A matrix exponential form for hitting probabilities and its application to a Markov modulated fluid queue with downward jumps

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Abstract

We consider a fluid queue with downward jumps, where the fluid flow rate and the downward jumps are controlled by a background Markov chain with a finite state space. It is shown that the stationary distribution of a buffer content has a matrix exponential form, and the exponent matrix is identified. We derive these results using time reversed arguments and the background state distribution at the hitting time concerning the corresponding fluid flow with upward jumps. This distribution was recently studied for a fluid queue with upward jumps under a stability condition. We give an alternative proof for this result, using the rate conservation law. This proof not only simplifies the proof, but also explains an underlying Markov structure and enables us to study more complex cases such that the fluid flow has jumps subject to a nondecreasing Levy process, a Brownian component, and countably many background states.

1. Introduction

Consider a net fluid flow whose rate is controlled by a continuous time Markov chain with a finite state space. The rate may take positive and negative values, where the negative rate is meant that a fluid is draining. This fluid flow is referred to as a *conventional Markov modulated fluid flow*, and the state of the Markov chain is referred to as a background state. This class of fluid flows have been extensively studied for queueing models (see, e.g., [2, 5, 8, 12, 16]). In this paper, we extend the fluid flow in such a way that an accumulated net fluid flow may have downward jumps, i.e., instantaneous draining, when the background state changes. The sizes of those jumps are independent, but their distributions may depend on the states before and after jumps, i.e., the state transitions. Those jumps may be considered as disasters. This extended model is referred to as a *Markov modulated fluid flow with downward jumps*. This type of models have been studied for conventional customer-based queues (see, e.g., [11]).

A primary concern of this paper is to derive the stationary distribution of the buffer content in a fluid queue whose net input is the Markov modulated fluid flow with downward jumps, provided a stability condition, i.e., the mean rate of the net input is negative.

It turns out that the stationary distribution has a matrix exponential form. Here, the matrix exponential form for a queue was firstly studied for the $GI/PH/1$ model by Sengupta [18]. We introduce it in a slightly general form in Section 2. This result extends the corresponding results for the Markov modulated fluid queues in [5] and [16] as well as the matrix exponential form for the attained waiting time process in [18].

To derive the stationary distribution, we reverse the time axis of the additive process that describes the Markov modulated fluid flow with downward jumps, then the stationary distribution is obtained by computing the probabilities that the time reversed process hits downward levels. Since this time reversed process is equivalent to the corresponding additive process for a Markov modulated fluid flow with upward jumps, we can apply recent results due to Takada [19], in which the buffer content process is studied for the Markov modulated fluid flow with upward jumps, and a matrix exponential form was obtained for the background state distribution at the hitting instants at downward levels, provided the mean rate of the additive process is negative. As we shall see, this mean rate is needed to be positive for the present fluid queue.

Thus, we need to slightly modify the Takada's [19] results. Not only because of this but also for further extensions, we here give an alternative proof for the results in [19], using the time dependent rate conservation law due to Miyazawa [13]. This approach simplifies the arguments in [19], and explains an underlying Markov structure. Furthermore, it is applicable to more general situations. We discuss the following cases: the flow rate is allowed to be zero, which is excluded in [19], the fluid flow includes either jumps subject to a nondecreasing Levy process or Brownian fluctuations, and the background states are countably many, where the Brownian case is considered in [5, 16] for the case of no jumps.

This paper is made up by five sections. In Section 2, we introduce an additive process for the net fluid flow with upward jumps, and discuss on the hitting time. In Section 3, the background state distribution at the hitting time, which is referred to as hitting probabilities, shown to have the matrix exponential form. Conditions to determine this matrix form is also derived. In Section 4, the Markov modulated fluid queue with downward jumps is considered. Finally, in Section 5, four extensions of the present fluid model are considered.

2. Additive process for a fluid flow with upward jumps

We first consider a fluid flow with upward jumps, which will be applied to get main results in Section 4. This fluid flow is generated by a continuous-time Markov chain $M(t)$ with a finite state space S, which is referred to as a background process. It is assumed that $M(t)$ is right-continuous. Denote cardinality of S by S. The transition rate matrix of this Markov chain is decomposed into $S \times S$-matrices C and D such that D is nonnegative and $(C+D)\mathbf{1} = \mathbf{0}$, where $\mathbf{1}$ is the S-dimensional vector all of whose entries are units, and $\mathbf{0}$ is the corresponding zero vector. The ij-th entries of C and D are denoted by C_{ij} and D_{ij}, respectively. Throughout the paper, we assume

(i) $C + D$ is irreducible.

Since the state space S is finite, this implies that there is a unique stationary distribution of $C + D$. State transitions of $M(t)$ due to D, called D-transitions, are accompanied with upward jumps, while no jumps are attached for transitions due to C. At D-transitions from state i to state j, the size of the upward jump is subject to distribution F_{ij}. Let

$$D_{ij}(x) = D_{ij}F_{ij}(x), \qquad x \geq 0,\ i, j \in S. \tag{2.1}$$

By $D(x)$, we denote the matrix whose ij entry is $D_{ij}(x)$. We further assume that the fluid flow has rate $v(i)$ while the background state is i. The $v(i)$ may take any positive and negative values except zero, i.e.,

(ii) $v(i) \neq 0$ for all $i \in S$.

This condition will be removed in Section 5.

Define additive process $Y(t)$ as

$$Y(t) = \int_{0+}^{t+} v(M(u))du + \int_{0+}^{t+} A_u(M(u-), M(u))N(du), \qquad t \geq 0, \tag{2.2}$$

where $A_u(i, j)$ is the jump size at time u when the background state changes from i to j, and N is a point process generated by all D-transitions. We extend $Y(t)$ for $t < 0$ as

$$Y(t) = -\int_{t+}^{0+} v(M(u))du - \int_{t+}^{0+} A_u(M(u-), M(u))N(du). \tag{2.3}$$

Assume that $A_u(i, j)$'s, which are subject to distribution F_{ij}, are independent for different u's. Clearly, $Y(t)$ is right-continuous in t and $Y(0) = 0$. We assume that $\{M(t)\}_{t=-\infty}^{+\infty}$ is a stationary process, so $\{Y(t)\}_{t=-\infty}^{+\infty}$ has stationary increments. We further assume that $E(|A_u(i, j)|) < \infty$ for all $i, j \in S$, which is equivalent to $E(|Y(1)|) < \infty$.

