A MARKOV MODULATED FLUID QUEUE WITH BATCH ARRIVALS AND PREEMPTIONS

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Abstract

We consider a preemptive priority fluid queue with two buffers for continuous fluid and batch fluid inputs. Those two types of fluids are governed by a Markov chain with a finite state space, called a background process, and the continuous fluid is preemptively processed over the batch fluid. The stationary joint distribution of the two buffer contents and the background state is obtained in terms of matrix transforms. Numerical computation algorithms are presented for various moments of the two buffer contents.

1. Introduction

Recently Takada [15] studied a fluid queue with a continuous fluid and batch fluid inputs, where the latter is meant that a batch of a fluid instantaneously arrives. Both fluids are governed by a Markov chain with a finite state space, called a background process. This model extends the conventional fluid queue with Markov modulations as well as the MAP/G/1 queue, where MAP is meant a Markovian Arrival process (see, e.g., [11]). Since the continuous and batch fluids have different characteristics, it may be naturally questioned how the system performance changes when they are processed differently. In this paper, we consider this problem when the continuous fluid is preemptively
processed over the batch fluid. This priority model have two buffers, and our primary concern is with their stationary joint distribution. The model may apply to telecommunication links with two different sources: one is voice and video data and the other is file transfers. Since voice and video data are limited in delay, it may be natural to preemptively process them over file transfers.

In the literature, priority queues have been mainly considered for customer-based conventional queues. Takine [18] is a good source of the recent development of such priority queues. One basic assumption in those literature is that a low priority class has identically distributed service times. This allows to apply the delay cycle approach (e.g., see [7]) or the embedded Markov chain approach at the departure instants (e.g., see [17]). Those approaches are applicable for the nonpreemptive service. In this paper, we are only concerned with the preemptive service, i.e., processing, and take a different approach. This approach allows the amounts of a batch fluid to depend on the background state transitions, which corresponds with service times in the conventional MAP.

For the conventional fluid queues, similar models have been recently studied by Kella [4] and Kroese and Scheinhardt [5]. Those papers study parallel or tandem queues, but they can be considered as priority queues as well, as remarked in Kella [4]. Thus, we extend those models so as to have batch fluids. The analytical approach has been employed in [4] and [5]. For instance, the eigen value analysis is essential in [4]. Since the amount of a batch fluid in our model has an arbitrary distribution depending on background state transitions, such an analytical approach would be much complicated. In this paper, we employ a different approach.

Our approach is based on the matrix exponential form, introduced by Sen-gupta [14] and developed for the conventional fluid queues by Asmussen [3] and Rogers [12] and for the MAP/G/1 queues by Asmussen [2] and Takine and his coworkers [16]. Recently those two classes of models are combined into a Markov modulated fluid queue with batch fluid arrivals in [15], and it is shown that the matrix exponential form is still useful to get the stationary distribution of the buffer content. A significant feature of this approach requires less analytic computations such as finding eigen values, i.e., more algorithmic. Nevertheless it allows to have analytically tractable expressions.

A key observation of our approach is that the matrix exponential form is found for the joint LST of the total amount of batch fluids arriving during each
preempted period and the background states before and after the period. This enables us to consider the batch fluid, i.e., low priority fluid as the MAP/G/1 type queues with varying service rate, and we can get the stationary joint distribution of the two buffer contents and the background state in terms of transforms. Of course, the problem is not yet completely solved since those transforms includes the distributions that are only obtained algorithmically as the Laplace Stieltjes transform, LST in short. However, cross moments of the two buffer contents, e.g., their covariance, can be iteratively computed. We exemplify it by some numerical computations.

This paper is organized by six sections. In Section 2, we introduce our fluid model, and derive some basic results. In Section 3, the key observation is presented. In Section 4, algorithms are given for computing various moments, and they are exemplified in Section 5. Finally, we give some remarks on extending the present approach in Section 6.

2. Background Markov chain and two buffer processes

We consider a fluid queue with two different inputs, a continuous fluid and a discontinuous fluid that instantaneously arrives at random times. We refer to them as continuous fluid and batch fluid, respectively. The continuous fluid has preemptive priority over the batch fluid, and they are separately buffered if the input rates exceed an available processing speed. Those buffers are referred to as continuous fluid and batch fluid buffers, respectively. We are interested in the stationary joint distribution of those buffer contents.

We assume that the two inputs and processing are governed by a finite state Markov chain and distributions for batch sizes. The state of this Markov chain is called a background state. Namely, let \( M(t) \) be a Markov chain with finite state space \( S \), and it has a transition rate matrix of the form \( C + D \), where \( D \) is a nonnegative matrix. It is assumed that \( C + D \) is a stochastic rate matrix, i.e., its row sums equal zero. For each \( i \in S \), a number \( v(i) \) is associated, which represents the net flow rate. That is, if the continuous fluid buffer is not empty, its content changes with rate \( v(i) \) under background state \( i \). Otherwise, the batch fluid buffer is decreased with rate \( |v(i)| \) if \( v(i) < 0 \). The batch fluid buffer is increased by an amount subject to distribution \( F_{ij} \) when the background state changes from \( i \) to \( j \), which occurs with rate \( D_{ij} \).
where $D_{ij}$ is $ij$-entry of $D$. It is assumed that the batch amount is independent of everything else. If both of the buffers are empty and if $v(i) < 0$, they remain to be empty. Throughout the paper, we assume that $v(i) \neq 0$ for all $i \in S$. This assumption can be removed, but computations become complicated (see [10] for such computations).

Note that we did not specify an input rate of the continuous fluid and the potential release rates for both fluids, where potential rate is meant the maximum rate that can process the fluids, e.g., the release rate when either one of the buffers is not empty. However, if those rates are given, it is easy to implement them in our model. For example, let be $a(i)$ and $b(i)$ be the input and potential release rates, respectively. We then define $v(i) = a(i) - b(i)$.

Let $X_1(t)$ and $X_2(t)$ be the contents of the continuous and batch fluid buffers, respectively, at time $t$. We assume that all the stochastic processes are right continuous with left-hand limits. From our assumptions, the right-hand derivatives of $X_1(t)$ and $X_2(t)$ are

$$
\frac{d^+}{dt}X_1(t) = v(M(t))1(X_1(t) > 0),
$$

$$
\frac{d^+}{dt}X_2(t) = v(M(t))1(X_1(t) = 0, X_2(t) > 0).
$$

Let $Y_1(t)$ be the accumulated net flow of the continuous fluid in the time interval $(0, t]$, and $Y_2(t)$ be the accumulated input of the batch fluid in the same interval. Namely,

$$
Y_1(t) = \int_0^t v(M(u))du,
$$

$$
Y_2(t) = \int_0^t B_u(M(u-), M(u))N_2(du),
$$

where $B_u(i, j)$ is the amount of the batch fluids when background process $M(t)$ jumps from $i$ to $j$ at time $u$, and $N_2$ is a counting process of the batch fluid arrival instants. By our assumption, $B_u(i, j)$’s are independent and subject to distribution $F_{ij}$. We also define the total buffer content and the total net input by

$$
X(t) = X_1(t) + X_2(t),
$$

$$
Y(t) = Y_1(t) + Y_2(t).
$$

Since $X_1(t)$ and $X(t)$ are reflected processes with the boundary at origin,
we have
\[ X_1(t) = X_1(0) + Y_1(t) - \inf_{0 \leq u \leq t} \min(0, X_1(0) + Y_1(u)), \]
\[ X(t) = X(0) + Y(t) - \inf_{0 \leq u \leq t} \min(0, X(0) + Y(u)). \]
See, e.g., [4] for this representation. Hence,
\[ X_2(t) = X_2(0) + Y_2(t) - \inf_{0 \leq u \leq t} \min(0, X_2(0) + Y_2(u)) \]
\[ + \inf_{0 \leq u \leq t} \min(0, X_1(0) + Y_1(u)). \]
Thus, \( X_1(t) \) and \( X_2(t) \) are completely determined by \( Y_1(t) \) and \( Y_2(t) \) for each initial values \( X_1(0) \) and \( X_2(0) \). Note that the total buffer content \( X_1(t) + X_2(t) \) jointly with \( M(t) \) is studied in [15] and [10]. Our interest of this paper is in the joint process \((M(t), X_1(t), X_2(t))\).

