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CLASSIFICATION OF ORTHOGONAL 24-RUN 2^5 FACTORIAL DESIGNS DERIVABLE FROM SATURATED TWO-SYMBOL ORTHOGONAL ARRAYS OF STRENGTH 2, SIZE 24 AND INDEX 6

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Abstract. Orthogonal 24-run 2^5 factorial designs derivable from saturated two-symbol orthogonal arrays of strength 2, size 24 and index 6 by selecting five columns are classsified computationally into 63 isomorphic classes with respect to the permutation of factors and levels within factors. Specific features of those 63 representative designs are considered.

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§0. Introduction

Orthogonal 2^m factorial designs have widely been used in factor screening and related experiments. Among others, such designs obtained by assigning factors to the appropriate columns of a saturated orthogonal array (or so-called an orthogonal table) have been recommended for practical use (see, e.g., Taguchi [6,7], Box and Hunter [1,2]). Such kind of saturated orthogonal arrays, however, have been restricted to those constructed by the standard orthogonal polynomial models. Moreover, the size or the number of runs of such a design is necessarily restricted to the power of two.

Although all of the saturated orthogonal arrays of size 4λ are isomorphic to each other with respect to the permutation of $4\lambda - 1$ columns (factors) and symbols (levels) in each case of $\lambda = 1,\ 2$ and 3, respectively, there are several isomorphic classes of saturated orthogonal arrays of size 4λ since Hadamard matrices of size 4λ which are not isomorphic to each other do exist for those integral $\lambda \geq 4$. In fact, as it has been shown in our previous papers (Yamamoto, Fujii, Hyodo and Yumiba [8,9,13]), there are 5, 3 and 130 isomorphic classes of orthogonal arrays in those cases of $\lambda = 4, 5$ and 6, respectively. The possibility, therefore, of obtaining so many useful orthogonal 2^m factorial designs from such saturated orthogonal arrays is expected as it has been illustrated in our preceding paper (Yamamoto, Fujii, Hyodo and Yumiba [10]). The results of the classification of orthogonal 2^5 and 2^6 factorial designs having 16 and 20 runs derivable from such representative saturated orthogonal arrays with respect to the permutation of factors and levels have been given in Yamamoto, Fujii, Hyodo and Yumiba [12]. Representative designs of those isomorphic classes and their characteristic vectors have been given there.

In this paper, results of the classification of all orthogonal 24-run 2^5 factorial designs derivable from two-symbol orthogonal arrays of size 24, strength 2, 23 (maximal) constraints and index 6 are given. Eventually, our number of isomorphic classes is just the same with that of the orthogonal arrays having size 24, 5 constraints and index 6 in general by Namikawa, Fujii and Yamamoto [4]. Our results of the classification, therefore, imply the classification of all orthogonal 24-run 2^5 factorial designs.

§1. 2^m factorial designs

Consider a 2^m factorial experiment with m factors, $F(1),\ F(2), \ldots,$ and $F(m)$, each at two levels 0 and 1. Let $\theta\{\phi\}$; $\theta\{i\}$; and, in general, $\theta\{K\}$ $K = \{i_1, i_2, \ldots, i_k\} \subset \Omega = \{1, 2, \ldots, m\}$, be various factorial effects called the general mean; the main effect of the factor $F(i)$; and the k-factor interaction of $k(2 \leq k \leq m)$ factors $F(i_1),\ F(i_2), \ldots,$ and $F(i_k)$, respectively.

Let T be a fraction of the 2^m factorial design of m factors composed of n binary assemblies $(j_1^{(\alpha)}, j_2^{(\alpha)}, \ldots, j_m^{(\alpha)})$ with $j_i^{(\alpha)} = 1$ or 0 for $i = 1, 2, \ldots, m$ and $\alpha = 1, 2, \ldots, n$, and suppose $\boldsymbol{y}(T)$ be the corresponding vector of observations, i.e.,

$$(1.1)\quad T = \begin{bmatrix} j_1^{(1)}, j_2^{(1)}, \ldots, j_m^{(1)} \\ \vdots \\ j_1^{(\alpha)}, j_2^{(\alpha)}, \ldots, j_m^{(\alpha)} \\ \vdots \\ j_1^{(n)}, j_2^{(n)}, \ldots, j_m^{(n)} \end{bmatrix} \text{ and } \boldsymbol{y}(T) = \begin{bmatrix} y(j_1^{(1)}, j_2^{(1)}, \ldots, j_m^{(1)}) \\ \vdots \\ y(j_1^{(\alpha)}, j_2^{(\alpha)}, \ldots, j_m^{(\alpha)}) \\ \vdots \\ y(j_1^{(n)}, j_2^{(n)}, \ldots, j_m^{(n)}) \end{bmatrix}.$$

The vector of observations of this design T is expressed as

$$\boldsymbol{y}(T) = E(T)\Theta + \boldsymbol{e}, \tag{1.2}$$

in terms of $E(T)$, Θ, and $\boldsymbol{e}$, where $E(T)$ is the design matrix whose $(\alpha, \theta\{K\})$ element lying in the row corresponding to the αth observation and the column corresponding to the factorial effect $\theta\{K\}$ is given by $\prod_{i\in K} d(j_i^{(\alpha)})$, Θ is the column vector of factorial effects, i.e.,

$$\begin{aligned}\Theta^t \;=\; &\Big(\theta\{\phi\}; \theta\{1\}, \ldots, \theta\{m\}; \theta\{1,2\}, \ldots, \theta\{m-1, m\}; \ldots;\\ &\theta\{1, 2, \ldots, k\}, \ldots, \theta\{m-k+1, m-k+2, \ldots, m\};\\ &\ldots; \theta\{1, 2, \ldots, m\}\Big),\end{aligned} \tag{1.3}$$

and $\boldsymbol{e}$ is the error vector with a usual assumption that the components are distributed independently with $N(0, \sigma^2)$.

Here, $d(j) = -1$ or 1 according as $j = 0$ or 1 (see, e.g., Yamamoto, Shirakura and Kuwada [15]).