This class of fluid flows appears in various queueing models. For instance, if $v(i) = -1$, then $Y(t)$ describes the workload in $MAP/G/1$ queue during busy periods. If no jumps are accompanied, then $Y(t)$ describes the conventional Markov modulated fluid flow. In this case, $v(i)$ represents the difference of the input and output rates, i.e., the net flow rate. In those queueing applications, the following hitting time frequently has a key role in deriving the stationary distributions of the system states.

$$\tau_x^- = \inf\{t > 0; x + Y(t) < 0\}, \qquad x \geq 0,$$

where we set $\tau_x^- = \infty$ if there is no t to attain the infimum. That is, τ_x^- is the first time when $Y(t)$ attains $-x$. Then, the queueing problem is reduced to compute the following probabilities.

$$R_{ij}(x) = P\left(M(\tau_x^-) = j | M(0) = i\right), \quad x \geq 0,\ i, j \in S, \tag{2.4}$$

where the event $M(\tau_x^-) = j$ includes the event $\tau_x^- < \infty$. By $R(x)$, we denote the $S \times S$ matrix whose ij entry is given by (2.4). The following lemma clarifies when (2.4) gives a defective distribution.

Lemma 2.1 For each $i \in S$ and each $x > 0$,

$$P\left(\tau_x^- < \infty \middle| M(0) = i\right) < 1, \tag{2.5}$$

if and only if $E(Y(1)) > 0$. If this is the case,

$$\lim_{x\to\infty} P\left(\tau_x^- < \infty \middle| M(0) = i\right) = 0. \tag{2.6}$$

Proof. Since $Y(t)$ is an additive process generated by the Markov chain, the law of large numbers leads to

$$\lim_{t\to\infty} \frac{Y(t)}{t} = E(Y(1)), \qquad a.s.\ P(\cdot|M(0) = i). \tag{2.7}$$

Suppose $E(Y(1)) > 0$, then (2.7) implies that $Y(t) \to +\infty$ as $t \to +\infty$. Hence, there are $t_0 > 0$ and $j \in S$ such that $P(\inf_{u>t_0}(x + Y(u)) > 0, M(t_0) = j) > 0$. By appropriately choosing the sample path of $M(t)$ in $(0, t_0]$, we have (2.5). The fact that $\inf_{t>0} Y(t) > -\infty$ implies (2.6). If $E(Y(1)) < 0$, (2.7) implies that $Y(t) \to -\infty$ as $t \to +\infty$, so (2.5) does not hold. It remains to consider the case that $E(Y(1)) = 0$. Let $T_n(i)$ be the n-th recurrence time of $M(t)$ to state i. Since the Markov chain has the stationary distribution, $E(T_n(i) - T_{n-1}(i)|M(T_{n-1}) = i) < \infty$ for $n \geq 0$, where $T_0 = 0$. Furthermore, $Y(T_n(i)) - Y(T_{n-1}(i))$ for $n \geq 1$ are independent and identically distributed under $M(0) = i$. Since S is finite, it is not hard to see that $E(Y(T_1(i))|M(0) = i) < \infty$. Hence, by the law of large numbers,

$$\lim_{n\to\infty} \frac{Y(T_n(i))}{n} = E(Y(T_1(i))), \qquad a.s.\ P(\cdot|M(0) = i).$$

However, from (2.7),

$$\lim_{n\to\infty} \frac{Y(T_n(i))}{n} = \lim_{n\to\infty} \frac{T_n(i)}{n} \frac{Y(T_n(i))}{T_n(i)} = E(T_1(i))E(Y(1)) = 0.$$

Hence, $E(Y(T_1(i)))$ must be 0. Since $Y(T_n(i))$ for $n \geq 1$ can be considered as a random walk on the line, it goes under level 0 with probability one from the well known behaviors of the random walk (see, e.g., Theorem 2 of section XII.2 of [9]). Hence (2.5) does not hold. This completes the proof. □

Remark 2.1 Obviously this lemma holds true for a similar additive process with upward and downward jumps, i.e., when $A_u(i, j)$ may take positive and negative values.

We call a square matrix *substochastic* (or *stochastic*) if all its entries are nonnegative such that their row sums are not greater than (or equal) 1. Similarly, a square matrix is said to be *subrate* (or *rate*) if it has negative diagonal elements and nonnegative off-diagonal elements such that that their row sums are not greater than (or equal) 0. Further, a square matrix is called *ML*, if all its off-diagonal entries are nonnegative. Thus, Lemma 2.1 tells that $R(\infty)$ is stochastic if and only if $E(Y(1)) \leq 0$.

Assuming $E(Y(1)) < 0$, Takada [19] recently showed that $R(x)$ has the following form:

$$R(x) = Ue^{xQ}V, \qquad x \geq 0,$$

where U and V are nonnegative matrices, and Q is a subrate matrix. We call this expression a matrix-exponential form. Note that the size of Q is not necessarily to be S, and preferable to be smaller for computations. This matrix-exponential form was firstly studied for $GI/M/1$ type queue by Sengupta [18]. They have also appeared in various scenarios. See [5, 16] for the conventional fluid queue and [3, 10] for the $MAP/G/1$ queue.

The matrix-exponential form of $R(x)$ may be obvious, since $\{M(\tau_x^-); 0 \leq x < \infty\}$ is a continuous-time Markov chain, which follows from the fact that $Y(t)$ is skip free in the downward direction, as noticed in [5, 16] for the fluid queue. The problem is to identify its transition rate matrix. This has been done in [19], using integral equations. In this paper, we give an alternative way for the identification, using differential equations based on the rate conservation law. This approach not only simplifies the derivation in [19], but also is applicable for more general situations, which will be discussed in Section 5.

3. Hitting probabilities

In this section, we consider the hitting probability matrix $R(x)$. In Takada [19] and Rogers [16], it is firstly assumed that $v(i)$ only takes either 1 and -1, then the general case is obtained through a random time change. We do not need this modification. Our approach is based on the rate conservation law. Let $\{\theta_t\}$ be an operator group on the underlying sample space Ω such that $M(t) \circ \theta_s = M(t+s)$. Then, it is easy to see that $Y(t) \circ \theta_s = Y(t+s) - Y(s)$. That is, the probability law is unchanged under the time shift by θ_t. Note that the point process N is consistent with θ_t, i.e., its distribution is unchanged under a time shift by θ_t. Let $\lambda = E(N(0,1])$ and let P_0 be the Palm distribution with respect to N. We shall use the following rate conservation law.

