An $M/G/1$ Queue with Markov Dependent Exceptional Service Times

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Abstract

This paper considers an $M/G/1$ queue in which service time distributions in each busy period change according to a finite state Markov chain, embedded at the arrival instants of customers. It is assumed that this Markov chain has an upper triangular transition matrix. Applying the regenerative cycle approach with respect to a busy period, we obtain the Laplace-Stieltjes transform, i.e., LST, of the stationary waiting time distribution in a certain parametric form. We give a procedure to determine those parameters. Some detailed calculations and numerical results are presented as well.

1. Introduction

We consider the following modification of the $M/G/1$ queue. Similar to the standard $M/G/1$ queue, the system has a single server and Poisson arrivals. The service discipline is FCFS (First Come First Served), and the buffer size is infinity. However, the distributions of service times change according to a finite state Markov chain embedded at the arrival instants of customers that restarts every time when the busy period begins. It is assumed that this Markov chain has no cyclic path and it reaches an absorbing state with probability one. Service times are conditionally independent given a sample path of the Markov chain. We refer to this model as an $M/G/1$ queue with Markov dependent exceptional service.

This queueing model is an extension of the models with exceptional service studied in the literature (see, e.g., [1, 4, 6, 8, 9, 11]). Those exceptional service models assume that the service time distribution of a customer is determined by the number of customers who have been served before his service in the current busy period. The present model relaxes this fixed order and fixed number of the exceptional service time distributions by using the finite Markov chain (precisely, the fixed number is relaxed in [8], but the service time
The exceptional service model is intended to incorporate the changes of service times at the beginning of each busy period. The present model is more flexible in applications than the conventional exceptional service model. For instance, the server may need more or less experience by a chance to be settled down in each busy period. This situation is not covered by the conventional model.

The aim of this paper is to compute the stationary distributions of the waiting time and some other characteristics in transform expressions for the $M/G/1$ queue with Markov dependent exceptional service. We assume that this Markov chain has an upper triangular transition matrix. Similar to Kengaku and Miyazawa [6], we employ the regenerative cycle approach for the waiting time in a busy period, and get the Laplace-Stieltjes transforms, i.e., LST's, of the stationary distributions with some unknown parameters. However, determination of the parameters is quite different. We consider the waiting time process under the finite Markov chain that governs the change of service time distributions. So the expressions are given in vector-matrix form. Unlike the approach of [6], the parameters are determined by null points of the determinant of a matrix function. The stationary distributions in the conventional models are obtained as the limits of those in the present model. Hence, the present approach also gives an alternative derivation of the stationary distributions in the conventional models.

This paper is organized by five sections. In Section 2, we describe the model, and discuss a general expression of the LST of the stationary waiting time distribution. In Section 3, a procedure to determine the parameters is given. The stationary distributions of the sojourn time and the queue length are considered in Section 4. Finally, in Section 5, closed form formulas are obtained for two special cases, and the effect of Markov dependence is considered when there are two types of service time distributions. Some numerical results are presented for the latter as well.

2. Waiting time process on a busy period

Consider a single server queue with Poisson arrivals with rate $\lambda$. Customers are served in a First-Come First-Served manner, and their service times are independent. However, unlike the standard $M/G/1$ queue, the service time distributions of customers are Markov dependent in each busy period, which will be specified below. We assume that each busy period starts independent of the previous history of the system. So we only consider a single busy period. Let $N$ be the total number of customers who compose one busy period. For an integer $1 \leq n \leq N$, we will use the following notation.

$W_n$ : the waiting time of the $n$-th arriving customer in the current busy period.

$S_n$ : the service time of the $n$-th arriving customer in the current busy period.

$J_n$ : the type of the service time distribution of the $n$-th arriving customer in the current busy period. We shall call the type of the service time distribution the service type for short.

$T_n$ : the interarrival between the $n$-th customer and $(n + 1)$-th customer.
Assume that \( \{J_n\}_{n=1}^{\infty} \) is a Markov chain with a finite state space \( S = \{1, 2, \ldots, k\} \), and has a transition probability matrix \( P = \{p_{ij}\} \) with \( p_{kk} = 1 \), i.e., \( k \) is an absorbing state. We assume that all states in \( S \) eventually enter into state \( k \). Thus, the service type of \( n \)-th arriving customer depends only on the service type of \( (n-1) \)-th arriving customer through \( P \), and it becomes \( k \) eventually with probability one. This model is referred to as the \( M/G/1 \) queue with Markov dependent exceptional service. The model is an extension of the \( M/G/1 \) queue with exceptional service (see, e.g., [6]).

For integer \( n \geq 1 \) and \( i \in S \), let

\[
\begin{align*}
    f_{n,i}(\theta) &= E[e^{-\theta W_n}; N \geq n, J_n = i], \\
    g_i(\theta) &= E[e^{-\theta S_n}|J_n = i],
\end{align*}
\]

where \( \theta \geq 0 \), and let \( S_{<i>} \) be a random variable subject to the distribution of type \( i \) service, i.e., the distribution determined by \( g_i \). It is assumed that \( S_{<i>} \) has a finite expectation for each \( i \). Define

\[
\rho_i = \lambda E(S_{<i>}), \quad i = 1, 2, \ldots, k.
\]

We assume that the model is stable, i.e., there exists the stationary waiting time distribution. Clearly, this is equivalent to assume that

(a) \( \rho_k < 1 \).