The expectation of the αth observation of $\boldsymbol{y}(T)$ is expressed as:

$$\eta(j_1^{(\alpha)}, j_2^{(\alpha)}, \ldots, j_m^{(\alpha)}) = \sum_{u=0}^{m} \sum_{U\in\Omega(u)} \prod_{i\in U} d(j_i^{(\alpha)})\theta\{U\}, \tag{1.4}$$

where, $\Omega(u)$ denotes the collection of all subsets of $\Omega = \Omega(m)$ having the cardinality u each. In particular, $\Omega(0) = \phi$.

The column vector $\boldsymbol{d}(K)$ of the design matrix $E(T)$ corresponding to the factorial effect $\theta\{K\}$ is expressed as:

$$\boldsymbol{d}(K)^t = \left(\prod_{i\in K} d(j_i^{(1)}), \ldots, \prod_{i\in K} d(j_i^{(\alpha)}), \ldots, \prod_{i\in K} d(j_i^{(n)})\right). \tag{1.5}$$

In particular, $\boldsymbol{d}(\phi)^t = \boldsymbol{j}^t = (1, 1, \ldots, 1)$ for the general mean $\theta\{\phi\}$, and $\boldsymbol{d}(i)^t = (d(j_i^{(1)}), \ldots, d(j_i^{(\alpha)}), \ldots, d(j_i^{(n)}))$ for the main effect $\theta\{i\}$.

Definition 1.1. A column vector $\boldsymbol{d}(K)$ of the design matrix $E(T)$ is called the *loading vector* of a factorial effect $\theta\{K\}$.

Since $d(j) = -1$ or 1 according as $j = 0$ or 1, those loading vectors satisfy the following:

$$\boldsymbol{d}(U) * \boldsymbol{d}(V) = \boldsymbol{d}(U \triangle V), \tag{1.6}$$

where $\boldsymbol{x} * \boldsymbol{y}$ denotes the so-called *Schur product* of two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ and $U\triangle V$ denotes the *symmetric difference* of two subsets U and V of Ω.

Let $\|\boldsymbol{x}\|$ be a kind of magnitude called *spur* of the vector $\boldsymbol{x}$ being defined by the sum of its elements.

Definition 1.2. The spur of a loading vector $\boldsymbol{d}(K)$ is called the *loading coefficient* of the factorial effect $\theta\{K\}$ to the general mean $\theta\{\phi\}$ and is denoted by $\boldsymbol{\gamma}(K)$.

The normal equation for estimating Θ is given by

$$M(T)\Theta = E(T)^t \boldsymbol{y}(T), \tag{1.7}$$

where $M(T) = E(T)^t E(T)$ is the information matrix of the design T.

Under an a priori assumption that $p+1$ $(1 \leq p \leq m)$ or more higher order interactions can be assumed to be zero, the observation vector $\boldsymbol{y}(T)$ can be expressed as

$$\boldsymbol{y}(T) = E(p,T)\Theta(p) + \boldsymbol{e}, \tag{1.8}$$

in terms of $E(p,T)$, $\Theta(p)$, and $\boldsymbol{e}$, where $E(p,T)$ is a restricted design matrix composed of those column vectors of $E(T)$ corresponding up to p-factor interactions, $\Theta(p)$ is the vector of factorial effects up to p-factor interactions, and $\boldsymbol{e}$ is the error vector.

In such a situation, the normal equation for estimating $\Theta(q)$, a part of $\Theta(p)$ up to q-factor interactions $(q \leq p)$, is given by

$$M(q,T)\Theta(q) = E(q,T)^t \boldsymbol{y}(T), \tag{1.9}$$

where $M(q,T) = E(q,T)^t E(q,T)$ is a restricted information matrix called the *frontage* of the design T relative to $\Theta(q)$ and the remainder part of $M(p,T)$ is called the *profile* of the design T relative to $\Theta(p)$ (see Yamamoto, Fujii, Hyodo and Yumiba [10]).

Let $\varepsilon(U,V)$ be $(\theta\{U\},\theta\{V\})$ element of the information matrix lying in the row corresponding to $\theta\{U\}$ and the column corresponding to $\theta\{V\}$, respectively. Then, since $d(j) = \pm 1$, it is given by

$$\varepsilon(U,V) = \|\boldsymbol{d}(U) * \boldsymbol{d}(V)\| = \|\boldsymbol{d}(U \triangle V)\| = \gamma(U \triangle V). \tag{1.10}$$

This implies that the element $\varepsilon(U,V)$ is dependent on the design T through the loading coefficient $\gamma(U \triangle V)$ of the loading vector $\boldsymbol{d}(U \triangle V)$ corresponding to the factorial effect $\theta\{U \triangle V\}$.

Let $\gamma(T)$ be the first row vector of the information matrix $M(T)$, i.e.,

$$\boldsymbol{\gamma}(T) = (\boldsymbol{\gamma}_\phi(T), \boldsymbol{\gamma}_1(T), \ldots, \boldsymbol{\gamma}_k(T), \ldots, \boldsymbol{\gamma}_m(T)), \tag{1.11}$$

where $\boldsymbol{\gamma}_k(T) = (\gamma\{1,2,\ldots,k\}, \ldots, \gamma\{i_1,i_2,\ldots,i_k\}, \ldots, \gamma\{m-k+1, m-k+2,\ldots,m\})$ is the $(k+1)$st $\binom{m}{k}$ dimensional component vector of $\boldsymbol{\gamma}(T)$. Clearly, both $\boldsymbol{\gamma}_\phi(T)$ and $\boldsymbol{\gamma}_m(T)$ are scalars and $\boldsymbol{\gamma}_\phi(T) = \gamma(\phi) = n$. Hereafter, $\binom{n}{r}$ denotes the binomial coefficient with a usual convention.

Such a 2^m dimensional vector $\gamma(T)$ is called the *characteristic vector* of the information matrix $M(T)$ or the design T itself since it determines $M(T)$ completely (see Yamamoto, Fujii, Hyodo and Yumiba [11,12]). In fact, using (1.10), every row of $M(T)$ can be determined easily.