Lemma 3.1 (Theorem 2.1 in Miyazawa [13]) If a real valued process $Z(t)$ satisfies the following three conditions for each $t > 0$: (i) N includes all jump instants of $Z(t)$, (ii) $Z(t)$ has a right-hand derivative $Z'(t)$ except at the jump instants of N, and $Z(t)$ and $Z'(t)$ are bounded, (iii) $E_0((Z(t+) - Z(t-)) \circ \theta_{-t})$ is right-continuous, then we have

$$\frac{d}{dt}E(Z(t)) = E(Z'(t)) + \lambda E_0(Z(t+) \circ \theta_{-t} - Z(t-) \circ \theta_{-t}), \tag{3.1}$$

where E_0 represents the expectation concerning P_0.

We apply this lemma to

$$Z(t) = 1(M(0) = i, M(\tau^-_{x-\gamma(-t)}) = j) \circ \theta_t,$$

where $1(\cdot)$ is the indicator function of the statement "$\cdot$", and

$$\gamma(t) = \int_0^t v(M(u))du \ \left(= -\int_t^0 v(M(u))du\right) \qquad t \geq 0 \ (t < 0, \text{respectively}). \tag{3.2}$$

Note that, for $t \geq 0$,

$$\begin{aligned} M(\tau^-_{x-\gamma(-t)}) \circ \theta_t &= M(\inf\{u > 0; x - \gamma(-t) + Y(u) < 0\} \circ \theta_t + t) \\ &= M(\inf\{u + t > 0; x + \gamma(t) - Y(t) + Y(u+t) < 0\}) \\ &= M(\inf\{u \geq t; x + \gamma(t) - Y(t) + Y(u) < 0\}). \end{aligned} \tag{3.3}$$

This implies that $Z(t)$ is constant between jumps of $M(t)$, because $Y(t) - \gamma(t)$ only accumulate jumps. Hence, (i) is satisfied. (ii) and (iii) are obviously satisfied since $\gamma(t)$ is continuous. In particular, $Z'(t) = 0$ except at the jump instants, so $E(Z'(t)) = 0$. Since

$$\begin{aligned} Z(0+) &= 1(M(0) = i, M(\tau_x^-) = j), \\ Z(0-) &= 1(M(0-) = i, M(\tau_{x-Y(0-)}^-) = j) \\ &= 1(M(0-) = i, M(\tau_{x+A_0(M(0-),M(0))}^-) = j), \qquad a.s.\ P_0, \end{aligned}$$

we have

$$\lambda E_0(Z(0+)) = -C_{ii} P(M(0) = i) P_0(M(\tau_x^-) = j | M(0) = i), \tag{3.4}$$

$$\begin{aligned} \lambda E_0(Z(0-)) &= \lambda \sum_{k \in S} P_0(M(0-) = i, M(0) = k) \\ &\qquad \times P_0(M(\tau_{x+A_0(M(0-),M(0))}^-) = j | M(0-) = i, M(0) = k) \\ &= \sum_{k \neq i} C_{ik} P(M(0) = i) P_0(M(\tau_x^-) = j | M(0) = k) \\ &\quad + \sum_{k \in S} D_{ik} P(M(0) = i) P_0(M(\tau_{x+A_0(i,k)}^-) = j | M(0) = k). \end{aligned} \tag{3.5}$$

We next compute the left-hand side of (3.1). Since $|\gamma(-h)| \leq h \max(|v(i)|, |v(k)|)$ on the event $\{M(0) = i, M(-h) = k, N((-h, 0]) = 1\}$ and τ_{x+y}^- almost surely converges to τ_x^- as y goes to 0, we have

$$\begin{aligned} \lim_{h \downarrow 0} \Big| & P(M(0) = i, M(\tau_{x-\gamma(-h)}^-) = j | M(-h) = k, N((-h, 0]) = 1) \\ & - P(M(0) = i, M(\tau_x^-) = j | M(-h) = k, N((-h, 0]) = 1) \Big| \\ & \leq \lim_{h \downarrow 0} P(M(0) = i, M(\tau_{x-\gamma(-h)}^-) \neq M(\tau_x^-) | M(-h) = k, N((-h, 0]) = 1) = 0, \end{aligned}$$

where $N((-h, 0])$ is the number of transitions of the background process $\{M(t)\}$ during the time interval $(-h, 0]$. Since $P(N((-h, 0]) = 1) = O(h)$ and $P(N((-h, 0]) \geq 2) = o(h)$, the bounded convergence theorem yields

$$\begin{aligned} \lim_{h \downarrow 0} \frac{1}{h} \Big(& P(N((-h, 0]) \geq 1, M(0) = i, M(\tau_{x-\gamma(-h)}^-) = j) \\ & - P(N((-h, 0]) \geq 1, M(0) = i, M(\tau_x^-) = j) \Big) = 0. \end{aligned}$$

Hence, the left-hand side of (3.1) becomes

$$\begin{aligned} \frac{d}{dt} E(Z(t)) \Big|_{t=0} &= \frac{d}{dt} P(M(0) = i, M(\tau_{x-\gamma(-t)}^-) = j) \Big|_{t=0} \\ &= \frac{d}{dt} P(N((-t, 0]) = 0, M(0) = i, M(\tau_{x+v(i)t}^-) = j) \Big|_{t=0} \\ &= v(i) \frac{d}{dx} P(M(0) = i, M(\tau_x^-) = j). \end{aligned} \tag{3.6}$$

Let P_i be the conditional probability $P(\cdot | M(0) = i)$. Then, from the memoryless property of an exponential distribution, for any event A after time 0, we have that $P_0(A | M(0) = i) = P(A | M(0) = i)$. Thus, after substituting (3.4), (3.5) and (3.6) to (3.1) at $t = 0$, we divide the resulted formula by $P(M(0) = i)$, then we arrive at the following differential equations, which was aforementioned at the end of Section 2.