If \( W_n = W_{n-1} + S_{n-1} - T_{n-1} > 0 \) and \( N \geq n - 1 \), then \( N \geq n \), i.e., busy period doesn’t terminate. Using this fact, we have

\[
\begin{align*}
    f_{n,i}(\theta) &= E[e^{-\theta W_n}; N \geq n, J_n = i] \\
    &= \sum_j E[e^{-\theta(W_{n-1}+S_{n-1}-T_{n-1})}; N \geq n, J_{n-1} = j] p_{ji} \\
    &= \sum_j E\left[e^{-\theta(W_{n-1}+S_{n-1}-T_{n-1})}; N \geq n-1, W_{n-1} + S_{n-1} > T_{n-1}, J_{n-1} = j \right] p_{ji} \\
    &= \sum_j E\left[\int_0^{W_{n-1}+S_{n-1}} e^{-\theta(W_{n-1}+S_{n-1}-u)}e^{-\lambda u} du; N \geq n-1, J_{n-1} = j \right] p_{ji} \\
    &= \sum_j \frac{\lambda}{\lambda-\theta} \left[ f_{n-1,i}(\theta) g_j(\theta) - f_{n-1,i}(\lambda) g_j(\lambda) \right] p_{ji}.
\end{align*}
\]

Thus we have the following lemma.

**Lemma 2.1** For integer \( n \geq 2 \) and \( i \in S \),

\[
f_{n,i}(\theta) = \sum_{j=1}^{i} \frac{\lambda}{\lambda-\theta} \left[ f_{n-1,i}(\theta) g_j(\theta) - f_{n-1,i}(\lambda) g_j(\lambda) \right] p_{ji}.
\]
We next consider the stationary waiting time. Let us introduce the following notation.

\[ \psi_i(\theta) = \sum_{\ell=1}^{\infty} f_{\ell,i}(\theta), \quad i = 1, 2, \ldots, k, \]
\[ \Psi(\theta) = \sum_{j=1}^{k} \psi_j(\theta). \]

Since
\[ E[N] = \sum_{n=1}^{\infty} P(N \geq n) = \sum_{i=1}^{k} \sum_{n=1}^{\infty} f_{n,i}(0) = \Psi(0), \]
the following result is a direct consequence of the well known mean cycle formula for a regenerative process (see, e.g., [2]).

**Lemma 2.2** Let \( \phi(\theta) \) be the LST of the stationary waiting time distribution. Then \( \phi(\theta) \) is given by
\[ \phi(\theta) = \frac{1}{E[N]} \Psi(\theta) = \frac{\Psi(\theta)}{\Psi(0)}. \]

From Lemma 2.1, we have for \( i = 1, 2, \ldots, k, \)
\[ (\lambda - \theta) f_{n,i}(\theta) = \lambda \sum_{j=1}^{i} (f_{n-j,j}(\theta) g_j(\theta) - f_{n-j,j}(\lambda) g_j(\lambda)) p_{ji}, \quad n \geq 2. \tag{2.1} \]

Since the event \( \{W_1 = 0\} \) occurs w.p.1, we can assume that
\[ f_{1,i}(\theta) = 1[i = 1], \tag{2.2} \]
where \( 1[\cdot] \) is the indicator function of the statement \( \cdot \), i.e., it takes value 1 if the statement is true, and value 0 otherwise. In fact, if the initial background state has a distribution \( \{a_i\} \), i.e., \( f_{1,i}(\theta) = a_i \), then we can apply the following procedure. First compute \( f_{n,j} \), provided \( f_{1,j}(\theta) = 1[i = j] \). Denote those \( f_{n,j} \) by \( f_{n,j}^{(i)} \). Then, the desired \( f_{n,j}(\theta) \) is obtained as \( \sum_{i=1}^{k} a_i f_{n,j}^{(i)}(\theta) \).

We introduce the following vector and matrix notations.

\[ f_{n}(\theta) = [f_{n,1}(\theta), f_{n,2}(\theta), \ldots, f_{n,k}(\theta)], \]
\[ G(\theta) = \begin{bmatrix} g_1(\theta) & 0 & \cdots & 0 & 0 \\ 0 & g_2(\theta) & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & g_{k-1}(\theta) & 0 \\ 0 & \cdots & 0 & g_k(\theta) \end{bmatrix}. \]

That is, \( G(\theta) \) is the diagonal matrix whose \( i \)-th diagonal component is \( g_i(\theta) \). Then, (2.1) is rewritten as
\[ (\lambda - \theta) f_{n}(\theta) = \lambda f_{n-1}(\theta) G(\theta) P - \lambda f_{n-1}(\lambda) G(\lambda) P, \quad n \geq 2. \tag{2.3} \]
This together with (2.2) determines $f_n$ recursively, so, in principle, the stationary distribution of the waiting time is obtained by Lemma 2.2. However, such calculations may not be feasible, since the calculation of $f_n$ becomes highly complicated as $n$ is increased (see, e.g., [6]). We consider here an alternative way to get the stationary distribution under some extra assumptions. Before giving these assumptions, we continue the present settings in one more step.