The first member of the normal equation $M(T)\Theta = E(T)^t\boldsymbol{y}(T)$ is given by the spur of the Schur product of the loading vector $\boldsymbol{d}(\phi)$ and the observation vector $\boldsymbol{y}(T)$, i.e.,

$$\begin{aligned} n\theta\{\phi\} + \sum_{u=1}^{m} \sum_{U\in\Omega(u)} \gamma(U)\theta\{U\} &= \|\boldsymbol{d}(\phi) * \boldsymbol{y}(T)\| \\ &= \sum_{\alpha=1}^{n} y(j_1^{(\alpha)}, j_2^{(\alpha)}, \ldots, j_m^{(\alpha)}). \end{aligned} \tag{1.12}$$

In general, the member of the normal equation in the row corresponding to $\theta\{K\}$ is given by

$$n\theta\{K\} + \sum_{u=0}^{m} \sum_{K\neq U\in\Omega(u)} \gamma(K\triangle U)\theta\{U\} = \|\boldsymbol{d}(K) * \boldsymbol{y}(T)\|. \tag{1.13}$$

Definition 1.3. The linear equation (1.13) is called the *principal equation* for estimating the factorial effect $\theta\{K\}$.

The left hand member of the equation (1.12) may be regarded as an extension of the so-called *defining relation* introduced by Box and Hunter [1,2] in some sense. It, therefore, may be called a *defining formula.* The left hand member of (1.13) can easily be derived from that of (1.12) by multiplying $\theta\{K\}$ subject to the following symbolic operation '$\odot$', i.e.,

$$\theta\{K\} \odot \theta\{U\} = \theta\{K\triangle U\},$$

and vice versa.

The left hand member of the equation (1.13) may, therefore, be regarded as a derived relation introduced by Box and Hunter [1,2] in some sense and may be called a *derived formula.*

Clearly, (1.13) provides us BLUE of the $\theta\{K\}$ if $\gamma(K\triangle U) = 0$ and/or $\theta\{U\} = 0$ by assumption for every $U \neq K$.

Definition 1.4. In a fractional 2^m factorial design T having the characteristic vector $\boldsymbol{\gamma}(T)$, a factorial effect $\theta\{K\}$ is called

(a) *orthogonal* to the general mean $\theta\{\phi\}$ if $\gamma(K) = 0$,

(b) *confounded* or *aliased* (totally) with the general mean $\theta\{\phi\}$ if $|\gamma(K)| = \gamma(\phi) = n$, and,

(c) *partially confounded* or *partially aliased* with the general mean $\theta\{\phi\}$ if $0 < |\gamma(K)| < \gamma(\phi) = n$.

The fraction $|\gamma(K)|/n$ is called the *confounding coefficient* of $\theta\{K\}$ to $\theta\{\phi\}$.

With respect to the general formula (1.13),

Definition 1.5. In a fractional 2^m factorial design T having the characteristic vector $\boldsymbol{\gamma}(T)$, a factorial effect $\theta\{U\}$ is called

(a) *orthogonal* to a factorial effect $\theta\{K\}$ if $\gamma(K\triangle U) = 0$,

(b) *confounded* or *aliased* (totally) with a factorial effect $\theta\{K\}$ if $|\gamma(K\triangle U)| = n$, and,

(c) *partially confounded* or *partially aliased* with a factorial effect $\theta\{K\}$ if $0 < |\gamma(K\triangle U)| < n$.

The fraction $|\gamma(K\triangle U)|/n$ is also called the *confounding coefficient* of $\theta\{U\}$ to $\theta\{K\}$.

The following proposition is immediate from the results given in Yamamoto, Shirakura and Kuwada [15].

Proposition 1.6. *The component vectors of the characteristic vector $\boldsymbol{\gamma}(T)$ satisfy the following:*

(a) Both $\boldsymbol{\gamma}_1(T)$ and $\boldsymbol{\gamma}_2(T)$ are null vectors if and only if T is a two-symbol orthogonal array of strength 2.

(b) In general, every component vector $\boldsymbol{\gamma}_u(T)$ is a null vector for every $u = 1, 2, \ldots, t$ if and only if T is a two-symbol orthogonal array of strength t.

(c) Every component vector $\boldsymbol{\gamma}_u(T)$ is a γ_u multiple of the vector $(1, 1, \ldots, 1)$ for every $u = 1, 2, \ldots, t$ if and only if T is a two-symbol balanced array of strength t with index set $\{\mu_0, \mu_1, \ldots, \mu_t\}$, where

$$\gamma_u = \sum_{j=0}^{t}\sum_{p=0}^{u}(-1)^p \binom{u}{p}\binom{t-u}{j-u+p}\mu_j,\ u = 0, 1, \ldots, t,\ \textit{and},$$
$$\mu_v = \frac{1}{2^t}\sum_{i=0}^{t}\sum_{p=0}^{i}(-1)^p \binom{v}{i-p}\binom{t-v}{p}\gamma_i,\ v = 0, 1, \ldots, t.$$

§2. Orthogonal 24-run 2^5 factorial designs derivable from saturated two-symbol orthogonal arrays of size 24, strength 2 and index 6

It has been shown in our previous paper (Yamamoto, Fujii, Hyodo and Yumiba [13]) that there are 130 classes of two-symbol orthogonal arrays of strength $t = 2$, size $n = 24$, $m = 23$ constraints and index $\lambda = 6$, i.e., 2-OA(2,23,6)'s, isomorphic with respect to the permutation of columns (factors) and symbols (levels) within factors. The complete list of 130 representatives of such isomorphic classes of saturated orthogonal arrays can be seen in the above mentioned paper.

In order to classify all orthogonal 24-run 2^5 factorial designs derivable from saturated two-symbol orthogonal arrays of size 24, strength 2 and index 6 isomorphic with respect to the permutation of factors and levels within factors, all orthogonal designs obtained by assigning factors to the five columns of the above 130 representative arrays ($\binom{23}{5} \times 130$ in all) have been classified computationally into isomorphic classes. The concept of the modular representation of orthogonal arrays given in Yamamoto, Fujii, Namikawa and Mitsuoka [14] has play an important roll in our computation.

