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REMARKS ON COHOMOLOGY RINGS OF THE QUATERNION GROUP AND THE QUATERNION ALGEBRA

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Abstract. We will give a homomorphism of the cohomology rings $H^*(\Gamma, \Gamma) \to H^*(Q_8, {}_\psi\Gamma)$ induced by the ring homomorphism from the integral group algebra $\Lambda = \mathbb{Z}Q_8$ of the quaternion group Q_8 to the quaternion algebra $\Gamma = \Lambda e$ for a central idempotent e in $\mathbb{Q}Q_8$.

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Introduction

Let G be a finite group and e a central idempotent of $\mathbb{Q}G$. We set $\Lambda = \mathbb{Z}G$ and $\Gamma = \mathbb{Z}Ge$. The ring homomorphism $\varphi : \Lambda \to \Gamma, w \mapsto we$ derives a homomorphism $F^n : H^n(\Gamma, M) \to H^n(G, {}_\psi M)$ for any left Γ^{e}-module M, where ${}_\psi M$ denotes M regarded as a G-module using the ring homomorphism $\psi : \Lambda \to \Gamma^{\mathrm{e}}, x \mapsto xe \otimes \left(x^{-1}e\right)^\circ$ for $x \in G$. The map F^n induces the homomorphism of the cohomology rings $F^* : H^*(\Gamma, \Gamma) \to H^*(G, {}_\psi\Gamma)$. In the paper we determine F^* in the case G is the quaternion group Q_8 and Γ is the quaternion algebra over $\mathbb{Z}$.

In Section 1, as preliminaries, we give the above F^n on the cochain level explicitly for any finite group G following [CE] and [M]. Moreover we show that F^n preserves the cup product, hence that F^n yields the ring homomorphism F^*. In Section 2 we give the ring structure of $H^*(Q_8, {}_\psi\Gamma)$ (Theorem 1). In fact, we define a transformation map between the well known periodic resolution of period 4 and the standard resolution for Q_8, and compute the cup products of the generators of $H^*(Q_8, {}_\psi\Gamma)$. In Section 3, we determine the ring homomorphism F^* by investigating the images under F^1 of the generators χ, ξ and ω of $H^*(\Gamma, \Gamma) = \mathbb{Z}[\chi, \xi, \omega]/\left(2\chi, 2\xi, 2\omega, \chi^2 + \xi^2 + \omega^2\right)$ (Theorem 2).

§1. Preliminaries

Let G be a finite group and e a central idempotent of the group algebra $\mathbb{Q}G$. We set $\Lambda = \mathbb{Z}G$ and $\Gamma = \mathbb{Z}Ge$ in this section. Then we have a ring homomorphism $\varphi : \Lambda \to \Gamma$ by $\varphi(w) = we$ for $w \in \Lambda$. Let M be a left Γ^{e}-module, which is regarded as a left Λ^{e}-module using $\varphi^{\mathrm{e}} : \Lambda^{\mathrm{e}} \to \Gamma^{\mathrm{e}}$, hence it is denoted by ${}_{\varphi^{\mathrm{e}}}M$. Then we have a homomorphism

$$H^n(\Gamma, M) \to H^n(\Lambda, {}_{\varphi^{\mathrm{e}}}M)$$

for $n \geqslant 0$ (see [CE, Chapter IX, Section 5]). This is induced by the homomorphisms

$$\mathrm{Hom}_{\Gamma^{\mathrm{e}}}((X_\Gamma)_n, M) \xrightarrow{g_n^{\#}} \mathrm{Hom}_{\Gamma^{\mathrm{e}}}(\Gamma^{\mathrm{e}} \otimes_{\Lambda^{\mathrm{e}}} (X_\Lambda)_n, M) \xrightarrow{\sim} \mathrm{Hom}_{\Lambda^{\mathrm{e}}}((X_\Lambda)_n, {}_{\varphi^{\mathrm{e}}}M)$$