Let

$$\psi(\theta) = [\psi_1(\theta), \psi_2(\theta), \ldots, \psi_k(\theta)].$$

Then, summing (2.3) over $n \geq 2$ with (2.2) yields the following result.

**Lemma 2.3** For $\theta \geq 0$,

$$\psi(\theta)[(\lambda - \theta)I - \lambda G(\theta)P] = (\lambda - \theta)e_1 - \lambda \psi(\lambda)G(\lambda)P,$$

(2.4)

where $e_1$ is the unit row vector whose 1st component is one and the other components are zero.

Thus, $\psi(\theta)$ is obtained if $\psi(\lambda)$ is determined. In principle, this is routine. The standard way for it is to consider singular points of the matrix:

$$D(\theta) \equiv (\lambda - \theta)I - \lambda G(\theta)P,$$

and use them to determine $\psi(\lambda)$. However, the latter is not so easy in the present case, since it may be hard to identify $k$ independent equations to determine $\psi(\lambda)$. More precisely, it is not difficult to find $k$ singular points of $D(\theta)$, using the fact that, similar to the Perron-Frobenius Theorem for nonnegative matrices (see [10]), matrix of $G(\theta)P$ for each complex number $\theta$ satisfying $Re(\theta) > 0$ has $k$ eigenvalues whose absolute values are less than 1, but this yields $k^2$ equations of $\psi(\lambda)$, and it seems hard to find $k$ independent equations for $\psi(\lambda)$ among them.

Instead of considering the general $P$ matrix, we restrict our attention to the case that $P$ is upper triangular, i.e., we assume that

(b) The transition probability matrix $P$ has the following form:

$$P = \begin{bmatrix}
    p_{11} & p_{12} & \cdots & \cdots & p_{1k} \\
    p_{22} & p_{23} & \cdots & \cdots & p_{2k} \\
    \vdots & \vdots & \ddots & \ddots & \vdots \\
    \vdots & \vdots & \cdots & \cdots & \vdots \\
    p_{k-1,k-1} & p_{k-1,k} & & & p_{kk}
\end{bmatrix}$$

where $p_{kk} = 1$ and $p_{ii} < 1$ for $i = 1, 2, \ldots, k-1$.

This assumption is motivated by the application point of views, since the service time distributions may not go back to the earlier types in the progress of service. We shall see that this situation greatly simplifies analysis and enables us to get tractable solutions. The condition that $p_{ii} < 1$ for $i = 1, 2, \ldots, k$ is needed for $k$ to be only an absorbing state of the Markov chain with $P$. However, this condition is minor, and we can easily extend the present results below without the condition.
3. Stationary waiting time distribution

We derive the stationary waiting time distribution under assumptions (a) and (b) of Section 2. For this, we introduce some more notation. Denote the off diagonal $ij$-component of $D(\theta)$ by $d_{ij}(\theta)$, and the $i$-th diagonal component by $h_i(\theta)$. That is,

$$
\begin{align*}
  h_i(\theta) &= \lambda - \theta - \lambda g_i(\theta)p_{ii}, \quad i = 1, 2, \ldots, k, \\
  d_{ij}(\theta) &= -\lambda g_i(\theta)p_{ij}, \quad i \neq j, i, j = 1, 2, \ldots, k.
\end{align*}
$$

Since $D(\theta)$ is upper-triangular, we obviously have

$$
|D(\theta)| = h_1(\theta)h_2(\theta) \cdots h_{k-1}(\theta)h_k(\theta). \tag{3.1}
$$

Note that $h_i(\theta)$ is a concave function of $\theta$ for each $i$, and $h_i(0) > 0$ for $i \neq k$, while $h_k(0) = 0$ and $h_k'(0) < 0$. So, $h_i(\theta) = 0$ has only one root $\omega_i > 0$ for $i \neq k$ and $\omega_k = 0$ for $i = k$. Thus, $|D(\theta)|$ with $\theta \geq 0$ is zero only at $\theta = \omega_i$ $(1 \leq i \leq k)$.

Let $s_0(\theta)$ be the $i$-th component of vector $(\lambda - \theta)e_1 - \lambda\psi(\lambda)G(\lambda)$. Namely,

$$
\begin{align*}
  s_{01}(\theta) &= \lambda - \theta - \lambda\psi_1(\lambda)g_1(\lambda)p_{11}, \tag{3.2} \\
  s_{0i}(\theta) &= -\lambda \sum_{\ell=1}^{i-1} \psi_\ell(\lambda)g_\ell(\lambda)p_{i\ell}, \quad i = 2, 3, \ldots, k. \tag{3.3}
\end{align*}
$$

Since $s_{0i}(\theta)$ for $i \geq 2$ does not depend on $\theta$, we frequently write it as $s_{0i}$. We need one more notation.

$$
\begin{align*}
  s_{ij}(\theta) &= s_{i-1,j}(\theta) - \frac{d_{ij}(\theta)}{h_i(\theta)} s_{i-1,i}(\theta) \quad i = 1, 2, \ldots, k, \quad j = i + 1, \ldots, k. \tag{3.4}
\end{align*}
$$

Using these notation, we have the following lemma.

