_{\gamma_{\mathbf{x}}(i)}\pi(\mathbf{x}) &= \sum_{\mathbf{x}'\in\mathscr{X}}\sum_{i\in J(\mathbf{x}')}\sum_{\ell\notin U\subset J(\mathbf{x})} c_{\mathbf{x}'i}\mu^*_{\gamma_{\mathbf{x}'}(i)}\pi(\mathbf{x}')p((\mathbf{x}',i),(\mathbf{x},U)) \\
&\quad + \sum_{\mathbf{x}'\in\mathscr{X}}\sum_{i\in J(\mathbf{x}')}\sum_{\ell\in U\subset J(\mathbf{x})} c_{\mathbf{x}'i}\mu^*_{\gamma_{\mathbf{x}'}(i)}\pi(\mathbf{x}')p((\mathbf{x}',i),(\mathbf{x},U)) \\
&= \sum_{\mathbf{x}'\in\mathscr{X}}\sum_{i\in J(\mathbf{x}')}\sum_{\ell\notin U\subset J(\mathbf{x})} c_{\mathbf{x}'i}\mu^*_{\gamma_{\mathbf{x}'}(i)}\pi(\mathbf{x}')p((\mathbf{x}',i),(\mathbf{x},U)) \\
&\quad + c_{\mathbf{x}\ell}(\alpha_{\gamma_{\mathbf{x}}(\ell)}+\mu^*_{\gamma_{\mathbf{x}}(\ell)})\pi(\mathbf{x}).
\end{aligned}
$$

Substituting $c_{\mathbf{x}\ell}\alpha_{\gamma_{\mathbf{x}}(\ell)}\pi(\mathbf{x})$ from this equation into (87), we arrive at

$$
\begin{aligned}
c_{\mathbf{x}\ell}\theta\pi(\mathbf{x})\hat{H}_d(\theta) &= c_{\mathbf{x}\ell}\mu^*_d\pi(\mathbf{x}) + \sum_{i\in J(\mathbf{x})\setminus\{\ell\}} c_{\mathbf{x}i}\mu^*_{\gamma_{\mathbf{x}}(i)}\pi(\mathbf{x})\hat{H}_d(\theta) \\
&- \sum_{\mathbf{x}'\in\mathscr{X}}\sum_{i\in J(\mathbf{x}')}\sum_{\ell\notin U\subset J(\mathbf{x})} c_{\mathbf{x}'i}\mu^*_{\gamma_{\mathbf{x}'}(i)}\pi(\mathbf{x}')p((\mathbf{x}',i),(\mathbf{x},U))\hat{H}_d(\theta) \\
&- \sum_{\mathbf{x}'\in\mathscr{X}}\sum_{i\in J(\mathbf{x}')}\sum_{\ell\in U\subset J(\mathbf{x})} c_{\mathbf{x}'i}\mu^*_{\gamma_{\mathbf{x}'}(i)}\pi(\mathbf{x}')p((\mathbf{x}',i),(\mathbf{x},U))\hat{F}_d(\theta).
\end{aligned}
$$

This equation is identical with (72). Let $d = \gamma_{\mathbf{x}}(\ell)$, multiply both sides of it by $\prod_{i \in J(\mathbf{x}) \setminus \{\ell\}} \hat{H}_{\gamma_{\mathbf{x}}(i)}(\theta_i)$ and define distributions ν, ν_ℓ and constants λ_ℓ by

$$\hat{\nu}(\mathbf{x}, \theta_{\mathbf{x}}(J_{\mathrm{e}}(\mathbf{x}))) = \pi(\mathbf{x}) \prod_{i \in J(\mathbf{x})} \hat{H}_{\gamma_{\mathbf{x}}(i)}(\theta_i),$$
$$\hat{\nu}_\ell(\mathbf{x}, \theta_{\mathbf{x}}(J_{\mathrm{e}}(\mathbf{x}))) = \pi(\mathbf{x}) \prod_{i \in J(\mathbf{x}) \setminus \{\ell\}} \hat{H}_{\gamma_{\mathbf{x}}(i)}(\theta_i),$$
$$\lambda_\ell = \sum_{\mathbf{x} \in \mathscr{X}} c_{\mathbf{x}\ell} \mu^*_{\gamma_{\mathbf{x}}(\ell)} \pi(\mathbf{x}).$$

We then have the stationary equation (61). Hence, ν is the stationary distribution of the RGSMP by Theorem 10. □

There are a number of remarks on this theorem.

Remark 12. This theorem does not answer the uniqueness of the stationary distribution. However, the uniqueness can be considered through the irreducibility. In particular, for the macro state distribution, it is not hard to check the irreducibility from the global balance equation (76) similar to the irreducibility of a Markov chain with discrete state space $\mathscr{X}$.

Remark 13. Although at most one clock with non exponentially distributed lifetime is activated at each completion time, some clocks with exponentially distributed life times may be activated at the same instant. Thus, it is not necessary that $U = \{\ell\}$ in (76) and (77).

Remark 14. From the proof of Lemma 14, we can see that $\alpha_{\gamma_{\mathbf{x}}(\ell)} > 0$ if and only if $K_1(\mathbf{x}, \ell) > 0$, that is

$$\sum_{i \in J(\mathbf{x}) \setminus \{\ell\}} c_{\mathbf{x}i} \mu^*_{\gamma_{\mathbf{x}}(i)} \pi(\mathbf{x}) - \sum_{\mathbf{x}' \in \mathscr{X}} \sum_{i \in J(\mathbf{x}')} \sum_{\ell \notin U \subset J(\mathbf{x})} c_{\mathbf{x}'i} \mu^*_{\gamma_{\mathbf{x}'}(i)} \pi(\mathbf{x}') p((\mathbf{x}', i), (\mathbf{x}, U)) > 0.$$

By the global equation (76), this is equivalent to

$$\sum_{\mathbf{x}' \in \mathscr{X}} \sum_{i \in J(\mathbf{x}')} \sum_{\ell \in U \subset J(\mathbf{x})} c_{\mathbf{x}'i} \mu^*_{\gamma_{\mathbf{x}'}(i)} \pi(\mathbf{x}') p((\mathbf{x}', i), (\mathbf{x}, U)) - c_{\mathbf{x}\ell} \mu^*_{\gamma_{\mathbf{x}}(\ell)} \pi(\mathbf{x}) > 0.$$

This means that $\alpha_{\gamma_{\mathbf{x}}(\ell)} > 0$ holds if and only if the total activation rate of type $d = \gamma_{\mathbf{x}}(\ell)$ clock is greater than its total completion rate. Thus, α_d can be interpreted as an interruption rate.

Remark 15. We have not discussed how to compute the interruption rate α_d. In many cases, they are given as modeling parameters. If this is not the case, they would be determined by (80) although they are highly nonlinear equations.

In the rest of this section, we consider the case where there is no interruption, that is, $\alpha_d = 0$ for all $d \in D$. The following corollary is immediate from Theorem 11.