by means of the standard complex X_Λ and X_Γ of Λ and Γ respectively, where the above $g_n^{\#}$ is given by

$$\begin{aligned} &g_n : \Gamma^{\mathrm{e}} \otimes_{\Lambda^{\mathrm{e}}} (X_\Lambda)_n \to (X_\Gamma)_n, \\ &\quad (\gamma \otimes \gamma'^{\circ}) \otimes_{\Lambda^{\mathrm{e}}} \lambda_0[\lambda_1, \ldots, \lambda_n]\lambda_{n+1} \mapsto \gamma(\lambda_0 e)[\lambda_1 e, \ldots, \lambda_n e](\lambda_{n+1} e)\gamma'. \end{aligned}$$

Unless otherwise stated, in the rest of the paper, X_Λ and X_Γ denotes the standard complex of Λ and Γ respectively.

Next, we have an isomorphism

$$H^n(\Lambda, N) \xrightarrow{\sim} H^n(G, {}_\eta N) := \mathrm{Ext}^n_\Lambda(\mathbb{Z}, {}_\eta N)$$

for a left Λ^{e}-module N (see [M, Chapter X, Theorem 5.5]). In the above, ${}_\eta N$ denotes N regarded as a G-module using the ring homomorphism

$$\eta : \Lambda \to \Lambda^{\mathrm{e}}, \quad x \mapsto x \otimes \left(x^{-1}\right)^{\circ} \qquad \text{for} \quad x \in G.$$

The above map is induced by the homomorphism

$$\begin{aligned} &\mathrm{Hom}_{\Lambda^{\mathrm{e}}}((X_\Lambda)_n, N) \to \mathrm{Hom}_\Lambda((X_G)_n, {}_\eta N), \\ &\quad f \mapsto \left(x_0[x_1|\cdots|x_n] \mapsto f\left(x_0[x_1, \ldots, x_n](x_0 x_1 \cdots x_n)^{-1}\right)\right) \quad \text{for} \quad x_i \in G, \end{aligned}$$

where $(X_G)_n$ denotes $(X_\Lambda)_n \otimes_\Lambda \mathbb{Z}$ and the element $x_0[x_1|\cdots|x_n]$ denotes $x_0[x_1, \ldots, x_n] \otimes_\Lambda 1$ for $x_i \in G$.

Therefore, for any left Γ^{e}-module M, we have the homomorphism of cohomologies

$$F^n : H^n(\Gamma, M) \to H^n(G, {}_\psi M)$$

given by

$$\tilde{F}^n : \mathrm{Hom}_{\Gamma^{\mathrm{e}}}\left((X_\Gamma)_n, M\right) \to \mathrm{Hom}_\Lambda\left((X_G)_n, {}_\psi M\right),$$
$$\tilde{F}^n(f)(x_0[x_1|\cdots|x_n]) = f\left(x_0e[x_1e,\dots,x_ne]\,(x_0\cdots x_n)^{-1}\,e\right)$$

where ${}_\psi M$ denotes M regarded as a G-module using the ring homomorphism $\psi = \varphi^{\mathrm{e}} \circ \eta : \Lambda \to \Lambda^{\mathrm{e}} \to \Gamma^{\mathrm{e}}, x \mapsto xe \otimes \left(x^{-1}e\right)^\circ$ for $x \in G$.