**Lemma 3.1** For $\theta > 0$ and $\theta \neq \omega_i$ for $i = 1, 2, \ldots, k - 1$, we have

$$
\psi_i(\theta) = \frac{s_{i-1,i}(\theta)}{h_i(\theta)}, \quad i = 1, 2, \ldots, k. \tag{3.5}
$$

**Proof.** By (2.4), we have

$$
\psi(\theta)D(\theta) = s_0(\theta),
$$

where $s_0(\theta) = (s_{01}(\theta), s_{02}(\theta), \ldots, s_{0k}(\theta))$. From (3.4) and the Cramér formula, we have

$$
|D(\theta)|\psi_i(\theta) = \begin{vmatrix}
  d_{11}(\theta) & \cdots & d_{1k}(\theta) \\
  \vdots & & \vdots \\
  d_{i-1,1}(\theta) & \cdots & d_{i-1,k}(\theta) \\
  s_{01}(\theta) & \cdots & s_{0k}(\theta) \\
  d_{i+1,1}(\theta) & \cdots & d_{i+1,k}(\theta) \\
  \vdots & & \vdots \\
  d_{k1}(\theta) & \cdots & d_{kk}(\theta)
\end{vmatrix}.
$$


Hence, from (3.1), we have (3.5), since by its LST:
the stationary joint distribution of the waiting time and background state is determined
under the stability condition (a) and the upper-triangular assumption (b),
Theorem 3.1

Adding (1st row) to (i-th row),

Similarly, adding (1st row) to (i-th row), we eventually arrive at

Hence, from (3.1), we have (3.5), since \( h_i(\theta) \) vanishes only at \( \theta = \omega_i \) for \( \theta \geq 0 \).

The next lemma is an immediate consequence of (3.5) since \( \psi_i(\lambda) \) is well-defined in at \( \theta = \omega_i \).

**Lemma 3.2** If \( p_{ii} > 0 \) for \( i = 1, 2, \ldots, k - 1 \), then \( \psi_i(\lambda) \) for \( i = 1, 2, \ldots, k - 1 \) are uniquely determined by the equations:

\[
s_{i-1,i}(\omega_i) = 0, \quad i = 1, 2, \ldots, k.
\]  

**Theorem 3.1** Under the stability condition (a) and the upper-triangular assumption (b), the stationary joint distribution of the waiting time and background state is determined by its LST:

\[
\psi(\theta) = ((\lambda - \theta)e_1 - \lambda \psi(\lambda)G(\lambda)P)((\lambda - \theta)I - \lambda G(\theta)P)^{-1},
\]  

for \( \theta \geq 0 \) such that \( \theta \neq \omega_i \) for all \( i \), where \( \psi(\lambda) \) is determined in the following way.
(i) If all $p_{ii}$’s are positive, $\psi(\lambda)$ is determined by (3.6).

(ii) If $p_{ii} = 0$ for some $i$’s, we modify the transition probabilities in such a way that, for a sufficient small $\epsilon > 0$ and each $i$ among them, the $p_{ii}$ is increased by $\epsilon$ and $p_{ij}$ is decreased by $\epsilon$ for some $j \neq i$ satisfying $p_{ij} > 0$, keeping the modified matrix to be stochastic. Denote this modified matrix by $P_\epsilon$, and compute the corresponding $\psi_\epsilon(\lambda)$ by (3.6). Then,

$$\psi(\lambda) = \lim_{\epsilon \downarrow 0} \psi_\epsilon(\lambda).$$

(3.8)

**Proof.** By Lemma 2.3, case (i) is obtained. To see case (ii), we note that the single server queue with Markov dependent exceptional service in the steady state can be formulated as a single server queue with a stationary input, under the stability condition. Then, we can use the continuity property that the weak convergence of the stationary inputs together with the convergence of the expected unit input, i.e., the expected difference of the service time and inter-arrival time, implies the weak convergence of the stationary waiting times of the corresponding systems (see [3]). A slight modification is needed since we also have the background process, but it is clear that the same continuity property holds for the stationary joint distribution of the waiting time and background process. Since we are only concerned with $k$ service time distributions which have finite means, the input process of the modified system by the $\epsilon$ satisfies all the requirements for the continuity. Hence, we get (3.8).

From (3.2) and (3.3), we have

$$s_{01}(0) + \sum_{i=2}^{k} s_{0i} = \lambda - \lambda \sum_{i=1}^{k} \psi_i(\lambda)g_i(\lambda) = \lambda \psi(0)[I - G(0)P]u' = 0,$$

(3.9)

where the second equality follows from (2.4) with the vector $u = (1, \ldots, 1)$ and the transposition $'$. Since

$$s_{01}(\omega_1) = \lambda - \omega_1 - \lambda \psi_1(\lambda)g_1(\lambda)p_{11} = 0,$$

we have

$$\psi_1(\lambda) = \frac{\lambda - \omega_1}{\lambda g_1(\lambda)p_{11}} = \frac{g_1(\omega_1)}{g_1(\lambda)}.$$  

(3.10)