Corollary 10. Suppose the RGSMP satisfies (15a) and has no interruption. Then, the RGSMP is product form decomposable and satisfies (15b) and (15c) if and only if there exist the distribution π on $\mathscr{X}$ satisfying the global and local balances:

$$\sum_{i\in J(\mathbf{x})} c_{\mathbf{x}i}\mu_{\gamma_{\mathbf{x}}(i)}\pi(\mathbf{x}) = \sum_{\mathbf{x}'\in\mathscr{X}} \sum_{i\in J(\mathbf{x}')} \sum_{U\subset J(\mathbf{x})} c_{\mathbf{x}'i}\mu_{\gamma_{\mathbf{x}'}(i)}\pi(\mathbf{x}')p((\mathbf{x}',i),(\mathbf{x},U)), \tag{88}$$

$$c_{\mathbf{x}\ell}\mu_{\gamma_{\mathbf{x}}(\ell)}\pi(\mathbf{x}) = \sum_{\mathbf{x}'\in\mathscr{X}} \sum_{i\in J(\mathbf{x}')} \sum_{\ell\in U\subset J(\mathbf{x})} c_{\mathbf{x}'i}\mu_{\gamma_{\mathbf{x}'}(i)}\pi(\mathbf{x}')p((\mathbf{x}',i),(\mathbf{x},U)),\ \ell\in J_g(\mathbf{x}), \tag{89}$$

and the finite intensity condition:

$$\sum_{\mathbf{x}\in\mathscr{X}} \sum_{\ell\in J(\mathbf{x})} c_{\mathbf{x}\ell}\mu_{\gamma_{\mathbf{x}}(\ell)}\pi(\mathbf{x}) < \infty. \tag{90}$$

In this case, the stationary distribution ν is given by

$$\nu\Big(\mathbf{x}, \prod_{i\in J(\mathbf{x})} [0,u_i]\Big) = \pi(\mathbf{x}) \prod_{i\in J(\mathbf{x})} \mu_{\gamma_{\mathbf{x}}(i)} \int_0^{u_i} (1-F_{\gamma_{\mathbf{x}}(i)}(v))dv, \quad \mathbf{x}\in\mathscr{X}, u_i \geq 0. \tag{91}$$

Furthermore, not more than one clock is activated at once.

Note that the stationary distribution π of the macro states depend on F_d for $d\in D_g$ only through their means μ_d^{-1}. This stationary distribution is said to be insensitive with respect to F_d for $d\in D_g$.

Example 17. Consider the $M/G/1$ queue of Example 15. We have formulated it by the RGSMP. Since there is no interruption, we examine the product form decomposability by Corollary 10. The local balance condition (89) is

$$\mu_1\pi(1) = \lambda\pi(0) + \mu_1\pi(2), \qquad \mu_1\pi(n) = \mu_1\pi(n+1), \qquad n\geq 2.$$

Obviously, these are impossible. Hence, the $M/G/1$ queue with the first-come first-served discipline can not be product form decomposable. □

17 Applications to queues and their networks

How we can check the conditions in Theorem 11 and Corollary 10 to see the decomposability ? It is notable that we do not need to consider the RGSMP with generally distributed lifetimes. Namely, we only need to find the stationary distribution of the macro state which satisfies (76) and (77) (or (88) and (89)). In particular, if there is no interruption, it is sufficient to consider the RGSMP all of whose lifetimes are exponentially distributed. This greatly simplifies the verification of the decomposability.

In this section, we exemplify queues and their networks by applying Corollary 10 and Theorem 11 in this way. We first consider the following queueing system.

(17a) There are service positions numbered $1,2,\ldots$ to accommodate one customer in each position. Customers arrives subject to the Poisson process with rate λ with *i.i.d* amounts of work for service, whose distribution is denoted by F. This F is assumed to have a finite mean $\frac{1}{\mu}$.

(17b) An arriving customer who found n customers in the system gets into position ℓ with probability $\delta_{n+1,\ell}$ for $\ell = 1,2,\cdots,n+1$, and customers in positions $\ell,\ell+1,\cdots,n$ move to $\ell+1,\ell+2,\cdots,n+1$, respectively, where

$$\sum_{\ell=1}^{n+1} \delta_{n+1,\ell} = 1, \qquad n \geq 0.$$

Thus, if there are n customers in the system, positions $1,2,\cdots,n$ are occupied.

(17c) A customer in position ℓ is served at rate $c_{n,\ell}$ for $\ell = 1,2,\ldots,n$ when there are n customers in the system. Denote the total service rate in this case by $\sigma(n)$. That is,

$$\sigma(n) = \sum_{\ell=1}^{n} c_{n\ell}, \qquad n \geq 1.$$

If a customer in position ℓ leaves the system, customers in positions $\ell+1,\ell+2,\cdots,n$ move to $\ell,\ell+1,\cdots,n-1$.

This model is referred to as a packed positioning queue. For each t, denote the number of customers in system by $X(t)$ and the remaining work of the customer at position ℓ by $Y_\ell(t)$ for $\ell = 1,2,\ldots,X(t)$. Let $\mathbf{Y}(t) = (Y_1(t),\ldots,Y_{X(t)}(t))$. We show that the process $(X(t),\mathbf{Y}(t))$ is the RGSMP. Let

$$\mathscr{X} = \mathbb{N}_+, \quad D = \{e,g\}, \quad J_e(n) = \{0\}, \quad J_g(n) = \{1,2,\ldots,n\} \text{ for } n \in \mathscr{X},$$

where $\mathbb{N}_+ = \{0,1,2,\ldots\}$. The index functions are defined as $\gamma_{en}(0) = e$ and $\gamma_{gn}(\ell) = g$ for $\ell \geq 1$, where e and g represent exponential and general distributions, respectively.

Let $c_{n0} = \lambda$ for all $n \in \mathscr{X}$. We interpret $c_{n\ell}$ for $\ell \geq 1$ as the speed of a clock at the site ℓ under the macrostate n. Define transition probabilities by

$$\begin{aligned} &p((n,0),(n+1,\ell)) = \delta_{(n+1)\ell}, && (n \in \mathscr{X}, 1 \leq \ell \leq n+1),\\ &p((n,\ell),(n-1,\emptyset)) = 1, && (n \geq 1, 1 \leq \ell \leq n),\\ &\mu_e = \lambda, \qquad \mu_g = \mu. \end{aligned}$$

Thus, $(X(t),\mathbf{Y}(t))$ can be considered as the RGSMP. Hence, by Corollary 10, the RGSMP supplemented by the remaining service requirements is product-form decomposable if and only if

$$c_{n\ell}\mu\pi(n) = \lambda\delta_{n\ell}\pi(n-1) \qquad (n \geq 1, 1 \leq \ell \leq n). \tag{92}$$

From this, we see that

$$c_{n\ell} = \sigma(n)\delta_{n\ell}, \qquad n \geq 1, \ell = 1, 2, \ldots, n.$$

This service discipline is called symmetric by Kelly [18, 19].