Those results are summarized in the Table 1.

Table 1. Selected columns of 2-OA(2,23,6) and characteristic vectors of representative 24-run 2^5 designs
(Since γ_1 and γ_2 are null, they are omitted)
(Here '.' indicates '0' in this table)

Rep. des.	Rep. of 2-OA	Columns selected	ϕ	$\gamma\{\cdot\}$ 123	124	125	134	135	145	234	235	245	345	1234	1235	1245	1345	2345	12345	Resolution
(1)	[1]	1 2 3 4 5	24	24	.	.	.	.	24	.	.	.	.	.	.	.	.	24	.	
(2)	[1]	1 2 3 4 6	24	24	.	.	.	.	.	.	.	8	.	.	.	.	8	.	.	
(3)	[1]	1 2 4 6 8	24	.	.	.	.	.	.	8	8	8	8	.	.	.	.	8	.	V
(4)	[1]	1 2 4 6 9	24	.	.	.	.	.	.	8	.	.	.	.	8	8	8	.	8	V
(5)	[1]	1 2 4 7 9	24	.	.	.	.	.	.	.	.	8	8	8	8	.	.	8	.	V
(6)	[1]	1 2 5 7 9	24	.	.	.	.	.	.	8	8	8	.	.	.	.	8	.	8	V
(7)	[1]	1 3 5 7 9	24	.	.	.	.	.	.	.	.	.	.	8	8	8	8	8	.	V
(8)	[1]	2 3 4 5 6	24	.	.	.	.	8	.	.	.	8	.	24	.	.	.	.	.	IV
(9)	[1]	2 3 4 6 9	24	.	.	.	8	.	.	.	8	8	.	.	.	.	.	8	8	V
(10)	[1]	2 3 4 7 9	24	.	.	.	.	.	8	8	8	.	8	.	.	.	8	.	.	V
(11)	[1]	2 4 6 8 10	24	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	-16	
(12)	[1]	2 4 6 8 12	24	8	8	8	8	8	-8	8	-8	-8	8	8	-8	8	-8	8	.	
(13)	[1]	2 4 6 9 11	24	8	.	.	.	.	8	.	.	8	8	.	.	8	8	8	-16	
(14)	[1]	2 4 6 9 13	24	8	.	.	.	.	-8	.	.	-8	8	.	.	8	-8	8	.	V
(15)	[1]	2 4 7 9 11	24	.	.	.	8	8	8	8	8	8	.	8	8	8	.	.	.	
(16)	[1]	2 4 7 9 13	24	.	.	.	8	8	-8	8	-8	-8	.	8	-8	8	.	.	.	
(17)	[1]	2 5 7 9 11	24	8	8	8	8	8	8	.	.	.	.	.	.	.	.	8	-16	
(18)	[1]	2 5 7 9 13	24	8	8	8	8	8	-8	.	.	.	.	.	.	.	.	8	.	V
(19)	[2]	2 4 6 8 12	24	8	8	8	8	.	.	8	.	-8	.	8	-8	.	.	8	.	V
(20)	[2]	2 4 6 8 13	24	8	8	.	8	8	-8	8	-8	.	8	8	.	8	-8	.	.	