Furthermore F^n preserves the cup products, that is, the following diagram is commutative, which is shown by direct calculation on the cochain level:

$$\begin{array}{ccc} H^n(\Gamma, M) \otimes H^{n'}(\Gamma, M') & \xrightarrow{F^n \otimes F^{n'}} & H^n(G, {}_\psi M) \otimes H^{n'}(G, {}_\psi M') \\ \cup \downarrow & & \downarrow \cup_\Gamma \\ H^{n+n'}(\Gamma, M \otimes_\Gamma M') & \xrightarrow[F^{n+n'}]{} & H^{n+n'}\left(G, {}_\psi(M \otimes_\Gamma M')\right). \end{array}$$

In the above, $\cup_\Gamma$ denotes the map induced by the ordinary cup product $H^n(G, {}_\psi M) \otimes H^{n'}(G, {}_\psi M') \to H^{n+n'}(G, {}_\psi M \otimes {}_\psi M')$ and the left Λ-homomorphism ${}_\psi M \otimes {}_\psi M' \to {}_\psi(M \otimes_\Gamma M')$ given by $(a \otimes a' \mapsto a \otimes_\Gamma a')$. Hence F^n yields the ring homomorphism $F^* : H^*(\Gamma, \Gamma) \to H^*(G, {}_\psi \Gamma)$, where we set $H^*(-,-) = \bigoplus_{n \geqslant 0} H^n(-,-)$.

§2. $H^*(Q_8, {}_\psi \Gamma)$

Let G denote the quaternion group $Q_8 = \langle x, y | x^4 = 1, x^2 = y^2, yxy^{-1} = x^{-1}\rangle$. We set $e = (1 - x^2)/2 \in \mathbb{Q}G$. Then e is a central idempotent of $\mathbb{Q}G$ and $\mathbb{Q}Ge$ is the quaternion algebra over $\mathbb{Q}$, that is, $\mathbb{Q}Ge = \mathbb{Q}e \oplus \mathbb{Q}i \oplus \mathbb{Q}j \oplus \mathbb{Q}ij$ where we set $i = xe$ and $j = ye$. In the following, we set $\Lambda = \mathbb{Z}G$ and $\Gamma = \Lambda e = \mathbb{Z}e \oplus \mathbb{Z}i \oplus \mathbb{Z}j \oplus \mathbb{Z}ij$ the quaternion algebra over $\mathbb{Z}$. Let ${}_\psi \Gamma$ denote Γ regarded as a G-module using the ring homomorphisms $\psi : \Lambda \to \Gamma^{\mathrm{e}}; x \mapsto -i \otimes i^\circ, y \mapsto -j \otimes j^\circ$ as in Section 1.

We will determine the cohomology ring $H^*(G, {}_\psi \Gamma)$ using the fact that the integral complete cohomology ring $\hat{H}^*(G, \mathbb{Z})$ has an invertible element of degree 4 (and of order 8) (cf. [CE, Chapter XII, Sections 7 and 11]).

2.1. Module structure. The following periodic Λ-free resolution of $\mathbb{Z}$ of period 4 is well known (see [CE, Chapter XII, Section 7] or [T, Chapter 3, Periodicity]):

$$(Y, \delta) : \quad \cdots \to \Lambda^2 \xrightarrow{\delta_1} \Lambda \xrightarrow{\delta_4} \Lambda \xrightarrow{\delta_3} \Lambda^2 \xrightarrow{\delta_2} \Lambda^2 \xrightarrow{\delta_1} \Lambda \xrightarrow{\varepsilon} \mathbb{Z} \to 0,$$
$$\delta_1(z_1, z_2) = z_1(y-1) + z_2(x-1),$$
$$\delta_2(z_1, z_2) = \left(z_1(x-1) - z_2(y+1), z_1(xy+1) + z_2(x+1)\right),$$

$$\delta_3(z) = (-z(xy-1), z(x-1)),$$
$$\delta_4(z) = zN_G,$$

where Λ^2 denotes the direct sum $\Lambda \oplus \Lambda$ and N_G denotes $\sum_{w \in G} w$ $(\in \Lambda)$. Applying the functor $\mathrm{Hom}_\Lambda(-, {}_\psi\Gamma)$ on the sequence above, we have the following complex which gives $H^n(G, {}_\psi\Gamma)$, where we identify $\mathrm{Hom}_\Lambda(Y_0, {}_\psi\Gamma)$ with Γ, $\mathrm{Hom}_\Lambda(Y_1, {}_\psi\Gamma)$ with $\Gamma^2 := \Gamma \oplus \Gamma$ and so on:

$$\left(\mathrm{Hom}_\Lambda(Y, {}_\psi\Gamma), \delta^\#\right) : \quad \cdots \leftarrow \Gamma \xleftarrow{\delta_4^\#} \Gamma \xleftarrow{\delta_3^\#} \Gamma^2 \xleftarrow{\delta_2^\#} \Gamma^2 \xleftarrow{\delta_1^\#} \Gamma \leftarrow 0,$$
$$\delta_1^\#(\gamma) = ((y-1)\gamma, (x-1)\gamma),$$
$$\delta_2^\#(\gamma_1, \gamma_2) = ((x-1)\gamma_1 + (xy+1)\gamma_2, -(y+1)\gamma_1 + (x+1)\gamma_2),$$
$$\delta_3^\#(\gamma_1, \gamma_2) = -(xy-1)\gamma_1 + (x-1)\gamma_2,$$
$$\delta_4^\#(\gamma) = 2(1+x+y+xy)\gamma.$$

In the above, we note that $(y-1)\gamma = -j\gamma j - \gamma$ and so on. Therefore, the module structure of $H^n(G, {}_\psi\Gamma)$ is represented by the form of the subquotient of the complex $\mathrm{Hom}_\Lambda(Y, {}_\psi\Gamma)$ as follows:

$$H^n(G, {}_\psi\Gamma) = \begin{cases} \mathbb{Z}e & \text{for } n = 0 \\ \mathbb{Z}e/8 & \text{for } n \equiv 0 \mod 4,\ n \neq 0 \\ \mathbb{Z}(i,0)/2 \oplus \mathbb{Z}(0,j)/2 \oplus \mathbb{Z}(ij,ij)/2 & \text{for } n \equiv 1 \mod 4 \\ \mathbb{Z}(e,0)/2 \oplus \mathbb{Z}(0,e)/2 \oplus \mathbb{Z}(0,i)/2 & \\ \qquad \oplus\, \mathbb{Z}(j,j)/2 \oplus \mathbb{Z}(ij,0)/2 & \text{for } n \equiv 2 \mod 4 \\ \mathbb{Z}i/2 \oplus \mathbb{Z}j/2 \oplus \mathbb{Z}ij/2 & \text{for } n \equiv 3 \mod 4. \end{cases}$$

In the above, M/m denotes the quotient module M/mM for a $\mathbb{Z}$-module M and an integer m.

2.2. Product on $H^n(G, {}_\psi\Gamma)$. First, we give an initial part of a chain transformation lifting the identity map on $\mathbb{Z}$ between (Y, δ) in Section 2.1 and the standard complex (X_G, d_G), that is, $v_i : Y_i \to (X_G)_i$ $(0 \leqslant i \leqslant 4)$ and $u_i : (X_G)_i \to Y_i$ $(i = 0, 1)$ as follows:

$$v_0 = \mathrm{id};$$
$$v_1(1,0) = [y], \quad v_1(0,1) = [x];$$
$$v_2(1,0) = [x|y] + [xy|x], \quad v_2(0,1) = [x|x] - [y|y];$$
$$v_3(1) = -[xy|x|y] - [x|y|y] + [x|x|x] - [xy|xy|x];$$
$$v_4(1) = -[N_G|xy|x|y] - [N_G - 1|x|y|y] + [N_G - 1|x|x|x] - [N_G - 1|xy|xy|x] - [1|x|y|xy];$$

$$
\begin{aligned}
&u_0 = \mathrm{id};\\
&u_1 : [1] \mapsto (0,0), \quad [x] \mapsto (0,1), \quad [x^2] \mapsto (0, x+1), \quad [x^3] \mapsto (0, x^2+x+1),\\
&\qquad [y] \mapsto (1,0), \quad [xy] \mapsto (x,1), \quad [x^2y] \mapsto (x^2, x+1),\\
&\qquad [x^3y] \mapsto (x^3, x^2+x+1).
\end{aligned}
$$