Lemma 3.2 For $x > 0$ and $i, j \in S$, we have

$$v(i)\frac{d}{dx}P_i(M(\tau_x^-) = j) = -\sum_{k\in S}\Big(C_{ik}P_k(M(\tau_x^-) = j) + D_{ik}P_k(M(\tau^-_{x+A_0(i,k)}) = j)\Big). \tag{3.7}$$

We are now ready to compute $R(x)$. For convenience, let $S^+ = \{i \in S; v(i) > 0\}$ and $S^- = \{i \in S; v(i) < 0\}$. We use superscript $^-$ and $^+$ to indicate the sizes of vectors and matrices. For instance, C^{+-} is the $S^+ \times S^-$ matrix whose ij entry is C_{ij} for $i \in S^+$ and $j \in S^-$. Note that the ij-entry of $R(x)$ is written as $R_{ij}(x) = P_i\left(M(\tau_x^-) = j\right)$. Since the background state must be in S^- at time τ_x^-, we have

$$R^{++}(x) = 0^{++}, \qquad R^{-+}(x) = 0^{-+}.$$

Hence we only need to consider $R^{+-}(x)$ and $R^{--}(x)$. Obviously, $R^{--}(0) = I^{--}$, where I is the identity matrix.

Theorem 3.1 There exist minimal subrate matrix Q and substochastic matrix R^{+-} satisfying

$$-\begin{pmatrix} I^{--} \\ R^{+-} \end{pmatrix} Q = \Delta_v^{-1}\left(C\begin{pmatrix} I^{--} \\ R^{+-} \end{pmatrix} + \int_0^\infty D(du)\begin{pmatrix} I^{--} \\ R^{+-} \end{pmatrix} e^{uQ}\right), \tag{3.8}$$

where Δ_v is the $S \times S$ diagonal matrix whose i-th diagonal entry is $v(i)$, and we have

$$\begin{pmatrix} R^{--}(x) \\ R^{+-}(x) \end{pmatrix} = \begin{pmatrix} I^{--} \\ R^{+-} \end{pmatrix} e^{xQ}, \qquad x \geq 0, \tag{3.9}$$

PROOF. We first observe that, if $M(0) \in S^+$, then the additive process $Y(t)$ first returns to level 0 with a background state in S^- before it hits level $-x$, since there are no downward jumps. This implies

$$R^{+-}(x) = R^{+-}(0)R^{--}(x), \tag{3.10}$$

Since $Y(t)$ must firstly attain $-u$ before it attains $-(u+x)$, we have, using $D(x)$ of (2.1),

$$\begin{aligned} D_{ik}P_k(M(\tau^-_{x+A_0(i,k)}) = j) &= \int_0^\infty D_{ik}(du)R_{kj}(u+x), \\ &= \int_0^\infty D_{ik}(du)\sum_{\ell\in S^-} R_{k\ell}(u)R_{\ell j}(x) \end{aligned} \tag{3.11}$$

Thus, from Lemma 3.2 for $i, j \in S^-$, we have

$$\begin{aligned} -\Delta_v^{--}\frac{d}{dx}R^{--}(x) = \Big(&C^{--} + C^{-+}R^{+-}(0) \\ &+ \int_0^\infty D^{--}(du)R^{--}(u) + \int_0^\infty D^{-+}(du)R^{+-}R^{--}(u)\Big)R^{--}(x), \end{aligned} \tag{3.12}$$

where the derivative is taken in entry-wise for a matrix. Hence, if we define Q as

$$Q = C^{--} + C^{-+}R^{+-}(0) + \int_0^\infty D^{--}(du)R^{--}(u) + \int_0^\infty D^{-+}(du)R^{+-}R^{--}(u), \tag{3.13}$$

then we get

$$\frac{d}{dx}R^{--}(x) = QR^{--}(x).$$

Since $R^{--}(0) = I^{--}$, this differential equation has the solution

$$R^{--}(x) = e^{xQ}. \tag{3.14}$$

Similar to (3.12), we have, from Lemma 3.2,

$$\begin{aligned} -\Delta_v^{++}\frac{d}{dx}R^{+-}(x) &= \bigg(C^{+-} + C^{++}R^{+-}(0) \\ &\quad + \int_0^\infty D^{+-}(du)R^{--}(u) + \int_0^\infty D^{++}(du)R^{+-}R^{--}(u)\bigg)R^{--}(x). \end{aligned} \tag{3.15}$$

Substituting (3.10) and (3.14) into (3.12) and (3.15), we get (3.8) with $R^{+-} = R^{+-}(0)$. From (3.13), Q must be subrate. It remains to prove that (3.8) indeed has minimal solutions Q and R^{+-} which agree with Q of (3.13) and $R^{+-}(0)$, respectively. To this end, we inductively define sequences Q_n and R_n^{+-} as

$$\begin{aligned} Q_{n+1} &= (-\Delta_v^{--})^{-1}\bigg(C^{--} + C^{-+}R_n^{+-} \\ &\quad + \int_0^\infty D^{--}(du)e^{uQ_n} + \int_0^\infty D^{-+}(du)R_n^{+-}e^{uQ_n}\bigg), \end{aligned} \tag{3.16}$$

$$\begin{aligned} R_{n+1}^{+-} &= (\Delta_v^{++})^{-1}\bigg(C^{+-} + (\eta I^{++} + C^{++})R_n^{+-} + \int_0^\infty D^{+-}(du)e^{uQ_n} \\ &\quad + \int_0^\infty D^{++}(du)R_n^{+-}e^{uQ_n}\bigg)\Big(\eta I^{--} - Q_n\Big)^{-1}, \end{aligned} \tag{3.17}$$

where $Q_0 = C^{--}$ and $R_0^{+-} = 0^{+-}$, and $\eta \geq \max_{i\in S^+}|C_{ii}|$. Then, it can be proved in the exactly same way as in [19] that the iterations converge to right matrices. □

4. Markov modulated fluid queue with downward jumps

It is well known that the matrix exponential forms in Theorem 3.1 are used to get the stationary distributions of the buffer content process in the conventional Markov modulated fluid queue and the workload process in the $MAP/G/1$ queue. Those applications use Theorem 3.1 for the cases that $D = 0$ and $v(i) \equiv -1$, respectively. Recently, Takada [19] applies it to the Markov modulated fluid queues with upward jumps. In this case, the net fluid flow is described by $Y(t)$, provided $E(Y(1)) < 0$, i.e., the stability condition.

