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A deterministic method for computing Bertini type invariants of parametric ideals

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Let us start by recalling the classical theorem of Bertini. Let X be a smooth algebraic variety in a projective space $\mathbb{P}^n$. Let $H \subset \mathbb{P}^n$ be a hyperplane. Then, the theorem of Bertini says that the hyperplene section $X \cap H$ is smooth, if H is *general*. General objects are ubiquitous in many fields. In fact, especially in algebraic geometry, there are many properties, concepts and invariants that involve generality conditions. In this paper, w call such a kind of invariant Bertini type invariant. It is difficult to compute Bertini type invariants for singular varieties because of genericities. There are two major methods for computing Bertini invariants. One is the use of random numbers. The other method utilized tools from numerical algebraic geometry, the software Bertini developed by Daniel J. Bates et al [2]. Both methods are widely used, however, they are not deterministic.

We propose an alternative, deterministic method for computing Bertini type invariants. The key of our approach is the Gröbner basis computation with coefficients in the field of rational functions of new auxiliary indeterminates. For the case that a family of varieties or ideals depending on deformation parameters are given. Computing parameter dependency of Bertini type invariants is of fundamental importance. In this talk, we address such parametric cases and show that by utilizing the theory of comprehensive Gröbner systems, our approach can be extended to treat parametric cases.

Here we give an example to illustrate a role of auxiliary indeterminates in our approach.

Chern-Schwartz-MacPherson class

Let $\phi : \mathbb{P}^n \longrightarrow \mathbb{P}^n$ be a rational map. Let $g_i = \mathrm{card}(\phi^{-1}(\mathbb{P}^{n-i}) \cap \mathbb{P}^i)$, where $\mathbb{P}^{n-i}$ and $\mathbb{P}^i$ are *general* planes of dimension $n - i$ and i respectively. Then, $g = (g_0, g_1, g_2, \ldots, g_n)$ is called the projective degrees of the map ϕ. (See [6].)

Let $V = V(f)$ be a hypersurface of $\mathbb{P}^n$, where f is a defining polynomial of V. Let $c_{SM}(V)$ be the Chern-Schwartz-MacPherson class of V. (See [7].)

Theorem (P. Aluffi[1]) Let $g = (g_0, g_1, \ldots, g_n)$ be the projective degrees of the polar map

$\phi : \mathbb{P}^n \longrightarrow \mathbb{P}^n$ defined to be

$$\phi : p \mapsto (\frac{\partial f}{\partial z_0}(p), \frac{\partial f}{\partial z_1}(p), \cdots, \frac{\partial f}{\partial z_n}(p))$$

Then,

$$c_{SM}(V) = (1+h)^{n+1} - \sum_{j=0}^{n} g_j(-h)^j(1+h)^{n-j} \in \mathrm{A}_*(\mathbb{P}^n)$$

holds, where h is the class of a hyperplane defined by a general linear form and $\mathrm{A}_*(\mathbb{P}^n)$ is the Chow ring of the projective space $\mathbb{P}^n$.

The next result is due to M. Helmer [4].

Proposition (M. Helmer) Let $S = 1 - s\sum_{j=0}^{n} c_j \frac{\partial f}{\partial z_j},\ L_A = 1 - \sum_{j=0}^{n} r_j z_j$ and

$P_k = \sum_{j=0}^{n} a_{j,k}\frac{\partial f}{\partial z_j},\ L_k = \sum_{j=0}^{n} b_{j,k} z_j,\ k = 1, 2, \ldots, n.$

Assume that all the coefficients $a_{j,k}, b_{j,k}, c_j, r_j$ are *general*. Then the projective degrees $g = (g_0, g_1, g_2, \ldots, g_n)$ of the polar map defined in the above theorem are given by

$g_i = \dim_K(K[z_0, z_1, \ldots, z_n, s]/(P_1 + P_2 + \cdots + P_i + L_1 + L_2 + \cdots + L_{n-i} + L_A + S)).$

M. Helmer utilized probabilistic method and tools from numerical algebraic geometry for computing projective degrees and he obtained an algorithm for computing Chern-Schwartz-MacPherson classes [4].

Now, we regard $a_{j,k}, b_{j,k}, c_j, r_j$ as indeterminates and set $u = (a_{j,k}, b_{j,k}, c_j, r_j), j, k = 1, 2, \ldots, n$ and $x = (z_1, z_2, \cdots, z_n, s)$. Let $K(u)[x]$ denote the polynomial ring with coefficients in $K(u)$, where $K(u)$ is the fields of rational functions of u. We have the following result.

Proposition Let G_i be a Gröbner basis, in the ring $K(u)[x]$, of the ideal generated by $P_1, P_2, \ldots, P_i, L_1, L_2, \ldots, L_{n-i}, L_A$. Then, $g_i = \dim_{K(u)}(K(u)[x]/(G_i)), i = 1, 2, \ldots, n$ hold.

The above proposition togather with the theorem of Aluffi allow us to construct a deterministic method for computing Chern-Schwartz-MacPherson classes!!