Hence,

$$s_{01}(0) = \lambda - \lambda \psi_1(\lambda)g_1(\lambda)p_{11} = \omega_1,$$

(3.11)

so, from (3.9),

$$\sum_{i=2}^{k} s_{0i} = -s_{01}(0) = -\omega_1.$$  

(3.12)

These results will be convenient for deriving closed form formulas in Section 5.
4. The stationary sojourn time and queue length

We consider the stationary sojourn time and queue length, where the queue length is meant to include a customer being served. These distributions can be obtained in the same way as in [6]. We briefly discuss them. Let $U_n = W_n + S_n$, i.e., the sojourn time of the $n$-th arriving customer, and $U$ be a random variable subject to the stationary distribution of $\{U_n\}$. Let $L^+_n$ be the queue length just after the $n$-th customer completes his service, and $L^+$ be a random variable subject to the stationary distribution of $\{L^+_n\}$. Let $L$ be a random variable subject to the stationary queue length distribution. Then, the following results are obtained.

**Theorem 4.1** Under the assumptions of Theorem 3.1, we have

$$E[e^{-\theta U}] = \frac{1}{\Psi(0)} \sum_{i=1}^{k} \psi_i(\theta) g_i(\theta), \quad (4.1)$$

$$E[z^L] = E[z^L^+] = E[e^{-\lambda (1-z) U}]. \quad (4.2)$$

**Proof.** By the definitions,

$$E[e^{-\theta U_n}; N \geq n] = E[e^{-\theta (W_n + S_n)}; N \geq n]$$

$$= \sum_{i=1}^{k} E[e^{-\theta W_n}; N \geq n, J_n = i] E[e^{-\theta S_n} | J_n = i]$$

$$= \sum_{i=1}^{k} f_{n,i}(\theta) g_i(\theta).$$

This implies (4.1) since $E[e^{-\theta U}] = \sum_{n=1}^{\infty} E[e^{-\theta U_n}; N \geq n]/E[N]$. Similarly,

$$E[z^{L^+_n}; N \geq n] = \sum_{j=0}^{\infty} E \left[ e^{-\lambda (W_n + S_n)} \frac{(\lambda (W_n + S_n))^{j} z^j}{j!}; N \geq n \right]$$

$$= E[e^{-\lambda (1-z) (W_n + S_n)}; N \geq n]$$

$$= \sum_{i=1}^{k} f_{n,i}(\lambda (1-z)) g_i(\lambda (1-z))$$

implies the second equality of (4.2) since $E[z^{L^+}] = \sum_{n=1}^{\infty} E[z^{L^+_n}; N \geq n]/E[N]$. As is well known, the first equality is obtained from PASTA (see [12]) and the fact that the stationary queue length distribution observed by arriving customers is identical with the one observed by departing customers. \(\square\)

Note that (4.2) is known as the distributional law of Little’s formula (see [5]).

5. Special cases

In this section, we consider the two cases that $k = 2$ and $k = 3$, i.e., the service distributions are of two types and three types, respectively. We aim to derive closed form
expressions for the stationary waiting time distributions and their means, and consider the effect of the Markov dependence on the conventional exceptional model. Although the case of \( k = 2 \) is included in the case of \( k = 3 \), the former case is instructive to understand the structure of the model.

### 5.1 Two service types

Consider the model with two service types. The transition probability matrix \( P \) of the Markov chain \( \{J_n\} \) is

\[
P = \begin{bmatrix} p_{11} & p_{12} \\ 0 & p_{22} \end{bmatrix},
\]

where \( p_{22} = 1 \) and \( p_{12} = 1 - p_{11} \). Thus \( P \) is determined by one parameter \( p_{11} \). If \( p_{11} = 0 \), then the model reduces to the conventional exceptional model with one exceptional service. Assume that \( p_{11} > 0 \).

Since

\[
s_{12}(\theta) = s_{02} - \frac{d_{12}(\theta)}{h_1(\theta)} s_{01}(\theta),
\]

from Lemma 3.1, we have

\[
\Psi(\theta) = \psi_1(\theta) + \psi_2(\theta)
\]

\[
= s_{01}(\theta) \left\{ \frac{h_2(\theta) - d_{12}(\theta)}{h_1(\theta)h_2(\theta)} \right\} + s_{02} h_1(\theta).
\]

(5.1)

We next determine \( \Psi(\lambda) \). By Lemma 3.2, we need to solve

\[
s_{01}(\omega_1) = 0, \quad s_{12}(0) = 0.
\]

(5.2)

But, from (3.9), we already know that the first equation implies

\[
\psi_1(\lambda) = \frac{g_1(\omega_1)}{g_1(\lambda)}.
\]

On the other hand, the second equation of (5.2) becomes

\[
\lambda - \lambda \psi_1(\lambda) g_1(\lambda) - \lambda \psi_2(\lambda) g_2(\lambda) = 0,
\]

so we have

\[
\psi_2(\lambda) = \frac{\lambda - \lambda \psi_1(\lambda)}{\lambda g_2(\lambda)} = \frac{1 - g_1(\omega_1)}{g_2(\lambda)}.
\]

We now compute \( \Psi(0) \). To this end, we note that

\[
s_{01}(0) = \omega_1, \quad s_{02} = -\omega_1,
\]