We can assume without loss of generality that \( C + D \) is irreducible. Otherwise, we choose a irreducible subset of state space \( S \). Since \( S \) is finite, it has a stationary distribution, which is denoted by \( \pi \). Namely,
\[ \pi(C + D) = 0, \quad \pi e = 1. \]
where \( e \) is the vector all of whose entries are unity. Let \( \Delta_v \) be the diagonal matrix whose \( i \)-th diagonal entry is \( v(i) \). Let \( D(x) \) be \( S \times S \) matrix, i.e., a matrix whose entries are indexed by elements of \( S \times S \), whose \( ij \) entry is
\[ [D(x)]_{ij} = D_{ij}F_{ij}(x). \]
Using \( C \) and \( D(x) \), we specify the background process and associated jumps. It may be considered to combine them. To this end, we need dummy states since \( C + D(x) \), which has negative diagonal entries, is not appropriate for describing the jump distributions associated with the diagonal entries. Then, the description of the transition matrix becomes complicated, so combining the matrices is not very convenient. We also define an integration of a function \( \varphi \) with respect to \( D(x) \) by
\[ \left[ \int_0^\infty \varphi(x)D(dx) \right]_{ij} = \int_0^\infty \varphi(x)[D(dx)]_{ij}. \]
Then, it is intuitively clear that the stationary distribution of \((M(t), X_1(t), X_2(t))\) exists if and only if the mean net rate is negative, i.e.,
\[ E(Y(1)) = \pi \Delta_v e + \pi \int_0^\infty xD(dx)e < 0. \quad (2.1) \]
See, e.g., [10] for its verification for the total buffer content process \(X_1(t) + X_2(t)\), which verifies the statement above. We assume this stability condition throughout the paper.

Let \((M, X_1, X_2)\) be a random vector subject to the stationary distribution of \((M(t), X_1(t), X_2(t))\). We denote this distribution by vector \(G(x_1, x_2)\), i.e.,

\[
G(x_1, x_2)_i = P(X_1 \leq x_1, X_2 \leq x_2, M = i),
\]

and denote its Laplace-Stieltjes transform by vector \(g(\theta, \eta)\), where

\[
\hat{g}(\theta, \eta)_i = E(e^{-\theta X_1 - \eta X_2}; M = i).
\]

Let \(H(x) = G(0, x)\),

\[
\hat{h}(\eta) = \int_0^\infty e^{-\eta x} H(dx) \equiv g(\infty, \eta),
\]

\[
\hat{D}(\eta) = \int_0^\infty e^{-\eta x} D(dx),
\]

where integrations on a vector and matrix are defined component-wise. Then, we have the following result.

**Lemma 2.1**

\[
\hat{g}(\theta, \eta) \left( \theta \Delta_v - (C + \hat{D}(\eta)) \right) = \eta G(0, 0) \Delta_v + (\theta - \eta) \hat{h}(\eta) \Delta_v. \tag{2.2}
\]

**Proof.** We apply the rate conservation law (e.g., see [9]) to derive (2.2). To this end, we choose the stationary distribution for the distribution of \((M(0), X_1(0), X_2(0))\). Then, \(\{(M(t), X_1(t), X_2(t))\}\) becomes to be stationary. Define \(Z(t)\) as

\[
Z(t) = e^{-(\theta X_1(t) + \eta X_2(t))} 1(M(t) = i),
\]

for each \(i \in S\) and \(\theta, \eta \geq 0\). Let \(N\) be the point process that counts all jump instants of \(M\). Then, \(Z(t)\) has the right-hand derivative:

\[
Z'(t) = -\theta v(i) 1(X_1(t) > 0) Z(t)
\]

\[
-\eta v(i) 1(X_1(t) = 0, X_2(t) > 0, M(t) = i) e^{-\eta X_2(t)}
\]

\[
= -\theta v(i) (Z(t) - 1(X_1(t) = 0, M(t) = i) e^{-\eta X_2(t)})
\]

\[
-\eta v(i) (1(X_1(t) = 0, M(t) = i) e^{-\eta X_2(t)}
\]

\[
-1(X_1(t) = X_2(t) = 0, M(t) = i))
\]

\[
= -\theta v(i) Z(t) + (\theta - \eta) v(i) 1(X_1(t) = 0, M(t) = i) e^{-\eta X_2(t)}
\]

\[
+ \eta v(i) 1(X_1(t) = X_2(t) = 0, M(t) = i)
\]
for each \( t \), and its discontinuous points are all counted by the point process \( N \). Let \( E_N \) be the Palm expectation with respect to point process \( N \) and 
\[
\lambda = E(N(1)),
\]
then
\[
\begin{align*}
\lambda E_N(Z(0-)) &= -(C + \hat{D}(\eta))_i [\hat{g}(\theta, \eta)]_i, \\
\lambda E_N(Z(0+)) &= \sum_{j \in S} [C + \hat{D}(\eta)]_{ij} [\hat{g}(\theta, \eta)]_i.
\end{align*}
\]
Hence, applying the rate conservation law:
\[
E(Z'(0)) = \lambda E_N(Z(0-)) - E_N(Z(0+)),
\]
we have (2.2).

There are two unknown terms in (2.2), \( G(0, 0) \) and \( \hat{h}(\eta) \), where
\[
G(0, 0) = (0^+, G^- (0, 0)).
\]
The vector \( G^- (0, 0) \) can be identified using the recent result of [15], since it is the empty probability for the total buffer content \( X \). In fact, let \( \eta = \theta \) in (2.2), then
\[
\hat{g}(\theta, \theta) \left( \theta \Delta_v - (C + \hat{D}(\theta)) \right) = \theta G(0, 0) \Delta_v. \tag{2.3}
\]
This is the stationary equation for the total buffer content, whose LST is \( \hat{g}(\theta, \theta) \). A procedure to determine \( G^- (0, 0) \) is summarized in Appendix A. Thus, it remains to identify \( \hat{h}(\eta) \). This will be considered in the next section.

3. Accumulated batch fluid during a busy period

To compute \( \hat{h}(\eta) \), we first consider the batch fluid accumulated process \( Y_2(t) \) during the period when the continuous fluid buffer is not empty. Let
\[
\tau_x = \inf\{ t > 0 \mid x + Y_1(t) \leq 0 \}, \quad x \geq 0.
\]
Define \( S \times S \)-matrix \( R(y|x) \) and its LST \( \hat{R}(\eta|x) \) as
\[
[R(y|x)]_{ij} = P(Y_2(\tau_x) \leq y; M(\tau_x) = j| M(0) = i), \quad x, y \geq 0,
\]
\[
[\hat{R}(\eta|x)]_{ij} = E(e^{-\eta Y_2(\tau_x)}; M(\tau_x) = j| M(0) = i), \quad x, \eta \geq 0.
\]
Observe the sample path of \( Y_2(t) \) during the time interval \([0, \Delta t] \) for a small \( \Delta t > 0 \). We then can see that \( Y_2(\tau_x) \) approximately equals \( Y_2(\tau_x + Y_1(\Delta t)) \) plus
a batch amount arriving in the time interval \((0, \Delta t]\) for sufficiently small \(\Delta t\). Hence,

\[
[\hat{R}(\eta|x)]_{ij} = (1 + C_{ii} \Delta t)[\hat{R}(\eta|x + v(i) \Delta t)]_{ij} \\
+ \sum_{k \neq i, k \in S} C_{ik} \Delta t[\hat{R}(\eta|x + v(i) \Delta t)]_{kj} \\
+ \sum_{k \in S} [\hat{D}(\eta)]_{ik} \Delta t[\hat{R}(\eta|x + v(i) \Delta t)]_{kj} + o(\Delta t).
\]

Subtract \([\hat{R}(\eta|x + v(i) \Delta t)]_{ij}\) from both sides of the above equation and divide by \(\Delta t\). Then, letting \(\Delta t\) go to zero, we have

\[
-\Delta_v \frac{d}{dx} \hat{R}(\eta|x) = (C + \hat{D}(\eta)) \hat{R}(\eta|x).
\]  

(3.1)