Comprehensive Gröbner systems with auxiliary indeterminates

Let $x = \{x_1, x_2, \ldots, x_n\}, t = \{t_1, t_2, \ldots, t_m\}, u = \{u_1, u_2, \ldots, u_\ell\}$ and let $(K(u)[t][x]$ denote the ring of polynomials with coefficients in $K(u)[t]$. Here we regard x as main variables, t as parameters and u as auxiliary indeterminates.

Let $\overline{K(u)}$ be an algebraic closure of the field $K(u)$ of rational functions. For an arbitrary $\bar{t} \in \left(\overline{K(u)}\right)^m$, the specialization homomorphism

$$\sigma_{\bar{t}} : (K(u)[t])[x] \longrightarrow \overline{K(u)}[x]$$

is defined as the map that substitutes $\bar{t}$ into m variables t. For $G \subset (K(u)[t])\{x\}$, $\sigma_{\bar{t}}(G) = \{\sigma_{\bar{t}}(g) | g \in G\} \subset \overline{K(u)}\{x\}$. For $g_1, \ldots, g_r \in K(u)[t]$,

$$V_{\overline{K(u)}}(g_1, \ldots, g_r) = \left\{ \bar{t} \in \left(\overline{K(u)} \right)^m \mid g_1(\bar{t}) = \cdots = g_r(\bar{t}) = 0 \right\}.$$

We call an algebraic constructible set of the form $V_{\overline{K(u)}}(g_1, \ldots, g_r) \backslash V_{\overline{K(u)}}(g'_1, \ldots, g'_{r'})$, a stratum. Notations $A_1, A_2, \ldots, A_r$ are frequently used to represent strata.

Definition Fix a term ordering $\succ_x$ on $K[x]$. Let $F \subset (K(u)[t])[x]$, $A_1, \ldots, A_r \subset \left(\overline{K(u)} \right)^m$, $S_1, \ldots, S_r \subset (K(u)[t])[x]$. If a finite set $\mathcal{G} = \{(A_1, S_1), \ldots, (A_r, S_r)\}$ of pairs satisfies the properties such that (i) for $i \neq j$, $A_i \cap A_j = \emptyset$, and (ii) for all $\bar{t} \in A_i$ and $g \in S_i$, $ht_{\succ_x}(g) = ht_{\succ_x}(\sigma_{\bar{t}}(g))$ and $\sigma_{\bar{t}}(S_i)$ is a Gröbner basis of $\langle \sigma_{\bar{t}}(F) \rangle$ in $\overline{K(u)}[x]$, ($ht_{\succ_x}$ stands for the head term) then, $\mathcal{G}$ is called a comprehensive Gröbner system (CGS) of $\langle F \rangle$ over $\overline{K(u)}$ on $A_1 \cup \cdots \cup A_r$. We call a pair (A_i, S_i) segment of $\mathcal{G}$. We simply say that $\mathcal{G}$ is a comprehensive Gröbner system of $\langle F \rangle$ over $\overline{K(u)}$ if $A_1 \cup \cdots \cup A_r = \left(\overline{K(u)} \right)^m$.

We have the following

Proposition Let $V_{\overline{\mathbb{C}(u)}}(E)$ be a non-empty stratum in $\overline{\mathbb{C}(u)}^m$ where $E \subset \mathbb{C}(u)[t]$. Set $E' = \{hq \in \mathbb{C}[u][t] | \forall h \in E,\ q$ is the least common multiple of all denominators of coefficients in $\mathbb{C}(u)$ of $h\}$ and $T = \{c_\alpha | \sum c_\alpha u^\alpha \in E', c_\alpha \in \mathbb{C}[t]\} \subset \mathbb{C}[t]$. Then, $V_{\mathbb{C}}(E) = V_{\mathbb{C}}(T)$ in $\mathbb{C}^m$. (Notice that $V_{\overline{\mathbb{C}(u)}}(E) \cap \mathbb{C}^m = V_{\mathbb{C}}(E)$.)

The above proposition allows us to design a deterministic method for computing Bertini type invariants for parametric cases.

Keywords
comprehensive Gröbner system, parametric ideal, Chern-Schewartz-MacPherson class

References
[1] P. ALUFFI, Computing characteristic classes of projective schemes, *Jourrnal of Symbolic Computation* **35**, 3-19 (2003)
[2] D. J. BATES; J. D. HAUENSTEIN; A. J. SOMMESE; C. W. WAMPLER *Bertini: Software for numerical algebraic geometry*, 2013, http://bertini.nd.edu/
[3] C. HARRIS; M. HELMER, Segre class computation and practical applications, *Math. Comp.* **89**, 465–491 (2020).
[4] M. HELMER, Algorithms to compute the topological Euler characteristics, Chern-Schwartz-MacPhersoon class of projective varieties, *J. Symbolic Comput*. **73**, 120–138 (2016).
[5] C. JOST, Computing characteristic classes and the topological Euler characteristics of complex projective schemes, *J. Softw. Algebra Geom.* **7**, 31-39 (2015).
[6] K. NABESHIMA, Stability conditions of monomial bases and comprehensive Gröbner systems, *Lecture Notes in Comput. Sci.* **7442**, 248-259 (2012).
[7] M. H. SCHWARTZ. *Classes de Chern des Ensembles Analytiques*. Hermann Paris, 2000