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which follow from (3.11) and (3.12), and that

\[ h_1(0) = \lambda p_{12}, \quad h_2(0) = 0, \quad d_{12}(0) = -\lambda p_{12}, \]
\[ h_1'(0) = \rho_1, \quad h_2'(0) = \rho_2 - 1, \quad d_{12}'(0) = \rho_1 p_{12}. \]

Hence, we have, using l’Hospital’s rule,

\[
\Psi(0) = \lim_{\theta \to 0} \frac{\Psi(\theta)}{\phi(\theta)} = \frac{\lambda p_{12}(1 - \rho_2)}{\lambda p_{12} + \omega_1(\rho_1 - \rho_2)} \cdot s_{01}(0) \frac{h_2(\theta) - d_{12}(\theta) - \omega_1 h_1(\theta)}{h_1(\theta) h_2(\theta)}.
\]

Therefore, we have from (5.1) and (5.3),

\[
\phi(\theta) = \frac{\Psi(\theta)}{\Psi(0)} = \frac{\lambda p_{12}(1 - \rho_2)}{\lambda p_{12} + \omega_1(\rho_1 - \rho_2)} \cdot s_{01}(0) \frac{h_2(\theta) - d_{12}(\theta) - \omega_1 h_1(\theta)}{h_1(\theta) h_2(\theta)}.
\]

We compute the mean waiting time. From (5.4), we can get

\[
E[W] = -\phi'(0) = \frac{\lambda^2 p_{12} E[S_{<2>}] - 2(1 - \rho_1 p_{11})(1 - \rho_2)}{2 \lambda p_{12}(1 - \rho_2)} - \frac{1 + \rho_1 p_{12} - \rho_2 + \frac{\omega_1}{2}(E[S_{<1>}] - E[S_{<2>}])}{\omega_1(\rho_2 - \rho_1) - \lambda p_{12}}.
\]

See Appendix A of [7] for details of this computation. By \( W^* \) we denote the stationary waiting time when \( p_i = 0 \) for all \( i \), i.e., for the corresponding model with conventional exceptional service. Let us calculate \( E[W^*] \) from (5.5). To this end, let \( p_{11} \to 0 \), which implies \( p_{12} \to 1 \) and \( \omega_1 \to \lambda \). Hence, we have

\[
E[W^*] = \frac{\lambda^2 E[S_{<2>}] - 2(1 - \rho_2)}{2 \lambda (1 - \rho_2)} - \frac{1 + \rho_1 - \rho_2 + \frac{\omega_1}{2}(E[S_{<1>}] - E[S_{<2>}])}{\lambda(\rho_2 - \rho_1 - 1)} = \frac{\lambda E[S_{<2>}^2]}{2(1 - \rho_2)} + \frac{\lambda(\omega_1)}{2(1 + \rho_1 - \rho_2)} E[S_{<2>}] - E[S_{<1>}] - \lambda[S_{<2>}] - \lambda[S_{<1>}].
\]

This value is identical with \( E[W] \) of the conventional model with one exceptional service.

We are interested in the effect of the Markov dependence. To see this, we consider how \( E(W) \) changes when \( p_{11} \) is small. So, we expand (5.5) with respect to \( p_{11} \). Since \( p_{12} = 1 - p_{11} \) and, from \( h_1(\omega_1) = 0 \),

\[
\frac{d}{dp_{11}} \omega_1 = \frac{-\lambda g_1(\omega_1)}{1 + g_1'(\omega_1)p_{11}} = \lambda(1 - g_1(\lambda))p_{11} + o(p_{11}),
\]

\[ d_{12}(0) = -\lambda p_{12}, \]
\[ d_{12}'(0) = \rho_1 p_{12}. \]
we have

\[ E[W] = E[W^*] + p_{11}\Delta + o(p_{11}), \]

where \( o(s) \) means small order of \( s \), i.e., it is a function \( s \) such that \( \lim_{s \to 0} o(s)/s = 0 \), and

\[ \Delta = \frac{(\rho_1 - \rho_2)(g_1(\lambda) - 1 + \rho_1)}{\lambda(1 - \rho_2 + \rho_1)} + \lambda \frac{(E[S^2_{\leq 1}] - E[S^2_{\leq 2}])}{2(1 - \rho_2 + \rho_1)^2} (1 - g_1(\lambda)). \]

Since

\[ g_1(\lambda) = E(e^{-\lambda S_{\leq 1}}) \geq E(1 - \lambda S_{\leq 1}) = 1 - \rho_1, \]

we can find the following property of \( \Delta \).

(i) If \( \rho_1 \geq (\leq) \rho_2 \) and \( E[S^2_{\leq 1}] \geq (\leq) E[S^2_{\leq 2}] \), then \( \Delta \geq (\leq) 0 \).