This equation can be formally verified using the time-dependent rate conservation law as in [10] or the first step decomposition of \(\hat{R}(\eta|x)\), i.e., conditioning by the background state at the first transition instant of \(M(t)\) (e.g., see the first equalities of (B.4) and (B.5) in Appendix B). We omit such a verification since it is routine. Thus, we get \(\hat{R}(\eta|x)\) as the exponential matrix of \(-\Delta_v^{-1}(C + \hat{D}(\eta))\) multiplying with \(\hat{R}(\eta|0)\) from the left. However, this expression is not so useful, since it includes \(\hat{R}(\eta|0)\). To get the latter, one usually needs eigen values and eigen vectors of \(-\Delta_v^{-1}(C + \hat{D}(\eta))\). This problem is more difficult than in the case of the conventional fluid queue, since the matrix is a function of \(\eta\). To prevent the difficulty, we take an algorithmic approach based on Wiener-Hopf decomposition, which has been studied for the conventional fluid models (see [3, 12]). We first partition a \(S \times S\) matrix into the four \(S^+ \times S^+\) matrices, where

\[
S^+ = \{i \in S; v(i) > 0\}, \quad S^- = \{i \in S; v(i) < 0\},
\]

and let

\[
\hat{\Gamma}(\eta) = \Delta_v^{-1}(C + \hat{D}(\eta)), \quad \eta \geq 0.
\]  

(3.2)

Note that \(\hat{\Gamma}(\eta)\) is a subrate matrix, i.e., a square matrix all of whose off diagonal entries are nonnegative and all of its row sums are nonpositive. Then, (3.1) can be written as

\[
\frac{d}{dx} \hat{R}^{++}(\eta|x) = -\hat{\Gamma}^{++}(\eta)\hat{R}^{++}(\eta|x) - \hat{\Gamma}^{+-}(\eta)\hat{R}^{--}(\eta|x),
\]

(3.3)

\[
\frac{d}{dx} \hat{R}^{+-}(\eta|x) = \hat{\Gamma}^{-+}(\eta)\hat{R}^{++}(\eta|x) + \hat{\Gamma}^{-+}(\eta)\hat{R}^{--}(\eta|x),
\]

(3.4)
where $\hat{R}^+(\eta|x)$ and $\hat{R}^{++}(\eta|x)$ are zero matrices. Since $Y_1(t)$ is skip free, we have

$$\hat{R}^-(\eta|x) = \hat{R}^-(\eta|0)\hat{R}^-(\eta|x). \tag{3.5}$$

Substituting this into (3.4) and letting

$$Q^-(\eta) = \hat{\Gamma}^+(\eta)\hat{R}^-(\eta|0) + \hat{\Gamma}^{--}(\eta), \tag{3.6}$$

we get

$$\frac{d}{dx} \hat{R}^-(\eta|x) = Q^-(\eta)\hat{R}^-(\eta|x).$$

This and (3.5) yield

$$\hat{R}^-(\eta|x) = e^{Q^-(\eta)x}, \quad \hat{R}^+(\eta|x) = \hat{R}^+(\eta|0)e^{Q^-(\eta)x}.$$ 

Substituting these expressions into (3.3), we have

$$\hat{R}^+(\eta|0)(-Q^-(-\eta)) = \hat{\Gamma}^{++}(\eta)\hat{R}^+(\eta|0) + \hat{\Gamma}^{+-}(\eta). \tag{3.7}$$

**Lemma 3.1** For each $\eta \geq 0$, $Q^-(\eta)$ and $\hat{R}^+(\eta|0)$ are unique minimal solutions of (3.6) and (3.7) such that $Q^-(\eta)$ and $\hat{R}^+(\eta|0)$ are subrate and substochastic, respectively. In particular, $Q^-(0)$ is rate matrix, and $\hat{R}^+(0|0)$ is stochastic.

This lemma can be proved in a similar way to Theorem 4.1 of [15] (see also [3]). For this paper to be self-contained, we give a proof in Appendix B. This proof also gives a procedure to iteratively compute the minimal solutions $Q^-(\eta)$ and $\hat{R}^+(\eta|0)$ for each fixed $\eta \geq 0$ (see (B.1) and (B.2) in Appendix B). On the other hand, it is not easy to analytically get them as functions of $\eta$ except for $|S^+| = |S^-| = 1$. Namely, we need to solve the matrix equation:

$$(I^{++}, \hat{R}^+(\eta|0))\hat{\Gamma}(\eta) \begin{pmatrix} \hat{R}^+(\eta|0) \\ I^{--} \end{pmatrix} = 0^{+-}, \tag{3.8}$$

which is obtained from (3.6) and (3.7). However, we may not need full information on these functions. In particular, we are interested in cross moments $E(X_1^n X_2^m)$, for which we only need their partial derivatives at $\eta = 0$. The latter quantities can be algorithmically computed, which will be given in Section 4. We now derive the LST of $X_2(t)$ when $X_1(t) = 0$. 

Lemma 3.2 For \( \eta \geq 0 \), \( \hat{h}^- (\eta) \) is determined by,
\[
\hat{h}^- (\eta) \Delta_v^- (\eta I^- + Q^-(\eta)) = \eta H^- (0) \Delta_v^- .
\] (3.9)

Proof. Note that the \( i \)-th component of \( \hat{h}^- (\eta) \) is
\[
E \left( e^{-\eta X_2(t)} ; M(t) = i, X_1(t) = 0 \right),
\]
for the stationary \((X_1(t), X_2(t), M(t))\). Consider the process \((X_2(t), M(t))\) under \( X_1(t) = 0 \). This process is obtained by cutting out the time intervals in which \( X_1(t) > 0 \). The removed intervals accumulate batch fluid, and cause jumps for \( X_2(t) \). Consider such a time interval starting with background state \( i \in S^+ \) and ending with \( j \in S^- \). Then, the total jump sizes are subject to distribution \([R^+(x|0)]_{ij}\) if the interval starts with the transition due to the rate of \( C^- \), while they are subject to \([\int_0^x D^+(dy)R^+(x-y|0)]_{ij}\) if the interval starts with the transition due to the rate of \( D^- \). Hence, under \( X_1(t) = 0 \), \( X_2(t) \) decreases with rate \(|v^-(i)|\) when the background state is \( i \), and has jumps with rate
\[
\left[ D^- (x) + C^- R^+(x|0) + \int_0^x D^+(dy)R^+(x-y|0) \right]_{ij}, \tag{3.10}
\]
when the background state changes from \( i \) to \( j \) and jumps sizes are not greater than \( x \), while those changes without jumps have rate \([C^-]_{ij}\). Thus, the process \((X_2(t), M(t))\) under \( X_1(t) = 0 \) can be considered as a \( MAP/G/1 \) type queue with the jump matrix (3.10). Denote the LST of (3.10) by \( \hat{A}(\eta) \), then
\[
\hat{A}(\eta) = \hat{D}^- (\eta) + (C^- + \hat{D}^+(\eta))\hat{R}^+(\eta|0) \\
= -\hat{\Delta}_v^- \left( \hat{\Gamma}^- (\eta) + \hat{\Gamma}^+(\eta)\hat{R}^+(\eta|0) \right) - C^- \\
= -\hat{\Delta}_v^- Q^- (\eta) - C^-.
\]
A similar computation to Lemma 2.1 with \( \theta = \eta \) yields
\[
\hat{h}^- (\eta|0) \left( \eta \Delta_v^- - (C^- + \hat{A}(\eta)) \right) = \eta \hat{h}^- (\infty|0) \Delta_v^- ,
\]
where \([\hat{h}^- (\eta|0)]_i = E(e^{-\eta X_21(M = i)|X_1 = 0}) \). Hence,
\[
\hat{h}^- (\eta|0) \Delta_v^- (\eta I^- + Q^- (\eta)) = \eta \hat{h}^- (\infty|0) \Delta_v^- .
\]
Multiplying both sides of this equation with \( P(X_1 = 0) \), we get (3.9). \( \square \)

From Lemmas 2.1 and 3.2 and the fact that \( G(0,0) = (0^+, H^- (0)) \), we have the stationary joint distribution of \( X_1 \) and \( X_2 \) in the following form.
Theorem 3.1 The stationary joint distribution of buffer contents $X_1$ and $X_2$ is determined by

$$
\hat{g}(\theta, \eta) = \eta G(0, 0) \Delta_v \left\{ I + (\theta - \eta) (\eta I + Q(\eta))^{-1} \right\} \times (\theta \Delta_v - (C + \hat{D}(\eta)))^{-1}, \tag{3.11}
$$

for $\theta, \eta \geq 0$ except singular points, where

$$
Q(\eta) = \begin{pmatrix}
(1 - \eta) I^{++} & 0^{+-} \\
0^{-+} & Q^{--}(\eta)
\end{pmatrix}.
$$

Of course, the expression (3.11) is symbolic, and we need much effort to compute characteristics such as moments. In particular, $\hat{R}^{+ -}(\eta|0)$ and $Q^{--}(\eta)$ are only obtained as a limit of the iterations. In the next section, we give iteration algorithms to numerically compute various moments of joint buffer contents.