Thus, the packed positioning queue with Poisson arrivals and *i.i.d.* service requirements is product form decomposable if and only if its service discipline is symmetric. In this case, (92) uniquely determines the stationary distribution $\{\pi(n)\}$ as

$$\pi(n) = \pi(0)\frac{\lambda^n}{\mu^n \prod_{i=1}^{n} \sigma(i)}, \qquad n \geq 1.$$

where $\sigma(n) = \sum_{\ell=1}^{n} c_{n\ell}$, if

$$\sum_{n=0}^{+\infty} \frac{\lambda^n}{\mu^n \prod_{i=1}^{n} \sigma(i)} < +\infty.$$

This stationary distribution is insensitive with respect to the work for service. Furthermore, (92) implies

$$\sum_{\ell=1}^{n+1} c_{n\ell}\mu\pi(n+1) = \lambda\pi(n) \qquad n \geq 0.$$

Hence, by a similar time-reversed argument in Example 16, we can see that the departure process from this queue is also Poisson. This is generally known as quasi-reversibility (see [9] for its details).

Note that packing rule of service positions does not affect to get (92), that is, any reallocations are possible at arriving and departing instants if all positions are packed. In what follows, we refer to this model simply as a symmetric queue.

Remark 16. One might expect that the insensitivity of the queue length distribution implies the symmetric condition. But, this is not true. For example, assume that, for each n, $c_{n\ell} = \frac{1}{n}$ for $1 \leq \ell \leq n$ and $\delta_{n\ell} = 1$ only if $\ell = 1$. Then, the sample path of $\{X(t)\}$ is identical with that of the corresponding symmetric queue with $c_{n\ell} = \delta_{n\ell} = \frac{1}{n}$ for $1 \leq \ell \leq n$ since all customers in service have a same service rate after arriving of a new customer. Thus, (92) does not hold but the queue length distribution is still insensitive. This example is rather trivial, but shows the local balance (76) is indeed stronger than the insensitivity. □

We next consider the case where interruptions occur in the symmetric queue with Poisson arrivals. In addition to the assumptions (17a), (17b) and (17c), we assume the following condition.

(17d) Negative signals arrive according to the Poisson process with rate $\alpha\sigma(n)$, which is independent of everything else, and delete a customer in position ℓ with probability $\delta_{n\ell}$ when n customers are in the system.

The index for this signal is denoted by -1. That is, $J(n) = \{-1, 0, 1, 2, \ldots, n\}$ for $n \geq 0$. Suppose the local balance (77) holds. Since the general index g is only activated by arrivals, we have

$$(\mu^* + \alpha)\delta_{n\ell}\sigma(n)\pi(n) = \lambda\delta_{n\ell}\pi(n-1), \qquad n \geq 1,\ \ell = 1,2,\ldots,n,$$

where μ^* is given by (69) for $\alpha_d = \alpha$. Thus, the stationary distribution is given by

$$\pi(n) = \pi(0)\frac{\lambda^n}{(\mu^* + \alpha)^n \prod_{i=1}^n \sigma(i)}, \qquad n \geq 0,$$

where the stability condition $\sum_{n=0}^{\infty} \frac{\lambda^n}{(\mu^* + \alpha)^n \prod_{i=1}^n \sigma(i)} < \infty$ is assumed. Then, it is easy to see that this distribution satisfies the global balance (76):

$$(\lambda + (\mu^* + \alpha)\sigma(n))\pi(n) = \lambda\pi(n-1) + (\mu^* + \alpha)d(n+1)\pi(n+1), \quad n \geq 1.$$

Hence, (77) indeed holds, and we have the product form decomposability by Theorem 11.

Example 18. The symmetric queue can be generalized for multi-class queues and their networks. We show how to formulate multi-class symmetric queues by an RGSMP. Suppose there are T types of customers. Denote a set of their types $\{1,2,\cdots,T\}$ by $\mathscr{T}$. We assume that the arrival process of type i customers is Poisson with the rate λ_i and the arrival streams of different types of customers are independent. Now the macrostate needs to specify the configuration of customer types in positions. So far, we let

$$\mathscr{X} = \{\mathbf{x} = (t(1),t(2),\cdots,t(n)); n \geq 0, t(i) \in \mathscr{T}\}.$$

The site space is same as the packed positioning queue, but D is changed to $\{e,1,\cdots,T\}$, which means that different types of customers may have different service time distributions. Service discipline is also same as the packed positioning queue. We assume that the speeds of service and position selecting probabilities of arriving customers may depends on $n = |\mathbf{x}|$, so the total speed also only depends on n, which is denoted by $\sigma(n)$. Then, the local balance (76) becomes, for $\mathbf{x} = (t(1),\cdots,t(n))$ and $\mathbf{x} \ominus \mathbf{e}_\ell = (t(1),\cdots,t(\ell-1),t(\ell+1),\cdots,t(n))$,

$$c_{n\ell}\mu_{t(\ell)}\pi(\mathbf{x}) = \delta_{n\ell}\lambda_{t(\ell)}\pi(\mathbf{x} \ominus \mathbf{e}_\ell), \qquad \ell \in J_g(\mathbf{x}) \equiv \{1,2,\ldots,n\}. \tag{93}$$

Thus, if the queue is symmetric, i.e., if $c_{n\ell}$ is proportional to $\delta_{n\ell}$ concerning ℓ for each $n = |\mathbf{x}|$, then

$$\pi(\mathbf{x}) = \pi(\mathbf{0})\prod_{\ell=1}^{n} \frac{\lambda_{t(\ell)}}{\mu_{t(\ell)}\sigma(\ell)}, \qquad \mathbf{x} = (t(1),\cdots,t(n)),$$

gives a stationary distribution if the total sum of $\pi(\mathbf{x})$ over $\mathscr{X}$ is finite, where $\pi(\mathbf{0})$ is the normalizing constant. From (93), we again have the quasi-reversibility for each fixed type t as

$$\sum_{\ell=1}^{n+1} c_{n\ell}\mu_t \pi(\mathbf{x} \oplus \mathbf{e}_\ell(t)) = \lambda_t \pi(\mathbf{x}), \qquad \mathbf{x} \in \mathscr{X},$$

where $\mathbf{x} \oplus \mathbf{e}_\ell(t) = (t(1), \cdots, t(\ell-1), t, t(\ell+1), \ldots, t(n))$ for $\mathbf{x} = (t(1), \cdots, t(n))$. since $\sigma(n)\mu_{t(\ell)}\pi(\mathbf{x}) = \lambda_{t(\ell)}\pi(\mathbf{x}_\ell)$ by the symmetric condition. These are the well-known results originally obtained by [18] and Chandy, Howard and Towsley [8]. We here note that (93) fully verifies the insensitivity with respect to the distributions of the amount of work for all types due to Corollary 10. □

Consider an open or closed queueing network with multi-class Markovian routing whose nodes have symmetric service discipline in the sense of Example 18. Then, each node in separation with multi-class Poisson arrivals is quasi-reversible. Hence, from the product form solution for a quasi-reversible network (see, e.g., [9]), if all service requirement distributions are exponential and exogenous arrivals are subject to Poisson processes, then this queueing network has the product form stationary distribution for all type configurations over the network, and satisfies the local balance at each node for each type of customers.