In this section, we apply Theorem 3.1 to a Markov modulated queue with downward jumps. The downward jumps reduce the buffer content by their sizes if there is a sufficient fluid in the buffer, or empty the buffer otherwise. This model may be considered to modify the conventional fluid queue by disasters or negative arrivals. This buffer content process also includes the attained waiting time process in the $GI/PH/1$ type queue (see [18]) as a special case, i.e. $v(i) \equiv 1$.

Let us describe the Markov modulated queue with downward jumps. We use the same background process $M(t)$. However, the additive process $Y(t)$ is changed according to the downward jumps in the following way.

$$Y(t) = \begin{cases} \int_{0-}^{t} v(M(u))du - \int_{0-}^{t-} A_u(M(u-), M(u))N(du), & t \geq 0 \\ -\int_{t-}^{0-} v(M(u))du + \int_{t-}^{0-} A_u(M(u-), M(u))N(du), & t < 0. \end{cases}$$

Let $X(t)$ be the buffer content at time t generated by the process $\{Y(u)\}$. Then, if $X(0) = 0$, then it is well known that $X(t)$ is given by

$$X(t) = \sup_{0 \leq u \leq t} (Y(t) - Y(u)), \qquad t > 0.$$

As we have noted, we can assume without loss of generality that $\{M(t)\}$ is stationary, so $\{Y(t)\}$ has stationary increments. Under these assumptions, we have

$$\begin{aligned} (X(t), M(t)) &\sim \left(\sup_{0 \leq u \leq t} (Y(0) - Y(u-t)), M(0) \right) \\ &= \left(\sup_{0 \leq u \leq t} (-Y(-u)), M(0) \right), \end{aligned} \tag{4.1}$$

where $\sim$ stands for the equality in distribution. Let (X, M) be a random vector subject to the limiting distribution of $(X(t), M(t))$ as t goes infinity, which always exists in the sense that $X = \infty$ is allowed. Then, letting t tend to infinity in (4.1), we have

$$P(X > x, M = i) = P\left(\sup_{u \geq 0} (-Y(-u)) > x, M(0) = i \right), \qquad x \geq 0,\ i \in S. \tag{4.2}$$

Define the time reversed processes $\hat{Y}(t)$ and $\hat{M}(t)$ by

$$\hat{Y}(t) = Y(-t), \qquad \hat{M}(t) = M(-t), \qquad t \in \mathbb{R}.$$

Let $\hat{v}(i) = -v(i)$ for $i \in S$, and denote by row vector $\boldsymbol{\pi} \equiv \{\pi(i) : i \in S\}$ the stationary distribution of $M(t)$, which is assumed to exist (see (i) of Section 2). That is, $\boldsymbol{\pi}(C+D) = \mathbf{0}$, where $\mathbf{0}$ is the null vector. Let $\Delta_{\boldsymbol{\pi}}$ be the diagonal matrix whose i-th diagonal entry is $\pi(i)$. Then, it is easy to see that $\hat{Y}(t)$ and $\hat{M}(t)$ have the following properties.

(a) $\hat{M}(t)$ is a Markov chain with transition rate $\hat{C} + \hat{D}$ such that

$$\hat{C} = \Delta_{\boldsymbol{\pi}}^{-1} C' \Delta_{\boldsymbol{\pi}}, \qquad \hat{D} = \Delta_{\boldsymbol{\pi}}^{-1} D' \Delta_{\boldsymbol{\pi}},$$

where H' denotes the transpose of a matrix H.

(b) Let $\hat{F}(x) = [F(x)]'$ and $\hat{D}(x) = \Delta_{\boldsymbol{\pi}}^{-1} [D(x)]' \Delta_{\boldsymbol{\pi}}$. Then, $\hat{Y}(t)$ is given by

$$\hat{Y}(t) = \int_{0+}^{t+} \hat{v}(\hat{M}(u))du + \int_{0+}^{t+} \hat{A}_u(\hat{M}(u-), \hat{M}(u))\hat{N}(du), \qquad t \geq 0,$$

where $\hat{A}_u(i, j)$'s are independent for different u's and subject to distribution $\hat{F}_{ij}$, and $\hat{N}$ is a point process generated by all $\hat{D}$-transitions.

(c) $\hat{Y}(t)$ has stationary increments.

Thus $\hat{Y}(t)$ has upward jumps, so it is the same type of the additive process $Y(t)$ studied in Sections 2 and 3.

We now rewrite (4.2) using $\hat{Y}(t)$ and $\hat{\tau}_x^- \equiv \inf\{t > 0; x + \hat{Y}(t) < 0\}$.

$$\begin{aligned} P(X > x, M = i) &= \pi(i) P\left(x + \inf_{u \geq 0} \hat{Y}(u) < 0 \middle| \hat{M}(0) = i \right) \\ &= \pi(i) \sum_{j \in \hat{S}^-} P\left(\hat{M}(\hat{\tau}_x^-) = j | \hat{M}(0) = i \right), \quad x \geq 0,\ i \in S. \end{aligned} \tag{4.3}$$

Thus, by Lemma 2.1, $X(t)$ has the stationary distribution if and only if $E(\hat{Y}(1)) > 0$, i.e., $E(Y(1)) < 0$. We assume this stability condition below.

The relation (4.3) is well known for the conventional fluid queues (see, e.g., Asmussen [5] and Rogers [16]), and related results can be found in [6]. By (4.3), the problem to get the stationary distribution of $(X(t), M(t))$ is reduced to get the hitting probabilities. Hence, we can apply Theorem 3.1 with respect to $\hat{M}(t)$ and $\hat{Y}(t)$, using the notation: $\hat{S}^+ = S^-$ and $\hat{S}^- = S^+$.

