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Generic Gröbner basis of a parametric ideal and its application to a comprehensive Gröbner system

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We introduce a new computational method for stability conditions of Gröbner bases and a new algorithm for computing comprehensive Gröbner systems.

The concepts of a comprehensive Gröbner basis and a comprehensive Gröbner system were introduced by V. Weispfenning [5] as a special basis of a parametric polynomial system and has been regarded as one of the new most important tools to study parametric systems. After 2001, by utilizing results of M. Kalkbrener [1], new effective algorithms have been introduced in A. Suzuki and Y. Sato [4]; D. Kapur et al. [2]; K. Nabeshima [3] for computing comprehensive Gröbner systems in a commutative polynomial ring.

We remark that since algorithms, that are presented in [2,3], are generalizations of the Suzuki-Sato algorithm [4], thus all the first steps of the algorithms, in [2,3,4], for computing comprehensive Gröbner systems are the same "computing a Gröbner basis in a polynomial ring over a polynomial ring". Hence, the first steps also become the bottlenecks. Here we give a new method to become the substitute for computing the Gröbner basis.

We use the notation t as the abbreviation of m variables $t_1, \dots, t_m$ and the notation x as the abbreviation of n variables $x_1, \dots, x_n$. Let K and $\bar{K}$ be fields such that $\bar{K}$ is an algebraic closure field of K. Let $K[t][x]$ be a polynomial ring with coefficients in a polynomial ring $K[t]$. For $f_1, \dots, f_s \in K[x]$ (or $K[t][x]$), $\langle f_1, \dots, f_s\rangle = \{\sum_{i=1}^{s} h_i f_i | h_1, \dots, h_s \in K[x] (\text{or } K[t][x])\}$. A symbol $Term(x)$ means the set of terms of x. Fix a term ordering $\succ$ on $Term(x)$. Let $f \in K[x]$ (or $f \in K[t][x]$), then $ht(f), \text{ß}hm(f)$ and $hc(f)$ denote the head term, head monomial and head coefficient of f i.e. $hm(f) = hc(f)ht(f)$. For $F \subset K[x]$ (or $F \subset K[t][x]$), $ht(F) = \{ht(f)|f \in F\}$. For $g_1, \dots, g_r \in K[t]$, $\mathbb{V}(g_1, \dots, g_r) \subset \bar{K}^m$ denote the affine variety of $g_1, \dots, g_r$, i.e. $\mathbb{V}(g_1, \dots, g_r) = \{\bar{t} \in \bar{K}^m \mid g_1(\bar{t}) = \cdots = g_r(\bar{t}) = 0\}$. In this talk, we use an algebraically constructible set that has a form $\mathbb{V}(f_1, \dots, f_\ell) \backslash \mathbb{V}(f'_1, \dots, f'_{\ell'}) \subset \bar{K}^m$ where $f_1, \dots, f_\ell, f'_1, \dots, f'_{\ell'} \in K[t]$. For $\bar{t} \in \bar{K}^m$, the canonical specialization homomorphism $\sigma_{\bar{t}} : K[t][x] \to \bar{K}[x]$ (or $K[t] \to \bar{K}$) is defined as the map that substitutes t by $\bar{t}$ in $f(t, x) \in K[t][x]$. The image $\sigma_{\bar{t}}$ of a set $F \subset K[t][x]$ is denoted by $\sigma_{\bar{t}}(F) = \{\sigma_{\bar{t}}(f)|f \in F\} \subset \bar{K}[x]$.

We adopt the following as a definition of comprehensive Gröbner system.

Definition 1. *Fix a term ordering $\succ$ on Term(x). Let $F \subset K[t][x]$, $\mathbb{A}_1, \dots, \mathbb{A}_r \subset \overline{K}^m$,*

$G_1, \dots, G_r \subset K[t][x]$. *If a finite set* $\mathcal{G} = \{(\mathbb{A}_1, G_1), \dots, (\mathbb{A}_r, G_r)\}$ *of pairs satisfies the properties such that*

- *for* $i \neq j$, $\mathbb{A}_i \cap \mathbb{A}_j = \emptyset$, *and*
- *for all* $\bar{t} \in \mathbb{A}_i$ *and* $g \in G_i$, $ht(g) = ht(\sigma_{\bar{t}}(g))$ *and* $\sigma_{\bar{t}}(G_i)$ *is a Gröbner basis of* $\langle \sigma_{\bar{t}}(F) \rangle$ *in* $\overline{K}[x]$,

then, $\mathcal{G}$ *is called a comprehensive Gröbner system (CGS) of* $\langle F \rangle$ *over* $\overline{K}$ *on* $\mathbb{A}_1 \cup \dots \cup \mathbb{A}_r$. *We call a pair* $(\mathbb{A}_i, G_i)$ *segment of* $\mathcal{G}$. *We simply say that* $\mathcal{G}$ *is a comprehensive Gröbner system of* $\langle F \rangle$ *over* $\overline{K}$ *if* $\mathbb{A}_1 \cup \dots \cup \mathbb{A}_r = \overline{K}^m$.