This concludes that the stationary distribution is insensitive with respect to the distributions of the amount of work for all types of customers at each node. This result is usually verified by approximating such distributions by phase types of distributions or by assuming the densities of those distributions. We can again fully verify it by Corollary 10.

For those product form queueing networks, we can also consider the case that there are negative customers or negative signals at each node as in the condition (17d). Similar to the single node case, we can show the product form decomposability by Theorem 11. Of course, the macro state distribution can not be insensitive in this case.

18 Further insensitivity structure in RGSMP

The product form decomposable RGSMP has insensitive structure not only for the stationary distribution but also for other characteristics. A most prominent feature among them is the conditional mean actual lifetime of a clock given its nominal lifetime, where the nominal lifetime is meant the total amount of lifetime when the clock always advances with unit speed. In RGSMP, speeds of clocks may change, so the actual lifetimes are different from their nominal lifetimes in general. The actual lifetimes are interesting for us since they correspond with the sojourn times of customers in symmetric queues and their networks. We shall show that the mean total sojourn time of a customer arriving at a product form decomposable queueing network is proportional to his total work for service, and its coefficient can be computed.

We first consider the attained sojourn time of an arbitrary fixed clock of a fixed insensitive type $d \in D_g$ in RGSMS. Such a clock is called *tagged*. For this purpose, besides the initial distribution of the RGSMP $\{\mathbf{Z}(t)\}$ given in (91), we will consider a further initial distribution which will be specified below and which can be interpreted as a conditional version of that given in (91) under the condition that, at time zero, a new (tagged) clock of type $d \in D_g$ is activated.

Let τ^* denote the (total) nominal lifetime of the tagged clock. For $y \le \tau^*$, let T_y^* be the length of time required by the tagged clock to process y units of its nominal lifetime and let $\ell^*(t)$ denote the site at which this clock is at time $t \le T_y^*$. Then T_y^* is given by

$$T_y^* = \sup\left\{ t > 0 : \int_0^t c_{\mathbf{X}(u)\ell^*(u)} du < y \right\}. \tag{94}$$

Throughout this section we assume that the point process N generated by macro state transition instants has finite intensity λ.

We need further notation for describing various point processes arising in connection with stationary RGSMP. Let $N_{(d)}$ be the point process generated by all jump instants at which a new clock of type d is activated. Let $\lambda_{(d)}$ and $P_{(d)}$ denote the intensity of $N_{(d)}$ and the Palm distribution of P with respect to $N_{(d)}$, respectively. Note that $P_{(d)}$ can be interpreted as the conditional probability measure of P given that a clock of type d starts at time 0. For each site $s \in J$, we also introduce the point process N_s generated by all jump instants at which site s gets a new clock of type d, and denote its intensity and the corresponding Palm distribution by λ_s and P_s, respectively. Furthermore, we use the following notation:

$$J_d = \cup_{\mathbf{x} \in \mathscr{X}} \{s \in J(\mathbf{x}) : \gamma_{\mathbf{x}}(s) = d\}, \qquad \mathscr{X}_s = \{\mathbf{x} \in \mathscr{X} : s \in J(\mathbf{x})\}.$$

Since $N_{(d)} = \sum_{s \in J_d} N_s$, from the definition of Palm distribution, we have (see (14))

$$\lambda_{(d)} P_{(d)}(C) = \sum_{s \in J_d} \lambda_s P_s(C), \qquad C \in \mathscr{F}. \tag{95}$$

By $A^*(t)$ we denote the amount of the nominal lifetime of the tagged clock processed up to time t, i.e.

$$A^*(t) = \int_0^t c_{\mathbf{X}(u)\ell^*(u)} du$$

for every $t \ge 0$ with $A^*(t) < \tau^*$. For $t \ge T_{\tau^*}^*$, we put $A^*(t) \equiv \tau^*$. Furthermore, by $A_s^*(t)$ we denote the amount of the nominal lifetime that a clock of type d, which has been activated at time zero at site s, has consumed up to time $t \ge 0$. Since $P_s(N_{s'}(0) = 1) = \mathbf{1}_{\{s\}}(s')$ for $s, s' \in J$. Hence, (95) yields, for $u, y \ge 0$, $s' \in J_d$ and $C \in \mathscr{F}$,

$$\begin{aligned} \lambda_{(d)} P_{(d)}(A^*(u) < y, l^*(0+) = s', C) &= \sum_{s'' \in J_d} \lambda_{s''} P_{s''}(A^*(u) < y, l^*(0+) = s', C) \\ &= \lambda_{s'} P_{s'}(A_{s'}^*(u) < y, C) \end{aligned}$$

Thus, by summing up for all possible s' in the above equation, we get the following result.

Lemma 15. For $u, y \geq 0$ and $C \in \mathscr{F}$,

$$\lambda_{(d)} P_{(d)}(A^*(u) < y, C) = \sum_{s' \in J_d} \lambda_{s'} P_{s'}(A^*_{s'}(u) < y, C) \, .$$

We denote the nominal lifetime τ^* of the tagged clock by τ^*_s if the tagged clock is created at site s. For $s \in S$, $\mathbf{x} \in \mathscr{X}_s$ and $u, \mathbf{y}_\ell \geq 0$, define the event $C_{\mathbf{x}s}(u, \mathbf{y}_\ell) \in \mathscr{F}$ by

$$C_{\mathbf{x}s}(u, \mathbf{y}_\ell) \equiv \{X(u) = \mathbf{x}, R_{s'}(u) \leq y_{s'}(s' \in J(\mathbf{x}) \setminus \{s\})\} \, ,$$

where $\mathbf{y}_\ell = \{y_{s'}; s' \in J(\mathbf{x}) \setminus \{s\}\}$. Since the probability $P_{s'}(A^*_{s'}(u) < y, \ell^*(u) = s, C_{\mathbf{x}s}(u, \mathbf{y}_\ell) \,|\, \tau^*_{s'})$ does not depend on $\tau^*_{s'}$ on the set $\{\tau^*_{s'} \geq y\} \in \mathscr{F}$, we can write, for $0 \leq z \leq t^* - y$,

$$\begin{aligned} &P_{s'}(A^*_{s'}(u) < y, \ell^*(u) = s, C_{\mathbf{x}s}(u, \mathbf{y}_\ell) \,|\, \tau^*_{s'} = t^*) \\ &\qquad = P_{s'}(A^*_{s'}(u) < y, \ell^*(u) = s, C_{\mathbf{x}s}(u, \mathbf{y}_\ell) \,|\, \tau^*_{s'} = y + z) \, , \end{aligned} \tag{96}$$

where $t^* = \sup\{u : 1 - F_d(u) > 0\}$. Moreover, by Lemma 15, we have

$$\begin{aligned} &\lambda_{(d)} \int_0^\infty P_{(d)}(A^*(u) < y, \ell^*(u) = s, C_{\mathbf{x}s}(u, \mathbf{y}_\ell) |\ \tau^* = z) F_d(dz) \\ &= \sum_{s' \in J_d} \lambda_{s'} \int_0^\infty P_{s'}(A^*_{s'}(u) < y, \ell^*(u) = s, C_{\mathbf{x}s}(u, \mathbf{y}_\ell) |\ \tau^*_{s'} = z) F_d(dz) \, . \end{aligned} \tag{97}$$

We are now in a position to prove the next lemma.