Next, we calculate the products of the generators $A = (i,0), B = (0,j)$ and $C = (ij, ij)$ of $H^1(G, {}_\psi\Gamma)$ using the above chain transformations. These are obtained by means of the following homomorphisms:

$$
\begin{aligned}
\Gamma^2 \otimes \Gamma^2 &\xrightarrow{\alpha_1^{-1}\otimes\alpha_1^{-1}} \mathrm{Hom}_\Lambda(Y_1, {}_\psi\Gamma) \otimes \mathrm{Hom}_\Lambda(Y_1, {}_\psi\Gamma)\\
&\xrightarrow{u_1^{\#}\otimes u_1^{\#}} \mathrm{Hom}_\Lambda((X_G)_1, {}_\psi\Gamma) \otimes \mathrm{Hom}_\Lambda((X_G)_1, {}_\psi\Gamma)\\
&\xrightarrow{\cup^{1,1}} \mathrm{Hom}_\Lambda((X_G)_2, {}_\psi\Gamma)\\
&\xrightarrow{v_2^{\#}} \mathrm{Hom}_\Lambda(Y_2, {}_\psi\Gamma)\\
&\xrightarrow{\alpha_2} \Gamma^2,
\end{aligned}
$$

where α_1 denotes the isomorphism $\mathrm{Hom}_\Lambda(Y_1, {}_\psi\Gamma) \xrightarrow{\sim} \Gamma^2$ stated in Section 2.1, and so on. Let $\Delta_{k,l}$ denote the diagonal approximation giving the cup product $\cup^{k,l}$. Since

$$
\begin{aligned}
&u_1 \otimes u_1 \cdot \Delta_{1,1} \cdot v_2(1,0)\\
&\qquad = u_1 \otimes u_1 \cdot \Delta_{1,1}([x|y] + [xy|x])\\
&\qquad = u_1 \otimes u_1 ([x] \otimes x[y] + [xy] \otimes xy[x])\\
&\qquad = (0,1) \otimes x(1,0) + (x,1) \otimes xy(0,1),\\
&u_1 \otimes u_1 \cdot \Delta_{1,1} \cdot v_2(0,1)\\
&\qquad = (0,1) \otimes x(0,1) - (1,0) \otimes y(1,0)
\end{aligned}
$$

and also

$$
\alpha_1^{-1}(A)(z_1, z_2) = z_1 i, \quad \alpha_1^{-1}(B)(z_1, z_2) = z_2 j, \quad \alpha_1^{-1}(C)(z_1, z_2) = (z_1 + z_2) ij,
$$

it follows that the following equations hold in $H^2(G, {}_\psi\Gamma)$.

$$
\begin{aligned}
&A^2 = (0, e), \quad B^2 = (e, e), \quad C^2 = (e, 0),\\
&AB = BA = (ij, 0), \quad AC = CA = (j, j), \quad BC = CB = (0, i).
\end{aligned}
$$

This shows that $H^2(G, {}_\psi\Gamma)$ is generated by A, B and C. Note that $A^2 + B^2 + C^2 = 0$ in $H^2(G, {}_\psi\Gamma)$.

Similarly, we have the following equations in $H^3(G, {}_\psi\Gamma)$ by means of the cup product $\cup^{1,1}, \cup^{2,1}$ and v_3 stated above.

$$A^3 = B^3 = C^3 = 0, \quad ABC = 0,$$
$$AB^2 = AC^2 = i, \quad A^2B = BC^2 = j, \quad A^2C = B^2C = ij.$$

This shows that $H^3(G, {}_\psi\Gamma)$ is also generated by A, B and C.