(21)	[2]	2	4	6	9	12	24	8	.	8	.	.	-8	.	.	.	8	.	-8	8	-8	.	.	V
(22)	[2]	2	4	6	12	14	24	8	8	.	.	-8	8	.	.	8	8	-8	8	.	.	8	.	
(23)	[2]	2	4	6	12	15	24	8	8	8	.	.	.	.	-8	.	.	-8	.	8	-8	.	.	V
(24)	[2]	2	4	6	12	16	24	8	8	.	.	.	.	.	8	8	.	-8	-8	8	8	8	.	V
(25)	[2]	2	4	6	12	17	24	8	8	8	.	-8	8	.	.	.	-8	-8	.	.	.	.	.	V
(26)	[2]	2	4	6	13	15	24	8	.	8	8	.	8	-8	-8	8	8	.	.	.	.	8	.	V
(27)	[2]	2	4	7	8	14	24	.	8	.	.	.	.	.	-8	.	.	.	.	.	8	.	.	V
(28)	[2]	2	4	7	9	21	24	.	.	-8	8	8	-8	8	.	.	-8	8	-8	8	8	-8	.	
(29)	[2]	2	4	8	12	16	24	8	8	.	.	.	.	-8	.	8	8	.	.	8	.	.	8	V
(30)	[2]	2	4	8	12	17	24	8	8	8	.	8	8	-8	-8	.	.	.	-8	.	-8	8	-8	V
(31)	[2]	2	4	8	13	16	24	8	.	.	-8	.	8	.	.	.	.	8	.	.	-8	8	-8	V
(32)	[2]	2	4	9	12	16	24	.	8	.	-8	8	.	.	-8	8	.	8	-8	8	-8	8	-8	V
(33)	[11]	1	2	4	12	16	24	8	.	.	.	.	.	.	.	.	16	.	.	8	-8	-8	-8	V
(34)	[11]	1	2	4	12	18	24	8	.	.	.	.	.	.	.	.	8	.	.	8	16	.	.	V
(35)	[11]	1	2	4	12	21	24	8	.	.	.	.	.	.	.	.	8	.	.	8	.	.	16	V
(36)	[11]	1	2	4	13	16	24	8	.	.	.	.	.	.	.	8	-8	.	.	.	16	-8	-8	V
(37)	[11]	1	2	4	13	18	24	8	.	.	.	.	.	.	.	8	16	.	.	.	8	.	.	V
(38)	[11]	1	2	4	15	21	24	8	.	.	.	.	8	.	.	8	-16	.	.	-8	.	.	8	V
(39)	[11]	1	4	5	14	21	24	8	.	.	.	.	8	.	.	.	16	.	.	16	.	-8	.	
(40)	[11]	1	4	12	13	16	24	.	.	.	24	.	.	.	16	-8	.	.	-8	16	.	.	.	
(41)	[11]	1	4	12	14	16	24	.	.	.	.	.	8	.	16	-8	.	8	-8	-8	.	.	.	V
(42)	[11]	1	4	12	14	23	24	.	.	.	.	.	8	.	8	16	.	8	.	.	.	.	.	V
(43)	[11]	1	4	12	17	18	24	.	.	.	.	.	-8	-8	8	8	.	.	16	-8	.	.	.	V
(44)	[11]	1	4	14	15	21	24	.	.	.	-8	8	8	8	.	-16	.	-8	16	.	.	.	.	
(45)	[11]	1	4	14	15	23	24	.	.	.	-8	8	-8	8	16	.	.	-8	.	.	.	.	.	V
(46)	[11]	4	5	12	16	17	24	.	.	.	16	-8	.	-8	16	.	.	.	.	24	.	.	.	
(47)	[11]	4	5	12	16	20	24	.	.	.	16	.	8	-8	.	16	.	.	8	.	.	.	.	
(48)	[11]	4	5	13	14	23	24	.	.	.	8	.	16	8	16	.	.	8	8	8	.	.	.	
(49)	[11]	4	12	13	14	16	24	.	.	16	8	-8	-8	.	.	.	.	.	.	.	.	8	-8	V
(50)	[12]	1	6	7	12	14	24	24	.	.	.	.	8	.	.	-8	8	.	.	8	-8	8	.	
(51)	[12]	1	6	12	14	22	24	.	.	.	8	.	8	-8	8	8	.	8	.	8	-8	.	8	V
(52)	[12]	1	7	12	14	19	24	.	.	.	8	8	.	8	8	.	-8	-8	8	8	.	.	8	V
(53)	[12]	2	5	6	13	15	24	.	8	.	8	-8	.	.	-8	16	.	-8	8	.	-8	.	.	
(54)	[12]	2	5	6	13	21	24	.	8	.	8	8	.	.	-8	.	8	-8	8	.	.	8	8	V
(55)	[12]	2	5	6	15	17	24	.	.	8	-8	-8	.	-8	-8	.	8	8	.	16	.	.	.	
(56)	[12]	2	5	7	12	14	24	8	.	8	8	-8	.	-8	8	.	8	.	-8	16	.	.	8	
(57)	[12]	2	5	7	12	23	24	8	.	-8	8	.	.	-8	.	8	8	.	8	8	8	.	.	V
(58)	[12]	2	5	7	14	16	24	8	8	.	-8	-8	.	8	.	16	.	-8	-8	.	8	.	-8	
(59)	[12]	3	6	10	12	14	24	.	8	-8	-8	-8	.	8	.	-8	8	.	8	.	.	.	8	V
(60)	[12]	3	6	12	13	18	24	8	.	8	16	.	.	.	8	-8	.	.	.	8	-8	8	.	
(61)	[12]	7	12	14	21	23	24	8	8	8	8	8	8	.	.	.	.	.	.	.	.	24	.	
(62)	[14]	3	12	13	14	15	24	8	.	.	.	.	16	8	-8	.	.	-8	8	-8	8	8	8	
(63)	[30]	4	5	12	17	18	24	.	.	.	-8	.	16	16	8	.	.	.	16	-8	.	.	.	

Significant portions of our results given in the Table 1 will be listed in the following:

(a) The number of isomorphic classes of the orthogonal 24-run 2^5 factorial designs derived from saturated 2-OA(2,23,6) is 63.

(b) This number is just the same with that of all orthogonal 24-run 2^5 factorial designs or 2-OA(2,5,6)'s given in Namikawa, Fujii and Yamamoto [4]. This means that our 63 representative designs are the representatives of the isomorphic classes of all orthogonal 24-run 2^5 factorial designs.

(c) These representative designs are obtainable from only six saturated orthogonal arrays, i.e., [A1], [A2], [A11], [A12], [A14], and [A30] listed in the Appendix.

(d) The first saturated 2-OA(2,23,6), labeled [A1], yields (1) through (18) representatives of the isomorphic classes of orthogonal 24-run 2^5 factorial designs. In addition, the second [A2] yields (19) through (32) designs, the 11th [A11] yields (33) through (49) designs, the 12th [A12] yields (50) through (61) designs, the 14th [A14] yields (62) design and the 30th [A30] yields the last (63) representative design. No remaining saturated 2-OA(2,23,6) yields a new class of design.

(e) The selected five columns and the characteristic vectors of those 63 representative designs can be seen in this Table 1. The formulas (1.5) and (1.10) may be used for obtaining the normal equation of each design.

(f) Those 36 representative designs among 63 marked V are of resolution five, i.e., every effect up to two-factor interactions can be estimated under the assumption that three or more factor interactions are negligible.

(g) Although the design (8) is not of resolution V, it can be shown that this is a unique design of resolution IV among such a class of orthogonal designs.

(h) The design (7) is an orthogonal array of strength 3 and a simple balanced array of strength 5 having the index set {2,0,1,1,0,2}. The property of this design is main-effect optimal under the assumption that three or more factor interactions can be neglected.

(i) The design (7) is isomorphic to an example given in Hedayat [3]. Though the latter is not composed of a balanced array, it can be reduced to the former only by the symbol permutation within a factor.

(j) The design (35) is A-optimal among the resolution V orthogonal 24-run 2^5 factorial designs. This is superior to the A-optimal balanced 24-run 2^5 fractional factorial design of resolution V given in Srivastava and Chopra [5] in that our design (35) attains their minimums not only in the total sum of the variances of the estimates up to two-factor interactions but also in the partial sum of the variances of main effects and that of two-factor interactions, respectively.

§3. An illustrative example

As a guide to the use of the Table 1, the design (35) will be treated in this section.