(ii) If \( \rho_1 \geq (\leq) \rho_2 \) and \( E[S^2_{\leq 1}] \leq (\geq) E[S^2_{\leq 2}] \), then \( \Delta \geq (\leq) 0 \) if and only if

\[ 2(\rho_1 - \rho_2)(1 - \rho_2 + \rho_1) \left( \frac{\rho_1}{1 - g_1(\lambda)} - 1 \right) \geq (\leq) \lambda^2 \left( E[S^2_{\leq 2}] - E[S^2_{\leq 1}] \right). \]

The case (i) is reasonable, but the case (ii) is delicate. Some numerical examples are given in tables attached at the end of this paper, in which \( S_{\leq 1} \) and \( S_{\leq 2} \) are assumed to be deterministic for Tables 1 to 3 and hyper-exponential for Tables 4 to 6, where the hyper-exponential distributions have the following form:

\[ P(S_{\leq i} \leq x) = p_{\leq i>1}(1 - e^{-\mu_{\leq i>1}x}) + p_{\leq i>2}(1 - e^{-\mu_{\leq i>2}x}), \quad i = 1, 2, \]

with parameters \( p_{\leq i>1} = 0.4, p_{\leq i>2} = 0.6, p_{<i>1} = 0.3, p_{<i>2} = 0.7, \rho_2 = 0.6, \mu_{<i>1} = 1 \) and \( \mu_{<i>2} = \frac{7}{3} \). They show that \( E[W^*] + p_{11}\Delta \) gives good approximations for \( E[W] \) even for not very small \( p_{11} \).

### 5.2 Three service types

We consider the model with three service types, i.e., the case of \( k = 3 \). Because the computations are complicated, we here only present the resulting formulas (see Appendix B of [7] for their detailed computations).

\[
\Psi(\theta) = \frac{1}{h_1(\theta)h_2(\theta)h_3(\theta)} \left[ s_{01}(\theta) \{ h_2(\theta)h_3(\theta) - d_{12}(\theta)h_3(\theta) + d_{12}(\theta)d_{23}(\theta) - d_{13}(\theta)h_2(\theta) \} ight. \\
\left. + s_{02} \{ h_1(\theta)h_3(\theta) - d_{23}(\theta)h_1(\theta) \} + s_{03} h_1(\theta)h_2(\theta) \right]. \tag{5.6}
\]

\[
\psi_1(\lambda) = \frac{g_1(\omega_1)}{g_1(\lambda)},
\]

\[
\psi_2(\lambda) = \frac{g_2(\omega_2)}{g_2(\lambda)} \cdot \frac{\lambda p_{12} \{ g_1(\omega_2) - g_1(\omega_1) \}}{h_1(\omega_2)},
\]

\[
\psi_3(\lambda) = \frac{1 - \psi_1(\lambda)g_1(\lambda) - \psi_2(\lambda)g_2(\lambda)}{g_3(\lambda)}
\]

\[
= \frac{1}{g_3(\lambda)} \left[ 1 - g_1(\omega_1) - \frac{\lambda g_2(\omega_2)p_{12} \{ g_1(\omega_2) - g_1(\omega_1) \}}{h_1(\omega_1)} \right].
\]
\[ \Psi(0) = \frac{\omega_1 \{ (\rho_2 - \rho_3)p_{12} + (\rho_1 - \rho_3)p_{23} \} + s_02(p_{12} + p_{13})(\rho_2 - \rho_3) + \lambda(p_{12} + p_{13})p_{23}}{\lambda(p_{12} + p_{13})p_{23}(1 - \rho_3)} \]

\[ E[W] = \frac{1}{2\lambda^2 p_{12}p_{23}(1 - \rho_3)\Psi(0)} \times \left\{ \omega_1 \left( \lambda^2 E[S^2_{<1>}]p_{23} + \lambda^{2}E[S^2_{<2>}]p_{12} - \lambda^2 E[S^2_{<3>}]p_{23} + 2\rho_1 + 2\rho_1\rho_2(p_{12} - p_{22}) - 2\rho_3(1 - \rho_2p_{22} + \rho_1p_{12}) \right) \\
+ s_02(0) \left( \lambda^2(p_{12} + p_{13})\{E[S^2_{<2>}] - E[S^2_{<3>}]\} + 2(1 - \rho_1p_{11})(\rho_2 - \rho_3) \right) \\
- 2\lambda \left( \rho_3(p_{12} + p_{23}) + \rho_2(p_{13}p_{22} - p_{12}p_{23}) - \rho_1(p_{12}p_{23} + p_{13}p_{23}) - (p_{12} + p_{13} + p_{23}) \right) \right\} \\
- \frac{1 - \rho_1p_{11}}{\lambda(p_{12} + p_{13})} - \frac{1 - \rho_2p_{22}}{\lambda p_{23}} + \frac{\lambda E[S^2_{<3>}]}{2(1 - \rho_3)} \right) \right] \]  

(5.7)

Let \( p_{11}, p_{13}, \) and \( p_{22} \) tend to 0 in the above formula, then \( p_{12} \to 1, p_{23} \to 1, \omega_1 \to \lambda \) and \( \omega_2 \to \lambda. \) So we have \( s_{01}(0) = \lambda \) and

\[ \psi_1(\lambda) = 1, \quad \psi_2(\lambda) = -\lambda g_1(\lambda), \quad \psi_3(\lambda) = \frac{1 - g_1(\lambda) + \lambda g_1(\lambda)g_2(\lambda)}{g_3(\lambda)}, \quad \Psi(0) = \frac{1 + (\rho_1 - \rho_3) + (1 - g_1(\lambda))(\rho_2 - \rho_3)}{1 - \rho_3}. \]

Hence, from (5.7) we get

\[ E[W^*] = \frac{1}{1 + (\rho_1 - \rho_3) + (1 - g_1(\lambda))(\rho_2 - \rho_3)} \times \left[ \frac{\lambda}{\lambda} \left( (\rho_1 - 1 + g_1(\lambda))(\rho_2 - \rho_3) + \lambda(1 - g_1(\lambda)) \left[ E[S^2_{<2>}] - E[S^2_{<3>}] \right] \right) \right] + \frac{\lambda E[S^2_{<3>}]}{2(1 - \rho_3)}. \]

This agrees with the result in [6].