Moreover, we have the following equations in $H^4(G, {}_\psi\Gamma)$ by means of the cup products $\cup^{1,1}, \cup^{2,1}, \cup^{3,1}$ and v_4 stated above.

$$A^2B^2 (= A^2C^2 = B^2C^2) = 4D,$$

where D denotes e in $H^4(G, {}_\psi\Gamma)$. Since $\mathbb{Z}$ is a G-direct summand of ${}_\psi\Gamma$ using the embedding map $\mathbb{Z} \to {}_\psi\Gamma$ by $1 \mapsto e$, we have the following monomorphism of the complete cohomology rings.

$$\hat{H}^*(G, \mathbb{Z}) := \bigoplus_{r \in \mathbb{Z}} \hat{H}^r(G, \mathbb{Z}) \to \hat{H}^*(G, {}_\psi\Gamma) := \bigoplus_{r \in \mathbb{Z}} \hat{H}^r(G, {}_\psi\Gamma).$$

Since D above which is of order 8 in $H^4(G, {}_\psi\Gamma)$ is the image of an element of order 8 in $H^4(G, \mathbb{Z})$, invertible in $\hat{H}^*(G, \mathbb{Z})$, by the above map, it follows that D is also an invertible element in $\hat{H}^*(G, {}_\psi\Gamma)$.

Thus we have the following theorem.

Theorem 1. *The cohomology ring* $H^*(G, {}_\psi\Gamma)$ *is isomorphic to*

$$\mathbb{Z}[A, B, C, D]/(2A, 2B, 2C, 8D, A^2 + B^2 + C^2, A^3, B^3, C^3, ABC, A^2B^2 - 4D),$$

where $\deg A = \deg B = \deg C = 1$ *and* $\deg D = 4$.

By referring the module structure of $H^n(G, {}_\psi\Gamma)$ in Section 2.1, we know that the monomorphism of the ordinary cohomology rings $H^*(G, \mathbb{Z}) \to H^*(G, {}_\psi\Gamma)$ is induced by the map $X \mapsto A^2, Y \mapsto B^2, Z \mapsto D$ where X and Y denote certain generators of $H^2(G, \mathbb{Z})$ and Z denotes the element of order 8 in $H^4(G, \mathbb{Z})$ stated above. Hence we have the following corollary as an immediate consequence of the theorem, while the fact is already known in [A, Section 13].

Corollary. *The cohomology ring* $H^*(G, \mathbb{Z})$ *is isomorphic to*

$$\mathbb{Z}[X, Y, Z]/(2X, 2Y, 8Z, X^2, Y^2, XY - 4Z),$$

where $\deg X = \deg Y = 2$ *and* $\deg Z = 4$.

§3. The ring homomorphism $F^* : H^*(\Gamma,\Gamma) \to H^*(G, {}_\psi\Gamma)$

We use $W = (W_{p,q}; \delta', \delta'')$ stated in [S, Section 3.3] for a Γ^{e}-free complex of Γ giving $H^n(\Gamma, -)$. We remark that $W_{p,q} = \Gamma \otimes \Gamma$ for every p, q and

$$\delta' : W_{1,0} \to W_{0,0}, \quad [\cdot] \mapsto -j[\cdot]j - [\cdot];$$
$$\delta'' : W_{0,1} \to W_{0,0}, \quad [\cdot] \mapsto i[\cdot] - [\cdot]i,$$

where $[\cdot]$ denotes $1\otimes 1 \in \Gamma\otimes\Gamma$. Then an initial part of a chain transformation between the standard Γ^{e}-projective resolution (X_Γ, d_Γ) and W above is as follows:

$$t_0 = \mathrm{id} : (X_\Gamma)_0 \to W_0 = W_{0,0};$$
$$t_1 : (X_\Gamma)_1 \to W_1 = W_{0,1} \oplus W_{1,0},$$
$$[e] \mapsto (0,0), \quad [i] \mapsto ([\cdot], 0), \quad [j] \mapsto (0, [\cdot]j), \quad [ij] \mapsto ([\cdot]j, i[\cdot]j).$$