The design T of (35) can be obtained from the saturated array [A11] in the Appendix by arranging its 1st, 2nd, 4th, 12th and 21st columns. The design matrix $E(T)$ can be obtained from the T by arranging (i) a column $\boldsymbol{d}(\phi) = \boldsymbol{j}$, the loading vector corresponding to the general mean $\theta\{\phi\}$, (ii) every loading vector $\boldsymbol{d}(i)$ of the main effect $\theta\{i\}$ obtained by converting every 0 (indicated by '.' in this paper) in the column of T into -1, and then (iii) every loading vector $\boldsymbol{d}(K)$ corresponding to the factorial effect $\theta\{K\}$ obtained by calculating the Schur products of related $\boldsymbol{d}(i)$'s using the formula (1.5). The characteristic vector $\boldsymbol{\gamma}(T)$ in the Table 1 can be obtained easily from $E(T)$ by calculating the spur of every column vector.

The design T, the design matrix $E(T)$ and the characteristic vector $\gamma(T)$ of the design (35) are given in the following:

T $E(T)$

1	2	4	12	21	θ / α	ϕ	1	2	3	4	5	12	13	14	15	23	24	25	34	35	45	123	124	125	134	135	145	234	235	245	345	1234	1235	1245	1345	2345	12345
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	.	.	2	1	1	1	1	-1	-1	1	1	-1	-1	1	-1	-1	-1	-1	1	1	-1	-1	-1	-1	1	-1	-1	1	1	-1	-1	1	1	1	1
1	1	1	1	1	3	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	.	.	4	1	1	1	1	-1	-1	1	1	-1	-1	1	-1	-1	-1	-1	1	1	-1	-1	-1	-1	1	-1	-1	1	1	-1	-1	1	1	1	1
1	1	.	1	.	5	1	1	1	-1	1	-1	1	-1	1	-1	-1	1	-1	-1	1	-1	-1	1	-1	-1	1	-1	-1	1	-1	1	-1	1	-1	1	1	1
1	1	.	.	1	6	1	1	1	-1	-1	1	1	-1	-1	1	-1	-1	1	1	-1	-1	-1	-1	1	1	-1	-1	1	-1	-1	1	1	-1	-1	1	1	1
1	.	1	1	.	7	1	1	-1	1	1	-1	-1	1	1	-1	-1	-1	1	1	-1	-1	-1	-1	1	1	-1	-1	-1	1	1	-1	-1	1	1	-1	1	1
1	.	1	.	1	8	1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	-1	1	-1	-1	1	-1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	1
1	.	.	1	1	9	1	1	-1	-1	1	1	-1	-1	1	1	1	-1	-1	-1	-1	1	1	-1	-1	-1	-1	1	1	1	-1	-1	1	1	-1	-1	1	1
1	.	.	.	.	10	1	1	-1	-1	-1	-1	-1	-1	-1	-1	1	1	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1	1	1
1	.	.	1	.	11	1	1	-1	-1	1	-1	-1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	1	1	-1	1	1	-1	-1
1	.	.	.	1	12	1	1	-1	-1	-1	1	-1	-1	-1	1	1	1	-1	1	-1	-1	1	1	-1	1	-1	-1	-1	1	1	1	-1	1	1	1	-1	-1
.	1	1	1	.	13	1	-1	1	1	1	-1	-1	-1	-1	1	1	1	-1	1	-1	-1	-1	-1	1	-1	1	1	1	-1	-1	-1	-1	1	1	1	-1	1
.	1	1	.	1	14	1	-1	1	1	-1	1	-1	-1	1	-1	1	-1	1	-1	1	-1	-1	1	-1	1	-1	1	-1	1	-1	-1	1	-1	1	1	-1	1
.	1	.	1	1	15	1	-1	1	-1	1	1	-1	1	-1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	-1	1	-1	1	1	-1	1	-1	1
.	1	.	.	.	16	1	-1	1	-1	-1	-1	-1	1	1	1	-1	-1	-1	1	1	1	1	1	1	-1	-1	-1	1	1	1	-1	-1	-1	-1	1	-1	1
.	1	.	1	.	17	1	-1	1	-1	1	-1	-1	1	-1	1	-1	1	-1	-1	1	-1	1	-1	1	1	-1	1	-1	1	-1	1	1	-1	1	-1	1	-1
.	1	.	.	1	18	1	-1	1	-1	-1	1	-1	1	1	-1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	1	-1	-1	1	-1	1	1	-1	1	-1
.	.	.	1	.	19	1	-1	-1	-1	1	-1	1	1	-1	1	1	-1	1	-1	1	-1	-1	1	-1	1	-1	1	1	-1	1	1	-1	1	-1	-1	-1	1
.	.	.	.	1	20	1	-1	-1	-1	-1	1	1	1	1	-1	1	1	-1	1	-1	-1	-1	-1	1	-1	1	1	-1	1	1	1	1	-1	-1	-1	-1	1
.	.	1	1	1	21	1	-1	-1	1	1	1	1	-1	-1	-1	-1	-1	-1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	1	1	1	1	-1	-1	1
.	.	1	.	.	22	1	-1	-1	1	-1	-1	1	-1	1	1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	1	1	-1	1	-1	-1	1	-1	-1	1
.	.	1	1	1	23	1	-1	-1	1	1	1	1	-1	-1	-1	-1	-1	-1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	1	1	1	1	-1	-1	1
.	.	1	.	.	24	1	-1	-1	1	-1	-1	1	-1	1	1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	1	1	-1	1	-1	-1	1	-1	-1	1
					$\gamma(T)$:	(24	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	8	.	.	.	.	.	.	.	.	8	.	.	8	.	.	16)

The defining formula of this design may be expressed as:

$$24\{\mathbf{I}\} + 8\{\mathbf{123}\} + 8\{\mathbf{345}\} + 8\{\mathbf{1245}\} + 16\{\mathbf{12345}\}.$$

The coefficient matrices of the restricted normal equation, i.e., $M(2,T)\Theta(2) = E(2,T)^t\boldsymbol{y}(T)$, assuming that three or more higher order interactions are negligible, can be obtained using (1.10) as follows:

$M(2,T)$

	ϕ	1	2	3	4	5	12	13	14	15	23	24	25	34	35	45
ϕ	24	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1	.	24	.	.	.	.	.	.	.	.	8	.	.	.	.	.
2	.	.	24	.	.	.	.	8	.	.	.	.	.	.	.	.
3	.	.	.	24	.	.	8	.	.	.	.	.	.	.	.	8
4	.	.	.	.	24	.	.	.	.	.	.	.	.	.	8	.
5	.	.	.	.	.	24	.	.	.	.	.	.	.	8	.	.
12	.	.	.	8	.	.	24	.	.	.	.	.	.	.	.	8
13	.	.	8	.	.	.	.	24	.	.	.	.	.	.	.	.
14	.	.	.	.	.	.	.	.	24	.	.	.	8	.	.	.
15	.	.	.	.	.	.	.	.	.	24	.	8	.	.	.	.
23	.	8	.	.	.	.	.	.	.	.	24	.	.	.	.	.
24	.	.	.	.	.	.	.	.	.	8	.	24	.	.	.	.
25	.	.	.	.	.	.	.	.	8	.	.	.	24	.	.	.
34	.	.	.	.	.	8	.	.	.	.	.	.	.	24	.	.
35	.	.	.	.	8	.	.	.	.	.	.	.	.	.	24	.
45	.	.	.	8	.	.	8	.	.	.	.	.	.	.	.	24

$E(2,T)^t$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
ϕ	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
2	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1
3	1	1	1	1	-1	-1	1	1	-1	-1	-1	-1	1	1	-1	-1	-1	-1	-1	-1	1	1	1	1
4	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1
5	1	-1	1	-1	-1	1	-1	1	1	-1	-1	1	-1	1	1	-1	-1	1	-1	1	1	-1	1	-1
12	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	1	1	1	1	1	1
13	1	1	1	1	-1	-1	1	1	-1	-1	-1	-1	-1	-1	1	1	1	1	1	1	-1	-1	-1	-1
14	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1
15	1	-1	1	-1	-1	1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	1	-1	-1	1	-1	1
23	1	1	1	1	-1	-1	-1	-1	1	1	1	1	1	1	-1	-1	-1	-1	1	1	-1	-1	-1	-1
24	1	-1	1	-1	1	-1	-1	1	-1	1	-1	1	1	-1	1	-1	1	-1	-1	1	-1	1	-1	1
25	1	-1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	-1	1
34	1	-1	1	-1	-1	1	1	-1	-1	1	-1	1	1	-1	-1	1	-1	1	-1	1	1	-1	1	-1
35	1	-1	1	-1	1	-1	-1	1	-1	1	1	-1	-1	1	-1	1	1	-1	1	-1	1	-1	1	-1
45	1	1	1	1	-1	-1	-1	-1	1	1	-1	-1	-1	-1	1	1	-1	-1	-1	-1	1	1	1	1

The normal equation of estimation can be solved by the elementary transformation of rows as $M^*(2,T)\Theta(2) = E^*(2,T)^t\boldsymbol{y}(T)$. The coefficient matrices of the above equation and the variances of estimated effects after recovery from confounding are as follows:

$M^*(2,T)$

	ϕ	1	2	3	4	5	12	13	14	15	23	24	25	34	35	45
ϕ	24	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1	.	32	.	.	.	.	.	.	.	.	.	.	.	.	.	.
2	.	.	32	.	.	.	.	.	.	.	.	.	.	.	.	.
3	.	.	.	40	.	.	.	.	.	.	.	.	.	.	.	.
4	.	.	.	.	32	.	.	.	.	.	.	.	.	.	.	.
5	.	.	.	.	.	32	.	.	.	.	.	.	.	.	.	.
12	.	.	.	.	.	.	40	.	.	.	.	.	.	.	.	.
13	.	.	.	.	.	.	.	32	.	.	.	.	.	.	.	.
14	.	.	.	.	.	.	.	.	32	.	.	.	.	.	.	.
15	.	.	.	.	.	.	.	.	.	32	.	.	.	.	.	.
23	.	.	.	.	.	.	.	.	.	.	32	.	.	.	.	.
24	.	.	.	.	.	.	.	.	.	.	.	32	.	.	.	.
25	.	.	.	.	.	.	.	.	.	.	.	.	32	.	.	.
34	.	.	.	.	.	.	.	.	.	.	.	.	.	32	.	.
35	.	.	.	.	.	.	.	.	.	.	.	.	.	.	32	.
45	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	40

$E^*(2,T)^t$

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	Variance/σ^2
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0.04166667
1	1	1	1	2	2	2	2	1	1	1	1	-2	-2	-1	-1	-1	-1	-2	-2	-1	-1	-1	-1	0.04687500
1	1	1	1	2	2	-2	-2	-1	-1	-1	-1	2	2	1	1	1	1	-2	-2	-1	-1	-1	-1	0.04687500
1	1	1	1	-2	-2	3	3	-2	-2	-1	-1	3	3	-2	-2	-1	-1	-2	-2	1	1	1	1	0.05000000
1	-1	1	-1	1	-1	2	-2	2	-2	1	-1	2	-2	2	-2	1	-1	1	-1	1	-1	1	-1	0.04687500
1	-1	1	-1	-1	1	-2	2	2	-2	-1	1	-2	2	2	-2	-1	1	-1	1	1	-1	1	-1	0.04687500
1	1	1	1	3	3	-2	-2	-2	-2	-1	-1	-2	-2	-2	-2	-1	-1	3	3	1	1	1	1	0.05000000
1	1	1	1	-2	-2	2	2	-1	-1	-1	-1	-2	-2	1	1	1	1	2	2	-1	-1	-1	-1	0.04687500
1	-1	1	-1	2	-2	1	-1	2	-2	1	-1	-1	1	-2	2	-1	1	-2	2	-1	1	-1	1	0.04687500
1	-1	1	-1	-2	2	-1	1	2	-2	-1	1	1	-1	-2	2	1	-1	2	-2	-1	1	-1	1	0.04687500
1	1	1	1	-2	-2	-2	-2	1	1	1	1	2	2	-1	-1	-1	-1	2	2	-1	-1	-1	-1	0.04687500
1	-1	1	-1	2	-2	-1	1	-2	2	-1	1	1	-1	2	-2	1	-1	-2	2	-1	1	-1	1	0.04687500
1	-1	1	-1	-2	2	1	-1	-2	2	1	-1	-1	1	2	-2	-1	1	2	-2	-1	1	-1	1	0.04687500
1	-1	1	-1	-1	1	2	-2	-2	2	-1	1	2	-2	-2	2	-1	1	-1	1	1	-1	1	-1	0.04687500
1	-1	1	-1	1	-1	-2	2	-2	2	1	-1	-2	2	-2	2	1	-1	1	-1	1	-1	1	-1	0.04687500
1	1	1	1	-2	-2	-2	-2	3	3	-1	-1	-2	-2	3	3	-1	-1	-2	-2	1	1	1	1	0.05000000