Lemma 16. Assume that F_d is purely atomic and has a finite number of atoms, i.e. $F_d(x)$ is a step function with a finite number of jumps. Then, for every $\mathbf{x} \in \mathscr{X}$ and $s \in J(\mathbf{x})$ satisfying $\gamma_{\mathbf{x}}(s) = d$, for $0 \leq y \leq t^*$ and for $\mathbf{y}_\ell \geq 0$, we have

$$\begin{aligned} &\mu_d \pi(\mathbf{x}) y \prod_{s' \in J(\mathbf{x}) \setminus \{s\}} F^{(r)}_{\gamma_{\mathbf{x}}(s')}(y_{s'}) \\ &\qquad = \lambda_{(d)} \int_0^\infty P_{(d)}(A^*(u) < y, \ell^*(u) = s, C_{\mathbf{x}s}(u, \mathbf{y}_\ell) \,|\, \tau^* = t^*) du. \end{aligned} \tag{98}$$

Proof. Let $k_s(t)$ denote the site at which the clock being at time t at site s was originally activated. For $u \leq v$ let $A_s(u,v)$, $\ell_s(u,v)$ and $\tau_s(u,v)$ be the attained sojourn time, the site and the nominal lifetime, respectively, of a clock of type d at time v which started at site s at time u. Let $x \geq 0$, $s, s' \in J_d$, $\mathbf{x} \in \mathscr{X}_s$, and $\mathbf{y}_\ell \geq 0$ be arbitrary but fixed. Then, by the definition of $\ell_{s'}(u,v)$ and $N_{s'}$, we have

$$
\begin{aligned}
&P(R_s(0) > y, k_s(0) = s', C_{\mathbf{x}s}(0, \mathbf{y}_\ell)) \\
&\qquad = E\left(\int_{-\infty}^{0} \mathbf{1}_{\{R_s(0)>y, k_s(0)=s', C_{\mathbf{x}s}(0,\mathbf{y}_\ell), \ell_{s'}(u,0)=s\}} N_{s'}(du)\right) \\
&\qquad = E\left(\int_{-\infty}^{0} \mathbf{1}_{\{R_s(0)>y, C_{\mathbf{x}s}(0,\mathbf{y}_\ell), \ell_{s'}(u,0)=s\}} N_{s'}(du)\right).
\end{aligned}
$$

Moreover, note that, for $u < 0$, $A_{s'}(u,0) + R_s(0) = \tau_{s'}(u,0)$ on the set $\{\ell_{s'}(u,0) = s\}$, and that $A_{s'}(u,0) = A_{s'}(0,-u) \circ \theta_u$, $\tau_{s'}(u,0) = \tau_{s'}(0,-u) \circ \theta_u$ and $\ell_{s'}(u,0) = \ell_{s'}(0,-u) \circ \theta_u$. Then, the last term of the above formula becomes

$$
\begin{aligned}
&E\left(\int_{-\infty}^{0} \mathbf{1}_{\{A_{s'}(u,0)<\tau_{s'}(u,0)-y, C_{\mathbf{x}s}(0,\mathbf{y}_\ell), \ell_{s'}(u,0)=s\}} N_{s'}(du)\right) \\
&\qquad = E\left(\int_{-\infty}^{0} \mathbf{1}_{\{A_{s'}(0,-u)<\tau_{s'}(0,-u)-y, C_{\mathbf{x}s}(-u,\mathbf{y}_\ell), \ell_{s'}(0,-u)=s\}} \circ \theta_u N_{s'}(du)\right),
\end{aligned}
$$

which, by Lemma 6, equals

$$
\begin{aligned}
&\lambda_{s'} E_{s'}\left(\int_{-\infty}^{0} \mathbf{1}_{\{A_{s'}(0,-u)<\tau_{s'}(0,-u)-y, C_{\mathbf{x}s}(-u,\mathbf{y}_\ell), \ell_{s'}(0,-u)=s\}} du\right) \\
&\qquad = \lambda_{s'} E_{s'}\left(\int_{0}^{\infty} \mathbf{1}_{\{A_{s'}(0,u)<\tau_{s'}(0,u)-y, C_{\mathbf{x}s}(u,\mathbf{y}_\ell), \ell_{s'}(0,u)=s\}} du\right) \\
&\qquad = \lambda_{s'} \int_{0}^{\infty} P_{s'}\left(A_{s'}(0,u) < \tau_{s'}(0,u) - y, C_{\mathbf{x}s}(u,\mathbf{y}_\ell), \ell_{s'}(0,u) = s\right) du,
\end{aligned}
$$

where $E_{s'}$ denotes the expectation taken with respect to the Palm distribution $P_{s'}$. Thus, from the fact that

$$
A_{s'}(0,u) = A^*_{s'}(u), \qquad \tau_{s'}(0,u) = \tau^*_{s'}, \qquad \ell_{s'}(0,u) = s = \ell^*(u) \qquad P_{s'}\text{-a.s.}\,,
$$

we get

$$
\begin{aligned}
&P(R_s(0) > y, k_s(0) = s', C_{\mathbf{x}s}(0, \mathbf{y}_\ell)) \\
&= \lambda_{s'} \int_{0}^{\infty} \Big(\int_{y}^{t^*} P_{s'}(A^*_{s'}(u) < z - y, \ell^*(u) = s, C_{\mathbf{x}s}(u,\mathbf{y}_\ell) \,|\, \tau^*_{s'} = z) F_d(dz)\Big) du \\
&= \lambda_{s'} \int_{y}^{t^*} \Big(\int_{0}^{\infty} P_{s'}(A^*_{s'}(u) < z - y, \ell^*(u) = s, C_{\mathbf{x}s}(u,\mathbf{y}_\ell) \,|\, \tau^*_{s'} = t^*) du\Big) F_d(dz)\,, \qquad (99)
\end{aligned}
$$

where we have used (96) in the last equality of (99). Define a function H_d by

$$
H_d(y, \mathbf{y}_\ell) = \int_{0}^{\infty} P_{(d)}(A^*(u) < y, \ell^*(u) = s, C_{\mathbf{x}s}(u,\mathbf{y}_\ell) \,|\, \tau^* = t^*) du\,.
$$

Sum up both sides of (99) for all $s' \in J_d$, then (97) yields

$$
P(R_s(0) > y, C_{\mathbf{x}s}(0, \mathbf{y}_\ell)) = \lambda_{(d)} \int_{y}^{t^*} H_d(z - y, \mathbf{y}_\ell) F_d(dz)\,. \qquad (100)
$$