Applying the functor $\mathrm{Hom}_{\Gamma^{\mathrm{e}}}(-, \Gamma)$, we have

$$t_1^{\#} : \mathrm{Hom}_{\Gamma^{\mathrm{e}}}(W_1, \Gamma) \to \mathrm{Hom}_{\Gamma^{\mathrm{e}}}((X_\Gamma)_1, \Gamma).$$

Since the isomorphisms

$$\mathrm{Hom}_{\Gamma^{\mathrm{e}}}(W_1, \Gamma) \xrightarrow{\sim} \mathrm{Hom}_{\Gamma^{\mathrm{e}}}(W_{0,1}, \Gamma) \oplus \mathrm{Hom}_{\Gamma^{\mathrm{e}}}(W_{1,0}, \Gamma) \xrightarrow{\sim} \Gamma^{0,1} \oplus \Gamma^{1,0}$$

hold under the notation in [S, Section 3], it follows that $t_1^{\#}$ above is represented as follows:

$$t_1^{\#} : \Gamma^{0,1} \oplus \Gamma^{1,0} \to \mathrm{Hom}_{\Gamma^{\mathrm{e}}}((X_\Gamma)_1, \Gamma),$$
$$(z_1, z_2) \mapsto ([e] \mapsto 0, \quad [i] \mapsto z_1, \quad [j] \mapsto z_2 j, \quad [ij] \mapsto z_1 j + i z_2 j)\,.$$

Accordingly, $F^1 : H^1(\Gamma, \Gamma) \to H^1(G, {}_\psi\Gamma)$ stated in Section 1 is given on the cochain levels using $v_1^{\#}$ defined in Section 2.2 as follows:

$$\Gamma^{0,1} \oplus \Gamma^{1,0} \xrightarrow{t_1^{\#}} \mathrm{Hom}_{\Gamma^{\mathrm{e}}}((X_\Gamma)_1, \Gamma) \xrightarrow{\tilde{F}^1} \mathrm{Hom}_\Lambda((X_G)_1, {}_\psi\Gamma)$$
$$\xrightarrow{v_1^{\#}} \mathrm{Hom}_\Lambda(Y_1, {}_\psi\Gamma) \xrightarrow{\alpha_1} \Gamma^2,$$
$$(z_1, z_2) \mapsto (z_2, -z_1 i).$$

In particular, for the generators $\chi = (0, i), \xi = (ij, 0)$ and $\omega = (j, ij) (\in \Gamma^{0,1} \oplus \Gamma^{1,0})$ with $\deg\chi = \deg\xi = \deg\omega = 1$ of $H^*(\Gamma,\Gamma) = \mathbb{Z}[\chi,\xi,\omega]/(2\chi, 2\xi, 2\omega, \chi^2 + \xi^2 + \omega^2)$ (see [S, Section 3.4]), we have

$$F^1(\chi) = A, \quad F^1(\xi) = B, \quad F^1(\omega) = C \qquad \text{in } H^1(G, {}_\psi\Gamma).$$

Thus we have the following theorem.

Theorem 2. *The ring homomorphism $F^* : H^*(\Gamma,\Gamma) \to H^*(G, {}_\psi\Gamma)$ is induced by $F^1(\chi) = A$, $F^1(\xi) = B$ and $F^1(\omega) = C$. Hence* Ker F^* *is the ideal generated by χ^3, ξ^3, ω^3 and $\chi\xi\omega$, and, of course,* Im F^* *coincides with the canonical image of $\mathbb{Z}[A, B, C]$ in $H^*(G, {}_\psi\Gamma)$. In particular, F^n is an isomorphism for $0 \leqslant n \leqslant 2$ and the zero map for $n \geqslant 5$.*

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