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Appendix

Table of saturated orthogonal arrays giving 63 representative designs
('.' indicates '0' in this Table)

```
[A1]
11111111111111111111111
11111111111............
11111......111111......
1....1111..1111..11....
111..11....11......1111
111......11..11..1111..
1..1111........111111..
111....11......1111..11
1......111111..11..11..
1..11..11....11....1111
1....11..11..1111....11
1..11....1111....11..11
.1.1.1.1.1.1.1.1.1.1.1.
.1.1.1.1.1..1.1.1.1.1.1
.1.1..1.1.11.1.1..1.1.1
..1.11.1..11.1..11..1.1
.1..11..1.11..1.1.11.1.
.1..1.1.11..11..11.1..1
..11.1..1.1.1.11.1.1..1
.1..1.11..1.1.11.1..11.
..1.1.11.1.1..11..11..1
..11..11..1.11..1.11.1.
..1.11..11..11.1..1.11.
..11..1.11.1..1.11..11.
```

```
[A2]
11111111111111111111111
11111111111............
11111......111111......
1....1111..1111..11....
111..11....11......1111
111......11..11..1111..
1..1111........111111..
111....11......1111..11
1......111111..11..11..
1..11..11....11....1111
1....11..11..1111....11
1..11....1111....11..11
.1.1.1.1.1.1.1.1.1.1.1.
.1.1.1.1.1..1.1.1.1.1.1
..11.1..1.11.1.1..1.1.1
.1..1.11..11.1..11..1.1
.1.1..1.1.11..1.1.11.1.
..11..1.11..11..11.1..1
.1..11..1.1.1.11.1.1..1
..11..11..1.1.11.1..11.
..1.1.11.1.1..11..11..1
..1.11.1..1.11..1.11.1.
.1..1.1.11..11.1..1.11.
..1.11..11.1..1.11..11.
```

```
[A11]
11111111111111111111111
11111111111............
11.11.1....111...1..1.1
11.11.1.......111.11.1.
11...1..1.1111.11.1....
11...1..1.1...1..1.1111
1.11.1...1.11.11.1...1.
1.11.1...1...1..1.111.1
1.1...111..11.1...111..
1.1...111....1.111...11
1...1..1.1111...1..1.11
1...1..1.11..111.11.1..
.111...1..11.111...1..1
.111...1..1.1...111.11.
.11.1...11.1.11.1...11.
.11.1...11..1..1.111..1
.1...111.1.1.1...111.1.
.1...111.1..1.111...1.1
..1.111...11..1.111...1
..1.111...1.11.1...111.
...111.11..1...111.11..
...111.11...111...1..11
...1..1.1111...1..1.111
...1..1.111.111.11.1...
```

```
[A12]
11111111111111111111111
11111......111111......
11111111111............
111..11....11....1111..
111....11....11..11..11
111......11....11..1111
1..1111......1.1.1.1.11
1..11..11..1....1.1111.
1..11....11.1.1..11.1.1
1....1111...1.111...1.1
1....11..111.11.1.1..1.
1......1111111.1.1.1...
.1.1.1.1.1.1..11.1..11.
.1.1.1.1.1..11..1.11..1
.1.1..1.1.11..11..11..1
.1..11..1.1.11.1..1.11.
.1..1.11..1.1.1.11.1.1.
.1..1.1.11.1.1..11..1.1
..11.1..1.1..11.11.11..
..11..11..1111......111
..11..1.11..1..1111..1.
..1.1.11.1...111..111..
..1.11..11.11.1....1.11
..1.11.1..11...1111...1
```

```
[A14]
11111111111111111111111
11111......111111......
11111111111............
111..11....11....1111..
111....11....11..11..11
111......11....11..1111
1..1111......1.1.1.1.11
1..11..11..1....1.1111.
1..11....11.1.1..11.1.1
1....1111...1.111...1.1
1....11..111.11.1.1..1.
1......1111111.1.1.1...
.1.1.1.1.1.1..11.1..11.
.1.1.1.1.1..11..1.11..1
.1.1..1.1.1111......111
.1..11..1.1.1.11..11.1.
.1..1.11..1..1.1111.1..
.1..1.1.11.1..1.11.1..1
..11.1..1.1..11.11.11..
..11..11..11..11..11..1
..11..1.11..1..1111..1.
..1.1.11.1..111....111.
..1.11..11.1.1.1..1.1.1
..1.11.1..111...11...11
```

```
[A30]
11111111111111111111111
11111111111............
11.11.1....111...1..1.1
11.11.1.......111.11.1.
11...1..1.1111.11.1....
11...1..1.1...1..1.1111
1.11.1...1.11.1...111..
1.11.1...1...1.111...11
1.1...111..11.11.1...1.
1.1...111....1..1.111.1
1...1..1.1111...1..1.11
1...1..1.11..111.11.1..
.111...1..11.111...1..1
.111...1..1.1...111.11.
.11.1...11.1.11.1...11.
.11.1...11..1..1.111..1
.1...111.1.1.1...111.1.
.1...111.1..1.111...1.1
..1.111...11..1.111...1
..1.111...1.11.1...111.
...111.11..1...111.11..
...111.11...111...1..11
...1..1.1111...1..1.111
...1..1.111.111.11.1...
```