### 4. Cross moments of the buffer contents

In the previous sections, we derived the expression of $\hat{g}(\theta, \eta)$ in matrix forms. Since we can numerically evaluate $\hat{Q}^{--}(\eta)$ and $\hat{R}(\eta|0)$, we may be able to numerically invert the transform equation (3.11) at least for the marginal distributions, using techniques such as the finite Fourier transform. However, such computations would be much complicated. As the first step, we here give a feasible and fully guaranteed procedure to compute cross moments of $X_1$ and $X_2$. That is, an algorithm is not just given but mathematically verified.

In Section 4.1, we give an iteration algorithm to compute any orders of the derivatives of $Q^{--}(\eta)$ and $\hat{R}^{+ -}(\eta|0)$ at $\eta = 0$. These amounts are used to compute any order of the derivatives of $\hat{h}(\eta)$ at $\eta = 0$ in Section 4.2. Then we present an algorithm to compute cross moments of the buffer contents in Section 4.3. In Section 4.4, we give summary of the algorithms.

#### 4.1 Iteration algorithms for $Q^{--}(\eta)$ and $\hat{R}^{+ -}(\eta)$

We consider iteration algorithms to compute the $n$th derivatives of $Q^{--}(\eta)$ and $\hat{R}^{+ -}(\eta|0)$ at $\eta = 0$, which are used in Section 4.2. In what follows, we
use the definition of $\hat{\Gamma}(\eta)$ and $Q^{--}(\eta)$ of (3.2) and (3.6), respectively, and the expression of $\hat{\Gamma}^{--}(\eta|0)$ of (3.7). Define $Q_{(0)}$, $R_{(0)}$ and $\Gamma_{(0)}$ as

$$Q_{(0)} = Q^{--}(0), \quad R_{(0)} = \hat{\Gamma}^{--}(0|0), \quad \Gamma_{(0)} = \Delta_{1/|\nu|}(C + D).$$

Fix an $\xi$ such that $\xi > \max_i |C_{ii}/v(i)|$, which implies that the matrix $\xi I^{++} + \Gamma^{++}_{(0)}$ is nonnegative. We rewrite (3.7) as

$$\hat{\Gamma}^{++}(\eta|0)(\xi I^{--} - Q^{--}(\eta)) = (\xi I^{++} + \hat{\Gamma}^{++}(\eta))\hat{\Gamma}^{--}(\eta|0) + \hat{\Gamma}^{+-}(\eta). \quad (4.1)$$

From (3.6) and (4.1) we have,

$$Q_{(0)} = \Gamma^{++}_{(0)}R_{(0)} + \Gamma^{--}_{(0)}, \quad (4.2)$$
$$R_{(0)} = \left((\xi I^{++} + \Gamma^{++}_{(0)})R_{(0)} + \Gamma^{++}_{(0)}\right)(\xi I^{--} - Q_{(0)})^{-1}, \quad (4.3)$$

where the inverse of $\xi I^{--} - Q_{(0)}$ exists, since $Q_{(0)}$ is a ML-matrix, i.e., the matrix whose off diagonal entries are nonnegative (see [13]).

(Iteration for $n = 0$) Let $R^{(0)}_{(0)} = 0^{+-}$, and define inductively, for each $m \geq 1$,

$$Q^{(m)}_{(0)} = \Gamma^{--}_{(0)}R^{(m-1)}_{(0)} + \Gamma^{--}_{(0)}, \quad (4.4)$$
$$R^{(m)}_{(0)} = \left((\xi I^{++} + \Gamma^{++}_{(0)})R^{(m-1)}_{(0)} + \Gamma^{++}_{(0)}\right)(\xi I^{--} - Q^{(m)}_{(0)})^{-1}. \quad (4.5)$$

We next consider similar iterations for derivatives. To this end, we introduce the following notation. For $n \geq 1$,

$$R_{(n)} = (-1)^n \frac{d^n}{d\eta^n} \hat{\Gamma}^{+-}(\eta) \bigg|_{\eta=0}, \quad Q_{(n)} = (-1)^n \frac{d^n}{d\eta^n} Q^{--}(\eta) \bigg|_{\eta=0},$$
$$\Gamma_{(n)} = (-1)^n \Delta_{1/|\nu|} \frac{d^n}{d\eta^n} \left(C + \hat{D}(\eta)\right) \bigg|_{\eta=0},$$
$$U_{(n)}^{-} = \Gamma_{(n)}^{-} + \sum_{k=0}^{n-1} \binom{n}{k} \Gamma_{(n-k)}^{++} R_{(k)}^{-},$$
$$V_{(n)}^{+} = \Gamma_{(n)}^{+} + \sum_{k=1}^{n-1} \binom{n}{k} \left(\Gamma_{(n-k)}^{++} R_{(k)}^{+} + R_{(k)} R_{(n-k)}^{+}\right).$$

Then, from (3.6) and (4.1), we have, for $n \geq 1$,

$$Q_{(n)} = \Gamma^{++}_{(0)}R_{(n)} + U_{(n)}^{-}, \quad (4.6)$$
$$R_{(n)} = \left((\xi I^{++} + \Gamma^{++}_{(0)})R_{(n)} + R_{(0)} Q_{(n)} + V_{(n)}^{+}\right)(\xi I^{--} - Q_{(0)})^{-1}. \quad (4.7)$$
Note that $\Gamma(n)$ for $n \geq 1$ are nonnegative, so $U_{(n)}^-, V_{(n)}^+, Q_{(n)}$ and $R_{(n)}$ are all nonnegative for $n \geq 1$. We are now ready to present an iteration algorithm. Note that $U_{(n)}^-$ and $V_{(n)}^+$ are determined by $Q_{(k)}$ and $R_{(k)}$ for $k = 0, \ldots, n - 1$ if $\Gamma(k)$ for $k = 0, \ldots, n$ are given. Thus, from (4.6) and (4.7), we arrive at the following iterations.

(Iteration for $n \geq 1$) For each $n \geq 1$, suppose that $Q_{(k)}$ and $R_{(k)}$ for $k = 0, \ldots, n - 1$ are obtained. Let $R_{(n)}^{(0)} = 0^+$, and define inductively, for each $m \geq 1$,

$$Q_{(n)}^{(m)} = \Gamma_{(0)}^- R_{(n)}^{(m-1)} + U_{(n)}^-, \tag{4.8}$$

$$R_{(n)}^{(m)} = ((\xi I_{++,} + \Gamma_{(0)}^+)) (R_{(n)}^{(m-1)} + R_{(0)} Q_{(n)}^{(m)} + V_{(n)}^-) (\xi I_{--}^+ - Q_{(0)}^{-1}) \tag{4.9}$$

Similar to the nonnegativity of $U_{(n)}^-$, $V_{(n)}^+$, $Q_{(n)}$ and $R_{(n)}$ for $n \geq 1$, all the terms in the right-hand sides of (4.8) and (4.9) are nonnegative. So these iterations are tractable. Furthermore, the following facts guarantee their convergence.

**Theorem 4.1** Choose an $\xi > \max_i |C_{ii}/v(i)|$. Then, for each $n \geq 0$, $\{Q_{(n)}^{(m)}\}_{m \geq 1}$ and $\{R_{(n)}^{(m)}\}_{m \geq 0}$ are nondecreasing, and converge to $Q_{(n)}$ and $R_{(n)}$, respectively.