We convert the notation for $\hat{M}(t)$ and $\hat{Y}(t)$ to the one for the forward processes $M(t)$ and $Y(t)$. Since $+$ and $-$ are exchanged in the reversed time process, we have

$$\begin{aligned} \hat{C}^{--} &= (\Delta_{\boldsymbol{\pi}}^{++})^{-1}(C^{++})'\Delta_{\boldsymbol{\pi}}^{++}, \qquad & \hat{C}^{-+} &= (\Delta_{\boldsymbol{\pi}}^{++})^{-1}(C^{-+})'\Delta_{\boldsymbol{\pi}}^{--}, \\ \hat{C}^{+-} &= (\Delta_{\boldsymbol{\pi}}^{--})^{-1}(C^{+-})'\Delta_{\boldsymbol{\pi}}^{++}, \qquad & \hat{C}^{++} &= (\Delta_{\boldsymbol{\pi}}^{--})^{-1}(C^{--})'\Delta_{\boldsymbol{\pi}}^{--}. \end{aligned}$$

Similar expressions are also obtained for $\hat{D}(x)$ and Δ_v. Let

$$K = (\Delta_{\boldsymbol{\pi}}^{++})^{-1}(\hat{Q})'\Delta_{\boldsymbol{\pi}}^{++}, \qquad L^{+-} = (\Delta_{\boldsymbol{\pi}}^{++})^{-1}(\hat{R}^{+-})'\Delta_{\boldsymbol{\pi}}^{--},$$

where $\hat{Q}$ and $\hat{R}^{+-}$ correspond with Q and R^{+-} in (3.8) concerning $\hat{Y}(t)$ process. Then,

$$\hat{Q} = (\Delta_{\boldsymbol{\pi}}^{++})^{-1}(K)'\Delta_{\boldsymbol{\pi}}^{++}, \qquad \hat{R}^{+-} = (\Delta_{\boldsymbol{\pi}}^{--})^{-1}(L^{+-})'\Delta_{\boldsymbol{\pi}}^{++}.$$

Hence, (3.8) concerning $\hat{Y}(t)$ can be written as

$$K(L^{+-}, I^{++}) = \left((L^{+-}, I^{++})C + \int_0^\infty e^{xK}(L^{+-}, I^{++})D^{-+}(dx) \right)\Delta_v^{-1}, \tag{4.4}$$

where we have used the identity: $(\Delta_{\boldsymbol{\pi}}^{++})^{-1}e^{x(\hat{Q})'}\Delta_{\boldsymbol{\pi}}^{++} = e^{xK}$. From (3.9) and (4.3), we have

$$P(X > x, M = i) = \begin{cases} \pi(i)\left(e^{x\hat{Q}}\hat{\mathbf{1}}^-\right)_i, & i \in \hat{S}^-, \\ \pi(i)\left(\hat{R}^{+-}e^{x\hat{Q}}\hat{\mathbf{1}}^-\right)_i, & i \in \hat{S}^+, \end{cases}$$

where $\hat{\mathbf{1}}^- = \mathbf{1}^+$ is the S^+-column vector all of whose entries are units. Hence, using the relation that

$$e^{x\hat{Q}} = (\Delta_{\boldsymbol{\pi}}^{++})^{-1}\left(e^{xK}\right)'\Delta_{\boldsymbol{\pi}}^{++},$$

Theorem 3.1 leads to the following results.

Theorem 4.1 The stationary distribution of $X(t)$ exists if and only if $E(\hat{Y}(1)) > 0$, i.e., $E(Y(1)) < 0$. If this condition holds, we have, for $x \geq 0$,

$$P(X > x, M = i) = \left[\boldsymbol{\pi}^{+} e^{xK} \left(L^{+-}, I^{++}\right)\right]_i, \qquad i \in S. \tag{4.5}$$

where K and L^{+-} are minimal solutions of (4.4), and obtained as the limits of K_n and L_n^{-+}, respectively, that are inductively defined as

$$\begin{aligned} K_{n+1} &= \left(C^{++} + L_n^{+-}C^{-+} + \int_0^\infty e^{xK_n} D^{++}(dx) + \int_0^\infty e^{xK_n} L_n^{+-} D^{-+}(dx)\right)(\Delta_v^{++})^{-1}, && (4.6)\\ L_{n+1}^{+-} &= \left(\eta I^{++} - K_n\right)^{-1}\Big(L_n^{+-}(\eta I^{--} + C^{--})(-\Delta_v^{--})^{-1} + C^{+-}(-\Delta_v^{--})^{-1} \\ &\quad + \int_0^\infty e^{xK_n} D^{+-}(dx)(-\Delta_v^{--})^{-1} + \int_0^\infty e^{xK_n} L_n^{+-} D^{--}(dx)(-\Delta_v^{--})^{-1}\Big), && (4.7) \end{aligned}$$

where $\eta \geq \max_{i \in S^-} |C_{ii}|/\pi(i)$, $K_0 = C^{++}$ and $L_0^{+-} = 0^{+-}$.

Remark 4.1 (4.5) generalizes the matrix-exponential form in the literature in two ways. If there are no jumps, (4.5) is identical with (3,4) of [16]. If $v(i) \equiv 1$, i.e., $S = S^+$, then (4.5) agrees with Lemma 2.4 of Sengupta [18] (see [6] for recent study of this model), provided

$$F_{ij}(x) + B_{ij}(x) = -D_{ij}, \tag{4.8}$$

using the notation $B_{ij}(x)$ of [18]. In [18], the jump down behavior to the empty state is slightly relaxed, so (4.8) is weakened. Let us check this identity under (4.8). Let $\overline{\boldsymbol{G}}(x)$ be the row vector whose i-th entry is $P(X > x, M = i)$, and let

$$\boldsymbol{g}(x) = -\frac{d}{dx}\overline{\boldsymbol{G}}(x) = -\boldsymbol{\pi}^{+} K e^{xK}.$$

Multiplying (4.4) with $\boldsymbol{\pi}^{+}$ from the left and applying integration by part, we have

$$\begin{aligned} \boldsymbol{g}(0) &= \boldsymbol{\pi}^{+}(C^{++} + D^{++}) + \int_0^\infty \boldsymbol{\pi}^{+} e^{xK} D^{++}(x) dx \\ &= \int_0^\infty \boldsymbol{g}(x) D^{++}(x) dx, \end{aligned} \tag{4.9}$$

since $\boldsymbol{\pi}^{+}(C^{++} + D^{++}) = \boldsymbol{\pi}(C + D) = \mathbf{0}$. (4.9) is the equation in (d) of Lemma 2.4 of [18].

Since $C + D$ is irreducible, (4.4) implies that K is also irreducible. Hence, (4.5) concludes the following result.

Corollary 4.1 Under the assumption of Theorem 4.1, the stationary tail probability of the buffer content, i.e., $P(X > x)$, is exponential if and only if S^+ is a singleton, i.e., there is a single background state such that $v(i) > 0$.