Let I be a monomial ideal in $K[x]$. Then, the minimal basis of I is written as $MB(I)$. In [3], Nabeshima gives the following theorem.

Theorem 2 (Nabeshima [3]). *Let* G *be a Gröbner basis of an ideal* $\langle F \rangle \subset K[t][x]$ *w.r.t. a term order* $\succ$ *on Term*(x) *and* $MB(\langle ht(G) \rangle) = \{m_1, \dots, m_\ell\}$ *where* $F \subset K[t][x]$. *Suppose that* $G_i = \{f \in G | ht(f) = m_i\}$ *for each* $i \in \{1, \dots, \ell\}$. *Then,* $\forall \bar{a} \in \bar{K}^m \backslash \bigcup_{i=1}^{\ell} \mathbb{V}(ht(G_i))$, $\sigma_{\bar{a}}(G_1 \cup G_2 \cup \dots \cup G_\ell)$ *is a Gröbner basis of* $\langle \sigma_{\bar{a}}(F) \rangle$ *w.r.t.* $\succ$ *in* $\bar{K}[x]$.

In [2], Kapur-Sun-Wang give the following theorem.

Theorem 3 (Kapur-Sun-Wang [2]). *Using the same notation as in Theorem 2, let* $g_i \in G_i$ *and* $h_i = hc(g_i)$ *for each* $i \in \{1, \dots, \ell\}$. *Then,* $\forall \bar{a} \in \bar{K}^m \backslash \mathbb{V}(h_1 \cdots h_\ell)$, $\sigma_{\bar{a}}(\{g_1, \dots, g_\ell\}$ *is a minimal Gröbner basis of* $\langle \sigma_{\bar{a}}(F) \rangle$ *w.r.t.* $\succ$ *in* $\bar{K}[x]$.

The bottleneck of the both theorems above for getting the pairs $(\bar{K}^m \backslash \bigcup_{i=1}^{\ell} \mathbb{V}(ht(G_i)), \{G_1 \cup G_2 \cup \dots \cup G_\ell\})$ or $(\bar{K}^m \backslash \mathbb{V}(h_1 \cdots h_\ell), \{g_1, \dots, g_\ell\})$ is comptuing the Gröbner basis G of $\langle F \rangle$ in $K[t][x]$.

Method 1

Step 1: Computing a Gröbner basis G of $\langle F \rangle$ in $K[t][x]$.

Let $g = \sum_{i=1}^{r} c_{\alpha_i} x^{\alpha_i} \in K(t)[x]$ where $c_{\alpha_i} \in K(t)$, $\alpha_i \in \mathbb{N}^n$ and $K(t)$ is a field of rational functions. Then, $dlcm(g) = lcm(nd(c_{\alpha_1}), \dots, nd(c_{\alpha_r}))$ where $nd(c_{\alpha_i})$ is the denominator of c_{α_i}.

The following theorem is a main result that is utilized in the new algorithm for computing comprehensive Gröbner systems.

Theorem 4. *Using the same notation as in Theorem 2, let* G' *be a reduced Gröbner basis of* $\langle F \rangle$ *w.r.t.* $\succ$ *in* $K(t)[x]$, $G'' = \{dlcm(g) \cdot g | g \in G'\}$, $h = \prod_{g \in G''} hc(g)$ *and* S *a reduced Gröbner basis of the ideal quotient* $\langle F \rangle : \langle G'' \rangle$ *w.r.t. a block term order with* $x \gg t$ *in* $K[t, x]$. *Then,*
(1) $S \cap K[t] \neq \emptyset$,
(2) for all $\bar{a} \in \bar{K}^m \backslash ((S \cap K[t]) \cup \mathbb{V}(h))$, $\sigma_{\bar{a}}(G')$ *is the reduced Gröbner basis of* $\langle \sigma_{\bar{a}}(F) \rangle$ *w.r.t.* $\succ$ *in* $\bar{K}[x]$.

In order to obtain the pair $(\bar{K}^m \backslash ((S \cap K[t]) \cup \mathbb{V}(h)), G')$, we have to compute a Gröbner basis G' of $\langle F \rangle$ in $K(t)[x]$ and a reduced Gröbner basis of the ideal quotient $\langle F \rangle : \langle G'' \rangle$.

Method 2

Step 1: Computing a reduced Gröbner basis G' of $\langle F \rangle$ in $K(t)[x]$.
Step 2: Computing the reduced Gröbner basis of the ideal quotient $\langle F \rangle : \langle G'' \rangle$.

In this talk, we report the comparison between Method 1 and Method 2, too.

Corollary 5. *Using the same notation as in Theorem 4, let* $g \in S \cap K[t]$*. Then, for all* $\bar{a} \in \bar{K}^m \backslash \mathbb{V}(g \cdot h)$*,* $\sigma_{\bar{a}}(G')$ *is the reduced Gröbner basis of* $\langle \sigma_{\bar{a}}(F) \rangle$ *w.r.t.* $\succ$ *in* $\bar{K}[x]$*.*

We also give a new algorithm for computing comprehensive Gröbner systems.

Keywords

comprehensive Gröbner system, stability of Gröbner basis, ideal quotient

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