**Proof.** We prove the theorem by induction. We first consider the case that $n = 0$. This case has been already considered in the proof of Lemma 3.1 under the more general setting. That is, let $\eta = 0$ in Appendix B, then $Q_{(0)}^{(m)} = Q_{(0)}^{(m)}(0)$ and $R_{(0)}^{(m)} = R_{(0)}^{(m)}(0|0)$ verify the case for $n = 1$.

We next consider the case that $n \geq 1$. Suppose that the theorem holds true for $Q_{(k)}$ and $R_{(k)}$ for $k = 0, \ldots, n - 1$, and $Q_{(n)}^{(j+1)}$ and $R_{(n)}^{(j)}$ for $j = 0, \ldots, m - 1$. Then, (4.9) implies

$$R_{(n)}^{(m)} - R_{(n)}^{(m-1)} = ((\xi I_{++,} + \Gamma_{(0)}^+)) (R_{(n)}^{(m-1)} - R_{(n)}^{(m-2)})$$

$$+ R_{(0)} (\Gamma_{(0)}^- + \Gamma_{(0)}^+) (R_{(n)}^{(m-1)} - R_{(n)}^{(m-2)}) (\xi I_{--}^+ - Q_{(0)}^{-1}) \geq 0^+. \tag{4.9}$$

Similarly, we can verify $R_{(n)} \geq R_{(n)}^{(m)}$. From these facts and (4.8), we can see that $\{Q_{(n)}^{(m)}\}$ are nondecreasing and bounded by $Q_{(n)}$. Similar to the case that $n = 0$, we can construct lower bounds $\{R_{(n)}^{(m)}\}$ and $\{Q_{(n)}^{(m)}\}$ for $\{R_{(n)}^{(m)}\}$ and $\{Q_{(n)}^{(m)}\}$ by restricting the number of transitions that converge $Q_{(n)}$ and $R_{(n)}$. 
respectively. Note that this proof is easier compared with the case for $n = 0$ since the iteration is not needed for the exponent matrix $Q(0)$ for $n \geq 1$. Thus, the proof is completed.

4.2 Algorithm for derivatives of $\hat{h}(\eta)$ at $\eta = 0$

Denote the $m$-th derivatives of $\hat{D}(\eta)$ and $\hat{h}(\eta)$ by $\hat{D}_{(m)}(\eta)$ and $\hat{h}_{(m)}(\eta)$, respectively. In this section, we give algorithms to compute $\hat{h}_{(m)}(0)$ for $m \geq 0$, which will be used in the next subsection. We modify the algorithm in [16], which is used for the $MAP/G/1$ queue.

Suppose that we have computed $\hat{h}_{(k)}(0)$ for $k = 0, \ldots, m - 1$. In what follows, we give a procedure to compute $\hat{h}_{(m)}(0)$. We first differentiate both sides of (3.9) for $m$ times.

\[
\hat{h}_{(m)}(\eta)\Delta_\nu^-(\eta I - - + Q^- (\eta)) + m\hat{h}_{(m-1)}(\eta)\Delta_\nu^-(I^- + Q^- (\eta))
\]

\[
+1(m \geq 2) \sum_{k=0}^{m-2} \binom{m}{k} \hat{h}_{(k)}(\eta)\Delta_\nu^-(Q^-_{(m-k)}(\eta))
\]

\[
= 1(m = 0)\eta H^- (0)\Delta_\nu^- + 1(m = 1)H^- (0)\Delta_\nu^- .
\]

(4.10)

Let $\kappa^-$ be the stationary row vector of $-\Delta_\nu^- Q(0)$ such that $\kappa^- e^- = 1$. From (3.9) with $\eta = 0$, we can see that

\[
\hat{h}_{(0)}(0) = P(X_1 = 0)\kappa^-.
\]

We shall use (4.10) to inductively determine $\hat{h}_{(m)}(0)$. However, we can not directly do this computation since $Q(0)$ is singular. So, we introduce supplementary row vectors $\psi_{(m)}$, $y_{(m)}$ and real number $\beta_{(m)}$ for $m \geq 0$, defined as

\[
\psi_{(m)} = \hat{h}_{(m)}(0)\Delta_\nu^- Q(0),
\]

\[
y_{(m)} = \psi_{(m)} (e^- \kappa^- + \Delta_\nu^- Q(0))^{-1},
\]

\[
\beta_{(m)} = \hat{h}_{(m)}(0)e^- ;
\]

(4.11)

where $e^- \kappa^- + \Delta_\nu^- Q(0)$ is nonsingular (see Appendix B for its proof). Note that $\beta_{(0)} = P(X_1 = 0)$. From (4.11) and the fact that $\kappa^- \Delta_\nu^- Q(0) = 0$, it follows that

\[
\beta_{(m)} \kappa^- + \psi_{(m)} = \hat{h}_{(m)}(0) (e^- \kappa^- + \Delta_\nu^- Q(0)),
\]

\[
\beta_{(m)} \kappa^- = \beta_{(m)} \kappa^- (e^- \kappa^- + \Delta_\nu^- Q(0)) .
\]
Hence, we have
\[ \hat{h}^{\rightarrow}_{(m)}(0) = \beta_{(m)} \kappa^{\rightarrow} + y_{(m)}. \] (4.12)

Using the definition of \( \psi_{(m)} \) in (4.11), (4.10) with \( \eta = 0 \) yields
\[ \psi_{(m)} = -m \hat{h}^{\rightarrow}_{(m-1)}(0) \Delta_{\nu}^{\rightarrow}\left(I^{\rightarrow} - Q^{(1)}\right) - 1(m \geq 2) \sum_{k=0}^{m-2} (-1)^{m-k} \binom{m}{k} \hat{h}^{\rightarrow}_{(k)}(0) \Delta_{\nu}^{\rightarrow}Q^{(m-k)} + 1(m = 1)H^{-}(0)\Delta_{\nu}^{\rightarrow}. \] (4.13)

Thus, \( \psi_{(m)} \) and \( y_{(m)} \) are obtained through \( \hat{h}^{\rightarrow}_{(k)}(0) \) for \( k = 0, \ldots, m - 1 \). Differentiating (3.9) for \( m + 1 \) times at \( \eta = 0 \) and post-multiplying \( e^{\rightarrow} \) to both sides of the resulted formula, we get
\[
\beta_{(m)} = \frac{1}{(m + 1)\kappa^{\rightarrow}\Delta_{\nu}^{\rightarrow}\left(I^{\rightarrow} - Q^{(1)}\right)e^{\rightarrow}} \times \left(- \sum_{k=0}^{m-1} (-1)^{m+1-k} \binom{m+1}{k} \hat{h}^{\rightarrow}_{(k)}(0) \Delta_{\nu}^{\rightarrow}Q^{(m+1-k)}e^{\rightarrow} - (m + 1)y_{(m)}\Delta_{\nu}^{\rightarrow}\left(I^{\rightarrow} - Q^{(1)}\right)e^{\rightarrow} + 1(m = 0)H^{-}(0)\Delta_{\nu}^{\rightarrow}e^{\rightarrow}\right),
\] (4.14)

for \( m \geq 0 \). Hence, \( \beta_{(m)} \) is determined by \( \hat{h}^{\rightarrow}_{(k)}(0) \) for \( k = 0, \ldots, m - 1 \). Thus, \( \hat{h}^{\rightarrow}_{(m)}(0) \) is obtained by (4.12), which is determined through \( \hat{h}^{\rightarrow}_{(k)}(0) \) for \( k = 0, \ldots, m - 1 \).