Discrete-time analogs of Corollary 4.1 but only for its sufficiency can be found in queues with negative customers and batch departures (see, e.g., [7]). Let $-\alpha$ be the Perron-Frobenius eigen value of K, i.e., the eigen value such that its real part is maximum, and let $\boldsymbol{\ell}^+$ and $\boldsymbol{\nu}^+$ be the corresponding right and left eigen vectors, respectively. These vectors are normalized in such a way that $\boldsymbol{\nu}^+\boldsymbol{\ell}^+ = 1$. It is well known that $\alpha > 0$, $\boldsymbol{\ell}^+$ and $\boldsymbol{\nu}^+$ are positive vectors, and

$$e^{xK} = e^{-\alpha x}(1 + o(1))\boldsymbol{\ell}^+\boldsymbol{\nu}^+, \qquad \text{as } x \to \infty.$$

See Theorem 2.7 of [17]. Hence, we get

Corollary 4.2 Under the assumption of Theorem 4.1, we have

$$\lim_{x\to\infty} P\left(X > x, M = i\right) e^{\alpha x} = \left[\boldsymbol{\pi}^+\boldsymbol{\ell}^+\boldsymbol{\nu}^+\left(L^{+-}, I^{++}\right)\right]_i, \qquad i \in S. \tag{4.10}$$

When there are no jumps, Corollary 4.2 is obtained in Corollary 4.9 of Asmussen [4].

5. More general models

The present approach in deriving the hitting probabilities is applicable for more general models. We here discuss four such models.

5.1 Zero flow rate

Consider the case that the flow rate $v(i)$ may be zero in the model of Section 2, i.e., assumption (ii) of Section 2 is removed. Asmussen [5] considered such a case for the conventional Markov modulated fluid queue. He showed that this case can be reduced to the non-zero rate case, if the transition rate matrix of $M(t)$ is appropriately modified and if R_{ij} is suitably defined for $i \in S^0, j \in S$, where $S^0 = \{i \in S; v(i) = 0\}$. However, such a modification becomes very complicated in the present fluid model. We here present a more direct way to accommodate this case.

Note that (3.7) is valid for the case that $v(i) = 0$. Hence, writing the corresponding equations to (3.12) and (3.15), we have similar results to Theorem 3.1. In what follows, C^{0-} and so on are submatrices of C whose entries are composed of $S^0 \times S^-$ indexes and so on.

Theorem 5.1 There exists a subrate matrix Q and substochastic matrices R^{0-} and R^{+-} such that, for $x > 0$,

$$(R^{--}(x), R^{0-}(x), R^{+-}(x))' = (I^{--}, R^{0-}, R^{+-})'e^{xQ}, \tag{5.1}$$

where Q, R^{0-} and R^{+-} are minimal solutions of the following equations.

$$Q = (-\Delta_v^{--})^{-1}\Big(C^{--} + C^{-0}R^{0-} + C^{-+}R^{+-} + \int_0^\infty D^{--}(du)e^{uQ}$$

$$
+\int_0^\infty D^{-0}(du)R^{0-}e^{uQ} + \int_0^\infty D^{-+}(du)R^{+-}e^{uQ}\Big), \quad (5.2)
$$

$$
\begin{aligned}
-C^{00}R^{0-} &= C^{0-} + C^{0+}R^{+-} + \int_0^\infty D^{0-}(du)e^{uQ} \\
&\quad + \int_0^\infty D^{00}(du)R^{0-}e^{uQ} + \int_0^\infty D^{0+}(du)R^{+-}e^{uQ}, \qquad (5.3)\\
-R^{+-}Q &= (\Delta_v^{++})^{-1}\Big(C^{+-} + C^{+0}R^{0+} + C^{++}R^{+-} + \int_0^\infty D^{+-}(du)e^{uQ} \\
&\quad + \int_0^\infty D^{+0}(du)R^{0-}e^{uQ} + \int_0^\infty D^{++}(du)R^{+-}e^{uQ}\Big). \qquad (5.4)
\end{aligned}
$$

Since C^{00} is invertible and $(-C^{00})^{-1}$ is nonnegative, we can apply a similar induction procedure to (3.16). If there are no jumps, it is not hard to see that Theorem 5.1 is identical with Corollary 3.1 of [5].

5.2 Markov modulated Levy upward jumps

In the Markov modulated fluid flow in Sections 2 and 3, upward jumps occur at the state transition instants of the background Markov chain. We can consider a more general situation for those jumps by allowing them at any time. For instance, for each $j \in S$, suppose that upward jumps occur subject to a pure jump Levy process $L_j(t)$ of bounded variations during the background state being j. For each j, this Levy process is assumed to be independent of everything else. Let μ_j be a canonical measure of $L_j(t)$ such that

$$
\log E(e^{isL_j(t)}) = t\int_{0+}^\infty \frac{e^{isx}-1}{x^2}\mu_j(dx), \qquad t \geq 0, s \in I\!R,
$$

where i is the imaginary number here. See [15] for details of the pure jump Levy process. Let $\nu_j(dx) = \mu_j(dx)/x^2$. This ν_j may not be a finite measure, but the bounded variation assumption implies

$$
\int_{0+}^\infty x\nu_j(dx) < \infty.
$$

Thus, we have additional jumps by the Levy process, modulated by the background Markov chain. Then, using Lemma 2.3 of [14], Lemma 3.2 can be modified to

Lemma 5.1 For $x > 0$ and $i, j \in S$, we have

$$
\begin{aligned}
v(i)\frac{d}{dx}P_i(M(\tau_x^-) = j) &= -\sum_{k\in S}\Big(C_{ik}P_k(M(\tau_x^-) = j) + D_{ik}P_k(M(\tau_{x+A_0(i,k)}^-) = j)\Big) \\
&\quad - \int_0^\infty \Big(P_i(M(\tau_{x+u}^-) = j) - P_i(M(\tau_x^-) = j)\Big)\nu_i(du). \qquad (5.5)
\end{aligned}
$$

Thus, we can get a similar result to Theorem 3.1. We only need to add the corresponding terms into the right-hand sides of (3.8). For instance, the term for (3.8) is

$$
-\Delta_v^{-1}\int_0^\infty \Delta_\nu(du)\begin{pmatrix} I^{--} \\ R^{+-} \end{pmatrix}\left(e^{uQ} - I^{--}\right),
$$

where $\Delta_\nu(du)$ is the diagonal matrix whose i-th diagonal entry is $\nu_i(du)$. Note that this integral can not be split into two integrals, since they may be infinity. However, this fact does not cause a big problem to get an analog of (3.16) if we suitable choose η such that R^{++}_{n+1} of (3.17) is nonnegative. This matrix exponential form can be applied to get the stationary distribution of the buffer content in a Markov modulated fluid queue with either upward Levy jumps or downward Levy jumps. The letter is a direct extension of the result in Section 4, while the former is obtained by the $MAP/G/1$ type approach in the literature (see, e.g., [19]).