On the other hand, from (91), the left-hand side of (100) becomes

$$\pi(\mathbf{x})\overline{F}_d^{(r)}(y) \prod_{s' \in J(\mathbf{x}) \setminus \{s\}} F_{\gamma_{\mathbf{x}}(s')}^{(r)}(y_{s'}) = \mu_d \pi(\mathbf{x}) \prod_{s' \in J(\mathbf{x}) \setminus \{s\}} F_{\gamma_{\mathbf{x}}(s')}^{(r)}(y_{s'}) \int_y^{t^*} (z-y) F_d(dz) \tag{101}$$

where $\overline{F}_d^{(r)}(y) = 1 - F_d^{(r)}(y)$. Because of our assumption on F_d, there exist a positive integer n, two sets of positive numbers $\{a_i; i = 1,2,\ldots,n\}$ and $\{p_i; i = 1,2,\ldots,n\}$ satisfying

$$F_d(y) = \sum_{i=1}^{n} p_i \mathbf{1}_{[a_i,\infty)}(y) .$$

Here, we can assume that a_i is increasing in i. Then, from (100), (101), we get, for $0 \le y \le t^*$,

$$\lambda_{(d)} \sum_{i=1}^{n} p_i H_d((a_i - y)^+, \mathbf{y}_\ell) = \mu_d \pi(\mathbf{x}) \prod_{s' \in J(\mathbf{x}) \setminus \{s\}} F_{\gamma_{\mathbf{x}}(s')}^{(r)}(y_{s'}) \sum_{i=1}^{n} p_i (a_i - y)^+ , \tag{102}$$

where $y^+ = \max(y,0)$. Finally, (102) implies that, for $0 \le y \le t^*$,

$$\lambda_{(d)} H_d(y, \mathbf{y}_\ell) = \mu_d \pi(\mathbf{x}) y \prod_{s' \in J(\mathbf{x}) \setminus \{s\}} F_{\gamma_{\mathbf{x}}(s')}^{(r)}(y_{s'}) . \tag{103}$$

This can be proved in the following way. Consider (102) for each sub-interval $(a_{i-1}, a_i]$, where $a_0 = 0$. First, from (102) for $y \in (a_{n-1}, a_n]$, we have (103) for $0 \le y \le a_n - a_{n-1}$. Then, from (102) for $y \in (a_{n-2}, a_{n-1}]$, we have (103) for $a_n - a_{n-1} \le y \le \min[a_n - a_{n-2}, 2(a_n - a_{n-1})]$. If $2(a_n - a_{n-1}) < a_n - a_{n-2}$, then, by using the equation just proved, we get (103) for $2(a_n - a_{n-1}) \le y \le \min[a_n - a_{n-2}, 3(a_n - a_{n-1})]$. We repeat the argument and eventually get (103) for $a_n - a_{n-1} \le y \le a_n - a_{n-2}$. In a similar way we inductively get (103) for all the sub-intervals. (103) is nothing but (98), and therefore the lemma is proved. □

Note that, by (94), T_y^* is defined for $0 \le y \le \tau^*$. Now, we extend T_y^* to the whole non-negative half-line by changing the nominal lifetime of the tagged clock to infinity, and denote T_y^* in this case by T_y^∞. Clearly $T_y^* = T_y^\infty$ for $0 \le y \le \tau^*$. The nondecreasing process $\{T_y^\infty; y \ge 0\}$ is called a *attained sojourn time process*.

Analogously, by $\ell^\infty(t)$ we denote the site at which the tagged clock is at time t when its nominal lifetime is changed to infinity. Under the assumption of Lemma 16, we consider a time change of the RGSMP $\{\mathbf{Z}(t)\}$ by $\{T_t^\infty\}$.

Definition 14. Let $\{T_t^\infty; t \ge 0\}$ be the attained sojourn time process for a fixed index $d \in D$. Let $\tilde{X}(t) = X(T_t^\infty)$, $\tilde{\ell}(t) = \ell^\infty(T_t^\infty)$ and $\tilde{R}_s(t) = R_s(T_t^\infty)$. We define a time-changed process $\{\tilde{\mathbf{Z}}(t); 0 \le t < \infty\}$ as

$$\tilde{\mathbf{Z}}(t) = (\tilde{X}(t), \tilde{\ell}(t), \tilde{R}_{s'}(t); s' \in J(\tilde{X}(t)) \setminus \{\tilde{\ell}(t)\}) .$$

This process is said to be a time changed RGSMP concerning the attained lifetime.

Note that $\{\tilde{\mathbf{Z}}(t)\}$ is a Markov process because we can trace its history by using analogous dynamics as for the strong Markov process $\{\mathbf{Z}(t)\}$ and by using the supplementary information $\tilde{\ell}(t)$, which indicates the site at which the tagged clock is at present. For $s \in J_d$, $\mathbf{x} \in \mathscr{X}_s$ and $t, \mathbf{y}_\ell \geq 0$, define the event $\tilde{C}_{\mathbf{x}s}(u, \mathbf{y}_\ell) \in \mathscr{F}$ by

$$\tilde{C}_{\mathbf{x}s}(t, \mathbf{y}_\ell) \equiv \{\tilde{X}(t) = \mathbf{x}, \tilde{R}_{s'}(t) \leq y_{s'}(s' \in J(\mathbf{x}) \setminus \{s\})\} .$$

Note that, if we put $v = A^*(u)$ on $\{l^*(u) = s\} \cap C_{\mathbf{x}s}(u, \mathbf{y}_\ell)$, then $dv = c_{\mathbf{x}s} du$ and $T^*_{A^*(u)} = u$ for $c_{\mathbf{x}s} > 0$ while $dv = 0$ for $c_{\mathbf{x}s} = 0$. Hence, by Fubini's theorem and by changing variables from u to $v = A^*(u)$, we have, for $0 \leq y \leq t^*$,

$$\begin{aligned}
& c_{\mathbf{x}s} \int_0^\infty P_{(d)}(A^*(u) < y, \ell^*(u) = s, C_{\mathbf{x}s}(u, \mathbf{y}_\ell) \,|\, \tau^* = t^*) du \\
&= E_{(d)} \left(\int_0^\infty \mathbf{1}_{\{A^*(u) < y, \ell^*(u) = s, C_{\mathbf{x}s}(u, \mathbf{y}_\ell)\}} c_{\mathbf{x}s} du \middle| \tau^* = t^* \right) \\
&= E_{(d)} \left(\int_0^\infty \mathbf{1}_{\{v < y, l^*(T^*_v) = s, C_{\mathbf{x}s}(T^*_v, \mathbf{y}_\ell)\}} dv \middle| \tau^* = t^* \right) \\
&= E_{(d)} \left(\int_0^y \mathbf{1}_{\{\ell^\infty(T^\infty_v) = s, C_{\mathbf{x}s}(T^\infty_v, \mathbf{y}_\ell)\}} dv \right) \\
&= E_{(d)} \left(\int_0^y \mathbf{1}_{\{\tilde{\ell}(v) = s, \tilde{C}_{\mathbf{x}s}(v, \mathbf{y}_\ell)\}} dv \right) = \int_0^y P_{(d)} \left(\tilde{\ell}(v) = s, \tilde{C}_{\mathbf{x}s}(v, \mathbf{y}_\ell) \right) dv, \quad (104)
\end{aligned}$$

where the expectation $E_{(d)}$ is taken with respect to $P_{(d)}$. Multiplying both sides of (98) by $c_{\mathbf{x}s} \lambda^{-1}_{(d)}$, substituting (104) into its right-hand side and differentiating it with respect to y, we get, for $0 \leq y \leq t^*$,

$$\frac{c_{\mathbf{x}s} \mu_d \pi(\mathbf{x})}{\lambda_{(d)}} \prod_{s' \in J(\mathbf{x}) \setminus \{s\}} F^{(r)}_{\gamma_{\mathbf{x}}(s')}(y_{s'}) = P_{(d)}(\tilde{\ell}(x) = s, \tilde{C}_{\mathbf{x}s}(y, \mathbf{y}_\ell)) . \quad (105)$$