### 4.3 Algorithm for cross moments

Let \( \hat{g}_{(n,m)}(\theta, \eta) \) be the \( n, m \)-th partial derivatives of \( \hat{g}(\theta, \eta) \) concerning \( \theta \) and \( \eta \), respectively. Note that the \( n, m \)-th cross moments of \( X_1 \) and \( X_2 \) are given by
\[
E[X_1^n X_2^m] = (-1)^{n+m} \hat{g}_{(n,m)}(0,0)e.
\]
For convenience, we put \( \alpha_{(n,m)} = \hat{g}_{(n,m)}(0,0)e. \)

Partially differentiate both sides of (2.2) for \( n \) and \( m \) times with respect to \( \theta \) and \( \eta \), respectively, then we have
\[
\hat{g}_{(n,m)}(\theta, \eta) \left( \theta \Delta_{\nu} - (C + \hat{D}(\eta)) \right) + 1(n \geq 1)n\hat{g}_{(n-1,m)}(\theta, \eta)\Delta_{\nu}
\]
From (4.17) and (4.18), we see that
\[ e^{\pi} \]  
where (4.16) and the definition of which implies
\[ \hat{\phi} = \hat{\phi} \]
Similar to the computation of \( \hat{h}^- \), we use supplementary vectors to inductively compute \( \hat{g}_{(n,m)}(0,0) \). To this end, define row vectors \( \phi_{(n,m)} \) and \( x_{(n,m)} \) as
\[ \phi_{(n,m)} = -\hat{g}_{(n,m)}(0,0)(C + D), \]
\[ x_{(n,m)} = \phi_{(n,m)}(e \pi - (C + D))^{-1}, \]
where \( e \pi - (C + D) \) is non-singular (see Appendix C). Then, using (4.15), \( \phi_{(n,m)} \) is computed by
\[ \phi_{(n,m)} = 1(m \geq 1) \sum_{k=0}^{m-1} \binom{m}{k} \hat{g}_{(n,k)}(0,0) \hat{D}_{(m-k)}(0) - 1(n \geq 1)n \hat{g}_{(n-1,m)}(0,0) \Delta_v \]
\[ -1(n = 0, m \geq 2)m \hat{h}_{(m-1)}(0) \Delta_v + (n = 1) \hat{h}_{(1)}(0) \Delta_v. \]
From (4.17) and (4.18), we see that \( \phi_{(n,m)} \) and \( x_{(n,m)} \) are determined by \( \hat{g}_{(k,\ell)}(0,0) \) for \( k = 0, \ldots, n \) and \( \ell = 0, \ldots, m \) except for \( (k, \ell) = (n, m) \). From (4.16) and the definition of \( \alpha_{(n,m)}, \)
\[ \alpha_{(n,m)} \pi + \phi_{(n,m)} = \hat{g}_{(n,m)}(0,0)(e \pi - (C + D)), \]
which implies
\[ \hat{g}_{(n,m)}(0,0) = \alpha_{(n,m)} \pi + x_{(n,m)}. \]
It remains to compute $\alpha_{(n,m)}$. For this, we compute the $(n + 1, m)$-th partial derivatives of (2.2) at $\eta = \theta = 0$, which is:

$$-\hat{g}_{(n+1,m)}(0,0)(C + D) = 1(m \geq 1) \sum_{k=0}^{m-1} \binom{m}{k} \hat{g}_{(n+1,k)}(0,0) \hat{D}_{(m-k)}(0) - (n + 1)\hat{g}_{(n,m)}(0,0)\Delta \nu$$

$$+ 1(n = 0)\hat{h}_{(m)}(0)\Delta \nu.$$  

Post-multiplying $e$ to both sides of the above equation, and substituting (4.19), we have

$$1(m \geq 1) \sum_{k=0}^{m-1} \binom{m}{k} \left( \hat{g}_{(n+1,k)}(0,0) \hat{D}_{(m-k)}(0)e \right)$$

$$- (n + 1)\alpha_{(n,m)} \pi \Delta \nu e - (n + 1)\mathbf{x}_{(n,m)} \Delta \nu e + 1(n = 0)\hat{h}_{(m)}(0)\Delta \nu e = 0.$$  

or equivalently,

$$\alpha_{(n,m)} = \frac{1}{(n + 1)\pi \Delta \nu} \left( 1(m \geq 1) \sum_{k=0}^{m-1} \binom{m}{k} \left( \hat{g}_{(n+1,k)}(0,0) \hat{D}_{(m-k)}(0)e \right) - (n + 1)\mathbf{x}_{(n,m)} \Delta \nu e + 1(n = 0)\hat{h}_{(m)}(0)\Delta \nu e \right). \quad (4.20)$$

Thus, $\alpha_{(n,m)}$ is determined by $\hat{g}_{(n+1,k)}$ for $k = 1, \ldots, m - 1$ and $\mathbf{x}_{(n,m)}$. We may obtain an similar algorithm for $\alpha_{(n,m)}$, using the $(n, m+1)$-th partial derivatives of (2.2) at $\eta = \theta = 0$. However, it is not helpful to use $(n + 1, m+1)$-th partial derivatives. This technique is an extension of the algorithm for computing $\beta_{(n)}$ in Section 4.2. We need to carefully choose the order of computations. An example of the order will be given in Section 4.4.

### 4.4 Summary of algorithm

An algorithm to compute $\hat{g}_{(n,m)}(0,0)$ is given as follows.

**Step 0.** Compute $G(0,0) = (0^+, H^-(0))$, which is given in [15].

**Step 1.** Compute $R_{(k)}$ and $Q_{(k)}$ for $k = 0, \ldots, m + 1$ by the iterations.

**Step 2.** Computation of $\hat{h}_{(k)}(0)$ for $k = 0, \ldots, m + 1$.

- **Step 2-1.** Compute stationary distribution $\kappa^-$ of $-\Delta_v^- Q_{(0)}$.
- **Step 2-2.** Compute $(e^- \kappa^- + \Delta_v^- Q_{(0)})^{-1}$.
- **Step 2-3.** Compute $\beta_{(0)}$ by (4.14) and put $\hat{h}_{(0)}(0) = \beta_{(0)} \kappa^-$. 


Step 2-4. For \( k = 1 \) to \( m + 1 \), Compute \( \psi(k), y(k), \beta(k) \) and \( \hat{h}^{-}(0) \) by (4.13), (4.11), (4.14) and (4.12) in this order.

Step 2-5. Put \( \hat{h}(k)(0) = \{0^+, \hat{h}^{-}(0)\} \).

Step 3. Computation of \( \hat{g}_{(k,\ell)}(0,0) \) for \( k = 0, \ldots, n \) and \( \ell = 0, \ldots, m \).

Step 3-1. Compute stationary distribution \( \hat{g}(0,0) = \pi \) of \( C + D \).

Step 3-2. Compute \( (e\pi - (C + D))^{-1} \).

Step 3-3. For \( k + \ell = 1 \) to \( n + m \) and \( \ell = 0 \) to \( \min(k + \ell, m) \), Compute \( \phi_{(k,\ell)}, x_{(k,\ell)}, \alpha_{(k,\ell)} \) and \( \hat{g}_{(k,\ell)}(0,0) \) by (4.18), (4.17), (4.20) and (4.19) in this order.

Table 4.1: Computation order of \( \hat{g}_{(k,\ell)}(0,0) \).

<table>
<thead>
<tr>
<th>( \ell ) \backslash k</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>7</td>
<td>11</td>
<td>15</td>
<td>19</td>
<td>23</td>
<td>27</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>5</td>
<td>8</td>
<td>12</td>
<td>16</td>
<td>20</td>
<td>24</td>
<td>28</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>9</td>
<td>13</td>
<td>17</td>
<td>21</td>
<td>25</td>
<td>29</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td>14</td>
<td>18</td>
<td>22</td>
<td>26</td>
<td>30</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

For example, the order of computations for \( n = 5 \) and \( m = 3 \) is given in Table 4.1.

5. Numerical examples

For convenience, we use the same data for examples in [15]. Those data assume that jump sizes are deterministic. \( B \) is a \(|S| \times |S|\)-matrix, whose \((i, j)\)th element is the jump size of the buffer process when the background state changes from \( i \) to \( j \). Define \( F_{ij}(x) \) by

\[
F_{ij}(x) = 1(B_{ij} \leq x).
\]

For \( k \geq 0 \), \( \hat{D}_{(k)}(\eta) \) are computed as

\[
[\hat{D}_{(k)}(\eta)]_{ij} = (-B_{ij})^k e^{-\eta B_{ij}} D_{ij}.
\]

The parameters of the numerical example in [15] are

\[
S^+ = \{a, b\}, \quad S^- = \{c, d\}, \quad S = S^+ \cup S^-,
\]
where the states are ordered as $a, b, c$ and $d$. For instance, $C_{aa} = -3$ and $C_{ad} = 1$. The $v(d)$ is varied so as to represent different situations, the heavy traffic case ($-E(Y(1))$ close to zero) and the light traffic case ($-E(Y(1))$ far from zero). Various moments, covariances and correlation coefficients are given in Tables 5.1 and 5.2. We have checked that the results are fully compatible with those in [15] with respect to the mean and variance of $X_1 + X_2$.