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**References**


\[
\begin{array}{cccc}
\rho_1 & E[W] & E[W^+] & \Delta & E[W^+] + p_{11}\Delta \\
0.3 & 0.2254979246 & 0.2571428571 & -0.0531971356 & 0.2411837165 \\
0.5 & 0.3743082782 & 0.3888888889 & -0.0251952670 & 0.3813303088 \\
0.7 & 0.5268074641 & 0.5090909091 & 0.0313928604 & 0.5185087672 \\
0.9 & 0.6839717307 & 0.6230769231 & 0.1102503435 & 0.6561520261 \\
\end{array}
\]

Table 1: Deterministic service with \( \rho_2 = 0.6, \ p_{11} = 0.3 \)

\[
\begin{array}{cccc}
\rho_1 & E[W] & E[W^+] & \Delta & E[W^+] + p_{11}\Delta \\
0.3 & 0.2571428571 & -0.0531971356 & 0.2518231436 \\
0.5 & 0.3888888889 & -0.0251952670 & 0.3863693622 \\
0.7 & 0.5090909091 & 0.0313928604 & 0.5122301951 \\
0.9 & 0.6230769231 & 0.1102503435 & 0.6341019574 \\
\end{array}
\]

Table 2: Deterministic service with \( \rho_2 = 0.6, \ p_{11} = 0.1 \)

\[
\begin{array}{cccc}
\rho_1 & E[W] & E[W^+] & \Delta & E[W^+] + p_{11}\Delta \\
0.3 & 0.2562492022 & -0.0531971356 & 0.2566108858 \\
0.5 & 0.3885006994 & -0.0251952670 & 0.3886369362 \\
0.7 & 0.5095436365 & 0.0313928604 & 0.5094048377 \\
0.9 & 0.6245875853 & 0.1102503435 & 0.6241794265 \\
\end{array}
\]

Table 3: Deterministic service with \( \rho_2 = 0.6, \ p_{11} = 0.01 \)

\[
\begin{array}{cccc}
\rho_1 & \mu_{<1>1} & \mu_{<1>2} & E[W] & E[W^+] & \Delta & E[W^+] + p_{11}\Delta \\
0.3 & 2.0 & 6.0 & 0.7220281104 & 0.8058608059 & -0.1364797596 & 0.7649168780 \\
0.5 & 2.0 & 2.0 & 1.0418146033 & 1.0924369748 & -0.0813148789 & 1.0680425111 \\
0.7 & 2.0 & 1.2 & 1.4085908032 & 1.3968253968 & 0.0216702169 & 1.4033264619 \\
0.9 & 2.0 & 0.857 & 1.7978218679 & 1.7106610222 & 0.1533066624 & 1.7566530209 \\
\end{array}
\]

Table 4: Hyper-exponential service with \( \rho_2 = 0.6, \ p_{11} = 0.3 \)

\[
\begin{array}{cccc}
\rho_1 & \mu_{<1>1} & \mu_{<1>2} & E[W] & E[W^+] & \Delta & E[W^+] + p_{11}\Delta \\
0.3 & 2.0 & 6.0 & 0.7814856348 & 0.8058608059 & -0.1364797596 & 0.7922128299 \\
0.5 & 2.0 & 2.0 & 1.0781245887 & 1.0924369748 & -0.0813148789 & 1.0843054869 \\
0.7 & 2.0 & 1.2 & 1.3971212874 & 1.3968253968 & 0.0216702169 & 1.3970420990 \\
0.9 & 2.0 & 0.857 & 1.7343978802 & 1.7106610222 & 0.1533066624 & 1.7259916884 \\
\end{array}
\]

Table 5: Hyper-exponential service with \( \rho_2 = 0.6, \ p_{11} = 0.1 \)

\[
\begin{array}{cccc}
\rho_1 & \mu_{<1>1} & \mu_{<1>2} & E[W] & E[W^+] & \Delta & E[W^+] + p_{11}\Delta \\
0.3 & 2.0 & 6.0 & 0.8035535619 & 0.8058608059 & -0.1364797596 & 0.8044960083 \\
0.5 & 2.0 & 2.0 & 1.0910955610 & 1.0924369748 & -0.0813148789 & 1.0916238260 \\
0.7 & 2.0 & 1.2 & 1.3971212874 & 1.3968253968 & 0.0216702169 & 1.3970420990 \\
0.9 & 2.0 & 0.857 & 1.7285743090 & 1.7106610222 & 0.1533066624 & 1.7219408888 \\
\end{array}
\]

Table 6: Hyper-exponential service with \( \rho_2 = 0.6, \ p_{11} = 0.01 \)