Hence, $\{\tilde{\mathbf{Z}}(t); 0 \leq t < \infty\}$ is a stationary process. By summing up both sides of (105) for all possible $s, \mathbf{x}$, we get

$$\lambda_{(d)} = \mu_d \sum_{\mathbf{x} \in \mathscr{X}} \sum_{s \in J(\mathbf{x}) \cap J_d} c_{\mathbf{x}s} \pi(\mathbf{x}) .$$

Hence, the left-hand side of (105) can be expressed by

$$\pi^*(\mathbf{x}, s) \prod_{s' \in J(\mathbf{x}) \setminus \{s\}} F^{(r)}_{\gamma_{\mathbf{x}}(s')}(y_{s'}). \quad (106)$$

where $\pi^*(\mathbf{x}, s)$ is the probability distribution on $\{(\mathbf{x}, s); s \in J_d, \mathbf{x} \in \mathscr{X}_s\}$ defined as

$$\pi^*(\mathbf{x}, s) = \frac{c_{\mathbf{x}s} \pi(\mathbf{x})}{\sum_{\mathbf{x}' \in \mathscr{X}} \pi(\mathbf{x}') \sum_{s' \in J(\mathbf{x}') \cap J_d} c(s', \mathbf{x}')}. \quad (107)$$

Thus, we arrive at the following result.

Lemma 17. Under the assumption of Lemma 16, $\{\tilde{\mathbf{Z}}(t); t \geq 0\}$ is a stationary Markov process provided that the initial distribution of $\{\tilde{\mathbf{Z}}(t)\}$ is given by (106).

We now remove the assumption of Lemma 16. For this purpose, we will use a certain continuity property of Markov processes. Let F_d be a general lifetime distribution, and let $\{F_{d,n}\}$ be a sequence of distributions which satisfy the condition of Lemma 16 and which weakly converge to F_d. Let $\{\mathbf{Z}_n(t); t \geq 0\}$ and $\{\tilde{\mathbf{Z}}_n(t); t \geq 0\}$ be the processes corresponding to $\{\mathbf{Z}(t); t \geq 0\}$ and $\{\tilde{\mathbf{Z}}(t); t \geq 0\}$, respectively, for the RGSMP's with $F_{d,n}$ instead of F_d. We define the initial distribution of $\{\tilde{\mathbf{Z}}_n(t)\}$ by (106), in which F_d is replaced by $F_{d,n}$. By Lemma 17, $\{\tilde{\mathbf{Z}}_n(t)\}$ are stationary Markov processes. $\{\tilde{\mathbf{Z}}_n(t)\}$ and $\{\tilde{\mathbf{Z}}(t)\}$ are self-clocking jump processes as introduced in [26]. We apply Theorem 5.2 of [26] to those processes. Condition (i) of this theorem is clearly satisfied because of (90). The stationary one-dimensional distribution of $\{\tilde{\mathbf{Z}}_n(t)\}$ weakly converges to the left-hand side of (106), and the transition function at the jump instants of $\{\tilde{\mathbf{Z}}_n(t)\}$ satisfy conditions (ii) and (iii) of Theorem 5.2 of [26], which can easily be verified because we only change the lifetime distributions $F_{d,n}$ (see also Remark 5.2 of [26]). Thus, we get

Theorem 12. Assume that RGSMP is product form decomposable, and let $d \in D_g$ be fixed. Then, for a general lifetime distribution F_d, $\{\tilde{\mathbf{Z}}(t); t \geq 0\}$ is a stationary Markov process provided that the initial distribution of $\{\tilde{\mathbf{Z}}(t); t \geq 0\}$ is given by (106). Furthermore, combining (104) with (105) and (106), we also have (98), namely,

$$\frac{c_{\mathbf{x}s}\pi(\mathbf{x})y}{\sum_{\mathbf{x}' \in \mathscr{X}} \pi(\mathbf{x}') \sum_{s' \in J(\mathbf{x}') \cap J_d} c(s', \mathbf{x}')} \prod_{s' \in J(\mathbf{x}) \setminus \{s\}} F^{(r)}_{\gamma_{\mathbf{x}}(s')}(y_{s'})$$
$$= \int_0^\infty P_{(d)}(A^*(u) < y, \ell^*(u) = s, C_{\mathbf{x}s}(u, \mathbf{y}_\ell) \,|\, \tau^* = t^*) du. \quad (108)$$

We have the following verbal interpretation of Theorem 12. Under stationarity conditions, given we freeze a randomly chosen type-d clock once it has been started (i.e. putting its nominal lifetime equal to infinity), we observe a stationary process if we look at the remaining system at those times T_y^∞ when the frozen clock has consumed y units of resource, i.e. reached age $y \geq 0$. In particular, the distribution we see when the tagged clock has reached age y is the same for all y and hence, if we draw the age to be reached, blindly from some distribution, e.g. from F_d, we have the same distribution of the process at the time this (random) age is reached. Thus, the next corollary is a direct consequence of Theorem 12, i.e. the stationarity of $\{\tilde{\mathbf{Z}}(t)\}$.

Lemma 18. Under the conditions of Theorem 12, for $\mathbf{x} \in \mathscr{X}$, $s \in J_d$ and $y > 0$, we have

$$P_{(d)}(X(0) = \mathbf{x}, l(0) = s) = P_{(d)}(X(T^*_{\tau^*}-) = \mathbf{x}, l(T^*_{\tau^*}-) = s | \tau^* = y). \quad (109)$$

Note that formula (109) can be somewhat sharpened: In steady state, at the instants right after the starting of a randomly chosen type-d clock and right before expiring of *that same clock*, the joint distributions of the state $\mathbf{x}$, the site $s \in J(\mathbf{x}) \cap J_d$ on which that clock is found, and the residual lifetimes of the other clocks are both the same.

Theorem 12 also yields the following corollary because $\{T_y^\infty; x \geq 0\}$ is completely determined by $\{\tilde{\mathbf{Z}}(t); t \geq 0\}$ (see also Theorem 1 of [13]).

Corollary 11. Under the conditions of Theorem 12, the attained sojourn time process $\{T_y^\infty; y \geq 0\}$ has stationary increments.

Let $T_y^\infty(\mathbf{x}, s)$ denote the total sojourn time of the system in state $\mathbf{x} \in \mathscr{X}$ while the tagged clock is at site $s \in J(\mathbf{x}) \cap J_d$, until the tagged clock has processed y units of its nominal lifetime, where the nominal lifetime of the tagged clock is assumed to be infinity. Then we have the following result.