### Table 5.1: Moments and covariances of the buffer contents

<table>
<thead>
<tr>
<th>$v(d)$</th>
<th>Traffic</th>
<th>$E(X_1)$</th>
<th>$E(X_2)$</th>
<th>$Var(X_1)$</th>
<th>$Var(X_2)$</th>
<th>$Cov(X_1, X_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-2$</td>
<td>Heavy</td>
<td>1.545669</td>
<td>17.707186</td>
<td>3.786764</td>
<td>382.845227</td>
<td>4.439703</td>
</tr>
<tr>
<td>$-6$</td>
<td>-</td>
<td>0.885631</td>
<td>1.242796</td>
<td>1.572737</td>
<td>3.533477</td>
<td>1.072773</td>
</tr>
<tr>
<td>$-10$</td>
<td>Light</td>
<td>0.777286</td>
<td>0.833941</td>
<td>1.290275</td>
<td>1.801057</td>
<td>0.775628</td>
</tr>
</tbody>
</table>

### Table 5.2: Correlation coefficient of the buffer contents

<table>
<thead>
<tr>
<th>$v(d)$</th>
<th>$\rho(X_1, X_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-2$</td>
<td>0.0887</td>
</tr>
<tr>
<td>$-6$</td>
<td>0.0984</td>
</tr>
<tr>
<td>$-10$</td>
<td>0.1166</td>
</tr>
<tr>
<td>$-7$</td>
<td>0.2403</td>
</tr>
<tr>
<td>$-8$</td>
<td>0.3095</td>
</tr>
<tr>
<td>$-9$</td>
<td>0.3858</td>
</tr>
<tr>
<td>$-10$</td>
<td>0.4551</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$v(d)$</th>
<th>$\rho(X_1, X_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-7$</td>
<td>-0.4742</td>
</tr>
<tr>
<td>$-8$</td>
<td>-0.4886</td>
</tr>
<tr>
<td>$-9$</td>
<td>-0.4998</td>
</tr>
<tr>
<td>$-10$</td>
<td>-0.5088</td>
</tr>
<tr>
<td>$-10^3$</td>
<td>-0.5990</td>
</tr>
<tr>
<td>$-10^6$</td>
<td>0.6001</td>
</tr>
<tr>
<td>$-10^8$</td>
<td>0.6002</td>
</tr>
</tbody>
</table>

From Table 5.2, we can see that the correlation coefficient of $X_1$ and $X_2$ is increased as the relative processing speed $-v(d)$ under state $d$ is increased. This may be contrary to the intuition, but it may be because no batch fluid arrives during state $d$. 

$$
C = \begin{pmatrix}
-3 & 0 & 1 & 1 \\
0 & -2 & 1 & 0 \\
1 & 0 & -3 & 1 \\
0 & 1 & 1 & -2
\end{pmatrix}, \quad D = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
$$

$$
B = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0.5 & 0 & 0 \\
0.2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
$$

$v(a) = 2, v(b) = 1, v(c) = -2, v(d) = -2, -6, or -10.$
6. Concluding remarks

In this paper, we are only concerned with the priority rule that the continuous fluid is preemptively processed over the batch fluid. However, this classification of fluid types in the priority rule is not essential. In general, both of high and low priority types may be composed of continuous as well as batch fluids. In particular, the $MAP/G/1$ priority queue with two different types can be studied in the same line.

Another generalization is a priority queue with more than two types of inputs. This can be also studied in the same way. Of course, computations will be more complicated, but essentially the same approach would work. See Kella [4] for this class of extensions.

In this paper, we only considers time stationary characteristics. However, it is not difficult to get the corresponding characteristics at embedded instants such as the arrival instants of the batch fluid. To this end, we simply apply the rate conservation law, and get relationships among different time instants (see, e.g., [9]). Its details may be discussed elsewhere.

ACKNOWLEDGEMENTS

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References


A. Computation algorithm of $G^-(0, 0)$

An iterative algorithm to compute $G^-(0, 0)$ is given in [15], where the notation $F(0)$ is used for $G^-(0, 0)$ there. We here just give the algorithm for convenience. For a $\eta > \max_i |C_u/v(i)|$, define $Q_n^-$ and $R_n^+$ inductively, by

\[
R^-_0 = 0^-,
Q^-_n = - (\Delta^-)^{-1} \left\{ C^- + C^+ R^-_{n-1} + \int_0^\infty D^+ (dw) e^{Q^-_{n-1} w} + \int_0^\infty D^- (dw) e^{Q^-_{n-1} w} \right\}, \quad (A.1)
\]

\[
R^+_n = \Delta^+ (\Delta^-)^{-1} \left\{ C^+ + (\eta \Delta^+ + C^+) R^+_n + \int_0^\infty D^+ (dw) e^{Q^-_{n-1} w} + \int_0^\infty D^- (dw) e^{Q^-_{n-1} w} \right\} (\eta I^- - Q^-_{n-1})^{-1}. \quad (A.2)
\]

Then, $Q_n^-$ and $R_n^+$ converge to rate matrix $Q^-$ and stochastic matrix $\hat{R}^+$, respectively, as $n$ goes to infinity. Let $\kappa^-$ be the stationary vector of $Q^-$, then we have

\[
G^-(0, 0) = \mathcal{E}(Y(1)) \kappa^- (\Delta^-)^{-1}. \quad (A.3)
\]

The convergence of (A.1) and (A.2) is verified in Theorem 4.1 of [15] (see also Appendix B for a similar verification), and (A.3) is obtained in Section 5 of the same paper.

B. A proof of Lemma 3.1

The proof is composed of the following two parts. In the first part, we construct a nondecreasing sequence of matrices that are bounded by $\hat{R}^+(\eta|0)$. In the second part, we construct lower bounds of those matrices such that the lower bounds converges to $\hat{R}^+(\eta|0)$. Those lower bounds are obtained by restricting the number of transitions of the background Markov chain. We now give details of those two parts.
Fix an arbitrary $\eta \geq 0$, and choose a $\xi$ such that $\xi > \max_i |\hat{\Gamma}_{ii}(\eta)|$. Let $\hat{R}^{(0)}(\eta|0) = 0^{-+}$ and inductively define $Q^{(m)}(\eta)$ and $\hat{R}^{(m)}(\eta|0)$ for $m \geq 1$ as

\begin{align}
Q^{(m)}(\eta) &= \hat{\Gamma}^{-+}(\eta)\hat{R}^{(m-1)}(\eta|0) + \hat{\Gamma}^{--}(\eta), \\
\hat{R}^{(m)}(\eta|0) &= \left( (\xi I^{++} + \hat{\Gamma}^{++}(\eta))\hat{R}^{(m-1)}(\eta|0) + \hat{\Gamma}^{+-}(\eta) \right) \\
&\quad \left( (\xi I^{+-} - Q^{(m)}(\eta))^{-1} \right),
\end{align}

where $\xi I^{++} - Q^{(m)}(\eta)$ is nonsingular, and its inverse is nonnegative since $Q^{(m)}(\eta)$ is substochastic, which will be shown below. We prove by induction on $m$ that $Q^{(m)}(\eta)$ and $\hat{R}^{(m)}(\eta|0)$ are nondecreasing in $m$ and bounded by $Q^{--}(\eta)$ and $\hat{R}^{+-}(\eta|0)$, respectively.