5.3 Markov modulated Brownian component

Another extension is the case that the accumulated fluid process has a Brownian component $W(t)$ with zero mean and variance $\sigma^2(i)t$ under background state i, where it is assumed that $\sigma^2(i) > 0$ for all $i \in S$. Define additive process $Y^*(t)$ and $\gamma^*(t)$ as

$$Y^*(t) = Y(t) + W(t), \qquad \gamma^*(t) = \gamma(t) + W(t),$$

where $Y(t)$ is defined by (2.2) and (2.3) and $\gamma(t)$ by (3.2). Under each background state, $W(t)$ has increments that are independent of everything else. In this way, the fluid flow has the Brownian component, modulated by the background Markov chain. Let

$$Z(t) = 1(M(0) = i, M(\tau^-_{x-\gamma^*(-t)}) = j).$$

Then, we have an expression similar to (3.3) for this $Z(t)$. Hence, $Z(t)$ is again constant between jumps. Thus, we can apply (3.1) with $E(Z'(t)) = 0$. Since $E(Z(h))$ is computed as

$$\begin{aligned} &P(M(0) = i, M(\tau^-_{x-\gamma^*(-h)}) = j) \\ &\quad = \int_{-\infty}^{+\infty} P(M(0) = i, M(\tau^-_{x+v(i)h+y}) = j) \frac{1}{\sigma(i)\sqrt{2\pi h}} e^{-\frac{y^2}{2\sigma^2(i)h}} dy + o(h) \\ &\quad = \int_{-\infty}^{+\infty} P(M(0) = i, M(\tau^-_{x+v(i)h+\sigma(i)y\sqrt{h}}) = j) \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy + o(h) \end{aligned}$$

and, by Taylor expansion, which is valid by (5.8) below,

$$\begin{aligned} &P(M(0) = i, M(\tau^-_{x+v(i)h+\sigma(i)y\sqrt{h}}) = j) \\ &\quad = P(M(0) = i, M(\tau^-_x) = j) + (v(i)h + \sigma(i)y\sqrt{h}) \frac{d}{dx} P(M(0) = i, M(\tau^-_x) = j) \\ &\qquad + \frac{(v(i)h + \sigma(i)y\sqrt{h})^2}{2} \frac{d^2}{dx^2} P(M(0) = i, M(\tau^-_x) = j) + o(h), \end{aligned}$$

(3.1) at $t = 0$ becomes

$$\begin{aligned} &v(i) \frac{d}{dx} P_i(M(\tau^-_x) = j) + \frac{\sigma^2(i)}{2} \frac{d^2}{dx^2} P_i(M(\tau^-_x) = j) \\ &\quad = -\sum_{k \in S} \left(C_{ik} P_k(M(\tau^-_x) = j) + D_{ik} P_k(M(\tau^-_{x-A_0(i,k)}) = j) \right). \qquad (5.6) \end{aligned}$$

Note that we have not used any differential formula like Ito's formula in the derivation of (5.6). Since the Brownian sample path immediately hits the starting position, similar to (3.12), (5.6) yields

$$\Delta_v \frac{d}{dx} R(x) + \Delta_{\sigma^2/2} \frac{d^2}{dx^2} R(x) + \left(C + \int_0^\infty D(du) R(u) \right) R(x) = 0, \tag{5.7}$$

where $\Delta_{\sigma^2/2}$ is the diagonal matrix whose i-th diagonal entry is $\sigma^2(i)/2$. Since $M(\tau_x^-)$ is a Markov chain on S (see the end of Section 2), the solution of this equation has the following form for some $S \times S$ matrix Q:

$$R(x) = e^{xQ}. \tag{5.8}$$

Then, from (5.7), Q satisfies

$$\Delta_{\sigma^2/2} Q^2 + \Delta_v Q + \left(C + \int_0^\infty D(du) e^{uQ} \right) = 0. \tag{5.9}$$

We conjecture that (5.9) has a unique solution, but have not yet proved it. These results agree with those in [5] and [16] in the case that there are no jumps.

If $S_\sigma \equiv \{i \in S; \sigma^2(i) > 0\} \neq S$, we need to decompose S into S_σ, $S^- \setminus S_\sigma$ and $S^+ \setminus S_\sigma$. The matrix exponential form for $R(x)$ is similarly obtained. Since they are routine, we omit its derivation. When there are no jumps, the corresponding results but from the opposite direction can be found in [5].

5.4 Countably many background states

We finally consider the case that the background state space is countable. This case has been less studied because it is intractable for numerical computations. However, there are some applications for which the countable assumption is convenient (see [1]). So it may be interesting to see what conditions are needed to accommodate the case. We first assume the following conditions for the existence of the stationary version of the additive process $Y(t)$ and for a finite intensity of the point process N, which is necessary to define the Palm distribution P_0.

(i') $C + D$ has a stationary distribution, and its diagonal entries are bounded.

Note that Lemma 2.1 is still valid if $E(Y(1)) \neq 0$. If $E(Y(1)) = 0$, we need to carefully consider its proof. Then, we can see that the following condition is sufficient.

(ii') $\sup_{i \in S^-} |v(i)| < \infty$.

Concerning Theorem 3.1, for using (3.16), we need that $(-\Delta_v^{--})^{-1} C^{--}$ can be uniformized and that C_{ii} for $i \in S^+$ are bounded. The latter is assured by (i'), while the former requires

(ii") $\inf_{i \in S^-} |v(i)| > 0$.

Thus, we may conclude that all the results are still available under the reasonable bounded conditions when the background state space is countable.

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