Theorem 13. Under the conditions of Theorem 12, we get, for $y \geq 0$, and for $\mathbf{x} \in \mathscr{X}$ and $s \in J(\mathbf{x}) \cap J_d$,

$$E_{(d)}(T_y^\infty(\mathbf{x}, s)) = \frac{\pi(\mathbf{x})}{\sum_{\mathbf{x}' \in \mathscr{X}} \pi(\mathbf{x}') \sum_{s' \in J(\mathbf{x}') \cap J_d} \mathbf{c}_{\mathbf{x}'\mathbf{s}'}}\, y\,, \tag{110}$$

and, in particular, by summing up over all possible $\mathbf{x}$ and s,

$$E_{(d)}(T_y^\infty) = \frac{\sum_{\mathbf{x} \in \mathscr{X}} \pi(\mathbf{x}) |J(\mathbf{x}) \cap J_d|}{\sum_{\mathbf{x} \in \mathscr{X}} \pi(\mathbf{x}) \sum_{s \in J(\mathbf{x}) \cap J_d} c_{\mathbf{x}s}}\, y \tag{111}$$

where $|J(\mathbf{x}) \cap J_d|$ denotes the number of elements of the set $J(\mathbf{x}) \cap J_d$.

Proof. Since, for $0 \leq y \leq t^*$,

$$T_y^\infty(\mathbf{x}, s) = T_y^*(\mathbf{x}, s) = \int_0^\infty \mathbf{1}_{\{A^*(u) < x, \ell^*(u) = s, X^*(u) = g\}} du\,,$$

(108) of Theorem 12 yields (110) and (111). □

Remark 17. Note that the right-hand side of (110) does not depend on $s \in J(\mathbf{x}) \cap J_d$. Furthermore, if we sum (110) up over all $\mathbf{x}, s$ such that $c_{\mathbf{x}s} = 0$, we get a formula for the expected total time during which the tagged clock is interrupted (i.e. stands still) up to the time age y is reached.

In some systems, e.g., in the processor-sharing queue, many clocks of a given type d may run at the same time. Consider the time-changed process with respect to a clock of type d whose lifetime is infinite. Suppose that type-d clocks never run at zero speeds. Then the time-changed process is the RGSMP with macrostates $(\mathbf{x}, s)$, $s \in J_d$, $\mathbf{c} \in \mathscr{X}_s$, and where, in state $(\mathbf{x}, s)$, a clock $s' \in J(\mathbf{x}) \setminus \{s\}$ is running at the speed $c_{\mathbf{x}s'}/c_{\mathbf{x}s}$. Because (106) is the stationary distribution of this RGSMP, we can, for instance, consider the successive instants at which another type-d clock gets started, and study the corresponding time-changed process, all from the starting point of the first time-changed process. This new time-changed process lives on the states $[s', (\mathbf{x}, s)]$ with $s \in J(\mathbf{x}) \cap J_d$, $s' \in J(\mathbf{x}) \cap J_d$, $s' \neq s$. Let (107) be written as $\pi^*(\mathbf{x}, s) = \frac{1}{k^*} c_{\mathbf{x}s} \pi(\mathbf{x})$. Then the corresponding distribution for the second time-changed process is given by

$$\pi^{**}(s', (\mathbf{x}, s)) = \frac{1}{k^{**}} \frac{c_{\mathbf{x}s'}}{c_{\mathbf{x}s}} \pi^*(\mathbf{x}, s)$$

on account of (107) applied to the second time-changed process. So

$$\pi^{**}(s', (\mathbf{x}, s)) = \frac{c_{\mathbf{x}s'} \pi(\mathbf{x})}{k^* k^{**}} = \frac{c_{\mathbf{x}s'} \pi(\mathbf{x})}{\sum_{\mathbf{x}' \in \mathscr{X}} \pi(\mathbf{x}') \sum_{s'' \in J(\mathbf{x}') \cap J_d} c_{\mathbf{x}'s''}} .$$

This would be, in steady state, the probability that, right after the instant of birth of a type-d clock chosen at random while another type-d clock is already running (at site s), the state is $\mathbf{x}$ and that clock is sitting on s'.

Example 19 (Symmetric queue). Consider the symmetric queue of Section 17. Since there is only one type of customers, $|J(\mathbf{x}) \cap J_d| = n$ for $\mathbf{x} = n$. Hence, by Theorem 13, the conditional mean sojourn time of a customer who brings y amount of work is

$$E_d(T_y^\infty) = \frac{\sum_{n=1}^\infty n\rho^n \prod_{i=1}^n \sigma(i)^{-1}}{\sum_{n=1}^\infty \rho^n \prod_{i=1}^{n-1} \sigma(i)^{-1}} y,$$

where $\rho = \lambda/\mu$. In particular, $\sigma(n) = a$ for all $n \geq 1$ for some positive constant a, then

$$E_d(T_y^\infty) = \frac{\rho}{a(a-\rho)} y.$$

As is expected, the coefficient of the linear function is proportional to the mean queue length. For the product form decomposable network, similar results can be obtained for a given sequence of the amounts of work at visiting nodes of a tagged customer when his route is specified. □

19 Bibliographic notes

We briefly discuss about the literature in this chapter. Point processes and Palm measures are now standard in queueing books (see, e.g., [1]). In particular, Baccelli and Bremaud [2] is devoted to this topic and "Palm calculus" was coined there. Historically, the first comprehensive book on this topic for queues was written by Franken, König, Arndt and Schmidt [14]. However, point processes and Palm distributions are old stuff, going back to the ninety-sixties (see, e.g., [30, 21, 22]). There are some other approaches (see, e.g., [6]). The treatments of this topic from Section 2 to 7 are somehow different from the standard one as in [2]. We more emphasize the symmetric role of time stationary and Palm probability measures. This idea goes back to Miyazawa [23].

The materials in Sections 8 and 10 are taken from Miyazawa [24, 25]. The rate conservation law and their applications are surveyed in [28]. Example 8 is new. Piece-wise deterministic process (PDMP) in Section 12 was coined by Davis [10], and detailed in [11]. However, similar types of processes would have been considered long before since they are typical in queueing applications. Our treatments of PDMP is slightly different from those of Davis' as mentioned in Remark 5. Generalized semi-Markov process (GSMP) for the insensitivity in the same section has a long history. The earlier literature is Schassberger [31, 32] and Jansen König and Nawrotzki [15]. However, its limitation had been recognized (see, e.g., [3]). Schassberger [33] proposes "relabeling" to relax the limitation. Reallocatable GSMP (RGSMP) was introduced by Miyazawa[27]. It has a similar mechanism to Schassberger's, but allows interruptions.

The stationary equations in Section 12 is taken from Miyazawa [26], and those in Section 16 from Miyazawa [27]. Symmetric queue and their networks in Section 17 is due to Kelly [18, 19]. The locally balanced conditions and product form solutions are largely discussed in the queueing network literature (see, e.g., [4, 8, 9, 34] and references there). Section 18 is largely taken from Miyazawa, Schassberger and Schmidt [29], which generalizes the results in [12, 13].

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