Clearly, $\hat{R}^{+-}(\eta|0) \geq 0^{-+} = \hat{R}^{(0)}(\eta|0)$. Since $\hat{\Gamma}^{--}(\eta)$ is a substochastic matrix, $(\xi I^{+-} - \hat{\Gamma}^{--}(\eta))^{-1}$ is nonnegative. By the definition, $\hat{\Gamma}^{+-}(\eta)$ is nonnegative too. Hence, we have

\[
\hat{R}^{(1)}(\eta|0) = \hat{\Gamma}^{+-}(\eta)(\xi I^{+-} - \hat{\Gamma}^{--}(\eta))^{-1} \geq 0^{-+} = \hat{R}^{(0)}(\eta|0).
\]

Furthermore, $\hat{R}^{(1)}(\eta|0)e \leq 1$, i.e., $\hat{R}^{(1)}(\eta|0)$ is substochastic. Thus the induction holds for $m = 1$. Suppose that $\{R^{(k)}(\eta|0)\}_{k=0,1,\ldots,m-1}$ are substochastic, nondecreasing and bounded by $\hat{R}^{+-}(\eta|0)$. From (B.1) and the induction hypotheses, $\{Q^{(k)}(\eta)\}_{k=0,1,\ldots,m}$ are substochastic, nondecreasing and bounded by $Q(\eta)$. From these facts and (B.2),

\[
\hat{R}^{(m)}(\eta|0) - \hat{R}^{(m-1)}(\eta|0) \\
= \left( (\xi I^{++} + \hat{\Gamma}^{++}(\eta))\hat{R}^{(m-1)}(\eta|0) + \hat{\Gamma}^{+-}(\eta) \right) (\xi I^{+-} - Q^{(m-1)}(\eta))^{-1} \\
- \left( (\xi I^{++} + \hat{\Gamma}^{++}(\eta))\hat{R}^{(m-2)}(\eta|0) + \hat{\Gamma}^{+-}(\eta) \right) (\xi I^{+-} - Q^{(m-1)}(\eta))^{-1} \\
\geq \left( (\xi I^{++} + \hat{\Gamma}^{++}(\eta))(\hat{R}^{(m-1)}(\eta|0) - \hat{R}^{(m-2)}(\eta|0)) \right) (\xi I^{+-} - Q^{(m-1)}(\eta))^{-1} \geq 0^{+-}.
\]

Furthermore, from (B.2), $\hat{R}^{(m)}(\eta|0)$ is substochastic. Hence, $\{\hat{R}^{(m)}(\eta|0)\}_{m \geq 0}$ is substochastic, nondecreasing and bounded by $\hat{R}(\eta|0)$.

We next consider the lower bounds. Let $T_m$ be the $m$-th transition instant of the Markov chain $\{M(t)\}$, where $T_0 = 0$. For each $m \geq 0$, define the matrices $\tilde{R}^{(m)}(\eta|x)$ as

\[
[\tilde{R}^{(m)}(\eta|x)]_{ij} = E(e^{-\eta \tilde{Y}_2(\tau_x)}; M(\tau_x) = j, \tau_x \leq T_m | M(0) = i).
\]

Obviously, $\tilde{R}^{+-}(\eta|x) = 0^{+-}$ for $x \geq 0$. Furthermore, $\tilde{R}^{+-}(\eta|x)$ and $\tilde{R}^{--}(\eta|x)$ is nondecreasing in $m$, and converges to $\hat{R}^{+-}(\eta|x)$ and $\hat{R}^{--}(\eta|x)$,
respectively, as \( m \) goes to infinity. We show that
\[
\hat{R}_{(m)}^+(\eta|0) \leq \hat{R}^m(\eta|0), \quad \hat{R}_{(m)}^-(\eta|x) \leq e^{xQ^m(\eta)}, \quad m = 0, 1, \ldots, x \geq 0. \tag{B.3}
\]
Then, the proof is completed. Inequalities (B.3) can be verified by the same arguments in the proof of Theorem 4.1 of [15]. However, the situation is a little complicated in our case, so we outline a proof of (B.3).

We first note that (B.3) holds for \( m = 0 \). The first inequality becomes equality in this case, and
\[
\hat{R}_0^-(\eta|x) = 1(x = 0)I^- \leq e^{x\hat{\Gamma}^-}(\eta) = e^{xQ(\eta)}.
\]
Suppose that (B.3) holds true up to \( m \). We take conditional expectation of the expectation:
\[
\left[ \hat{R}_{(m+1)}^-(\eta|x) \right]_{ij} = E(e^{-\eta Y_2(\tau_x)}; \tau_x = j, \tau_x \leq m + 1|M(0) = i)
\]
given the state just after the first jump instant, which is subject to the exponential distribution with mean \( 1/|C_{ii}| \), if the instant is less than \( \tau_x \), and at \( \tau_x \) otherwise. Let
\[
\Delta_c = \text{diag}(\{|C_{ii}|/v(i); i \in S\}).
\]
We then have, using the induction assumptions,
\[
\hat{R}_{(m+1)}^-(\eta|x) = e^{-x\Delta_c^-} + \int_0^x e^{-y\Delta_c^-} dy \left( \hat{\Gamma}^- (\eta) + \hat{\Delta}_c^- \hat{R}_{(m)}^-(\eta|x-y) + \hat{\Gamma}^+ (\eta) \hat{R}_{(m)}^+(\eta|x-y) \right)
\]
\[
\leq e^{-x\Delta_c^-} + \int_0^x e^{-y\Delta_c^-} dy \left( \hat{\Gamma}^- (\eta) + \hat{\Delta}_c^- + \hat{\Gamma}^+ (\eta) \hat{R}_{(m)}^+(\eta|x-y) \right) e^{(x-y)Q(m)(\eta)}
\]
\[
\leq e^{-x\Delta_c^-} + \int_0^x e^{-y\Delta_c^-} dy \left( Q^{(m+1)}(\eta) + \Delta_c^- \right) e^{(x-y)Q^{(m+1)}(\eta)}
\]
\[
= e^{-x\Delta_c^-} + \int_0^x \frac{d}{dy} \left( -e^{-y\Delta_c^-} e^{(x-y)Q^{(m+1)}(\eta)} \right) dy
\]
\[
= e^{xQ^{(m+1)}(\eta)}, \tag{B.4}
\]
where (B.1) is used to get the second inequality. Thus we get the second inequality of (B.3) for \( m + 1 \). We next prove the first one.
\[
\hat{R}_{(m+1)}^+(\eta|0) = \int_0^\infty e^{-y\Delta_c^+} dy \left( \hat{\Gamma}^+ (\eta) + \Delta_c^+ \hat{R}_{(m)}^+(\eta|y) + \hat{\Gamma}^- (\eta) \hat{R}_{(m)}^-(\eta|y) \right)
\]
\[
\leq \int_0^\infty e^{-y\Delta^{++}} dy \left( (\hat{\Gamma}^{++}(\eta) + \Delta^{++}_\varepsilon) \hat{R}^{(m)}(\eta|0) + \hat{\Gamma}^{+-}(\eta) \right) e^{yQ^{(m)}(\eta)}
\]
\[
\leq \int_0^\infty e^{-y\Delta^{++}} dy \left( \Delta^{++}_\varepsilon \hat{R}^{(m+1)}(\eta|0) - \hat{R}^{(m+1)}(\eta|0) Q^m(\eta) \right) e^{yQ^{(m)}(\eta)}
\]
\[
\leq \int_0^\infty \frac{d}{dy} \left( -e^{-y\Delta^{--}} \hat{R}^{(m+1)}(\eta|0) e^{yQ^{(m)}(\eta)} \right) dy
\]
\[
= \hat{R}^{(m+1)}(\eta|0),
\]
where (B.2) is used to get the second inequality. Hence, we get the first inequality of (B.3) for \(m+1\). This completes the proof. \(\square\)

C. Nonsingularity of \((e^- \kappa^- + \Delta^{--}_\varepsilon Q^{--}(0))\)

Suppose a column vector \(u\) satisfies
\[
(e^- \kappa^- + \Delta^{--}_\varepsilon Q^{--}(0)) u = 0.
\]
(C.1)

The nonsingularity is obtained if we show that \(u = 0\). Pre-multiplying both sides of (C.1) with \(\kappa^-\), we have
\[
\kappa^- u = 0,
\]
(C.2)

since \(\kappa^-\) is the stationary vector of \(-\Delta^{--}_\varepsilon Q^{--}(0)\) and \(\kappa^- e^- = 1\). Hence, from (C.1),
\[
Q^{--}(0) u = 0.
\]

Since \(Q^{--}(0)\) is a rate matrix, i.e., an ML-matrix whose row sums all vanishes, \(u\) has to be a constant multiple of \(e^-\) by the Perron-Frobenius theorem. This constant must be zero by (C.2). Hence, we get \(u = 0\).