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FRACTIONAL 3^m FACTORIAL DESIGNS WITH SPECIAL REFERENCE TO 18-RUN ORTHOGONAL 3^4 FACTORIAL DESIGNS

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Abstract. Loading vectors and loading coefficients of the parameters of a 3^m factorial design and the characteristic vector of its information matrix are introduced. Specific properties of an orthogonal design derived from three-symbol orthogonal array of strength two are discussed. Orthogonal 18-run 3^4 factorial designs obtained respectively from the representatives of twelve isomorphic classes are reviewed and two designs among them are recommended for use from the practical point of view.

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§1. 3^m factorial designs

Consider a 3^m factorial experiment with m factors, $F(1), F(2), \ldots,$ and $F(m)$, each at three levels 0, 1 and 2. Let $(j_1, j_2, \ldots, j_m)$ be an assembly or a treatment combination of m factors each at three levels $j_p = 0, 1$ or 2 for every $p = 1, 2, \ldots, m$. Let $y(j_1, j_2, \ldots, j_m)$ and $\eta(j_1, j_2, \ldots, j_m)$ be the corresponding observation and the expectation of the assembly.

Let

$$(1.1) \qquad \boldsymbol{\eta}(Z) = \begin{bmatrix} \eta(0,\,0,\,\ldots,\,0,\,0) \\ \eta(0,\,0,\,\ldots,\,0,\,1) \\ \eta(0,\,0,\,\ldots,\,0,\,2) \\ \eta(0,\,0,\,\ldots,\,1,\,0) \\ \vdots \\ \eta(2,\,2,\,\ldots,\,2,\,0) \\ \eta(2,\,2,\,\ldots,\,2,\,1) \\ \eta(2,\,2,\,\ldots,\,2,\,2) \end{bmatrix} \text{ and } \Theta(Z) = \begin{bmatrix} \theta(0,\,0,\,\ldots,\,0,\,0) \\ \theta(0,\,0,\,\ldots,\,0,\,1) \\ \theta(0,\,0,\,\ldots,\,0,\,2) \\ \theta(0,\,0,\,\ldots,\,1,\,0) \\ \vdots \\ \theta(2,\,2,\,\ldots,\,2,\,0) \\ \theta(2,\,2,\,\ldots,\,2,\,1) \\ \theta(2,\,2,\,\ldots,\,2,\,2) \end{bmatrix}$$

be the vector of the expectation of possible 3^m assemblies and that of factorial effects based on the orthogonal polynomial models. They are linked to each other by

$$\Theta(Z) = \frac{1}{3^m} D'_{(m)} \boldsymbol{\eta}(Z), \tag{1.2}$$

where $D_{(m)} = D \otimes D \otimes \cdots \otimes D$ is the m-times Kronecker products of the matrix

$$D = \begin{bmatrix} d_{00} & d_{01} & d_{02} \\ d_{10} & d_{11} & d_{12} \\ d_{20} & d_{21} & d_{22} \end{bmatrix} = \begin{bmatrix} \boldsymbol{d}_0, & \boldsymbol{d}_1, & \boldsymbol{d}_2 \end{bmatrix} = \begin{bmatrix} 1 & -\sqrt{\frac{3}{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & -\sqrt{2} \\ 1 & \sqrt{\frac{3}{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}. \tag{1.3}$$

Of course, $\boldsymbol{d}'_0 = \boldsymbol{j}'_3 = (1, 1, 1)$, and $\boldsymbol{d}_0, \boldsymbol{d}_1$ and $\boldsymbol{d}_2$ satisfy $\boldsymbol{d}'_i \boldsymbol{d}_j = 3\delta_{ij}$ for Kronecker δ_{ij}, $i, j = 0, 1, 2$.

We may note that the definition of factorial effects here is designed to keep homoscedastic property among the BLUE's obtained under the complete 3^m factorial design in order to compare the effects by their location parameters only.

Solving (1.2), we have

$$\begin{aligned} &\boldsymbol{\eta}(Z) = D_{(m)} \Theta(Z), \text{ or} \\ &\eta(j_1, j_2, \ldots, j_m) = \sum_{\substack{i_p=0,1,2 \\ p=1,2,\ldots,m}} d_{j_1 i_1} d_{j_2 i_2} \cdots d_{j_m i_m} \theta(i_1, i_2, \ldots, i_m). \end{aligned} \tag{1.4}$$

Let $U^r = \{p | i_p = r\}$ be a subset of $\Omega = \{1, 2, \ldots, m\}$ with a superscript r in which the arguments i_p of $\theta(i_1, i_2, \ldots, i_m)$ are equal to r for $r = 0, 1$ and 2. Then the factorial effect $\theta(i_1, i_2, \ldots, i_m)$ can be expressed alternatively as $\theta(U^1 U^2)$ since $U^0 = \Omega - U^1 - U^2$. If both U^1 and U^2 are null, the parameter or factorial effect $\theta(0, 0, \ldots, 0, 0)$ is called the general mean and is denoted alternatively by $\theta(\phi)$. If $|U^1 \cup U^2| = 1$ and $U^1 \cup U^2 = \{p\}$, then the parameters $\theta(0, 0, \ldots, 1, \ldots, 0)$ and $\theta(0, 0, \ldots, 2, \ldots, 0)$ both having single nonzero argument in the pth position are called the linear and the quadratic main effects of the factor $F(p)$, respectively. They may be denoted alternatively by $\theta(p^1)$ and $\theta(p^2)$, respectively. If $|U^1 \cup U^2| = 2$ and $U^1 \cup U^2 = \{p, q\}$, then the parameter $\theta(i_1, i_2, \ldots, i_m)$ having two nonzero arguments i_p and i_q is called the linear $\times$ linear , the linear $\times$ quadratic, the quadratic $\times$ linear or the quadratic $\times$ quadratic two-factor interactions of the factors $F(p)$ and $F(q)$ according as (i_p, i_q) is equal to (1,1), (1,2), (2,1) or (2,2), respectively. Those two-factor interactions may be denoted alternatively by $\theta(p^{i_p} q^{i_q})$ for $i_p, i_q = 1$ or 2, respectively. In general, if $|U^1 \cup U^2| = k$, then the parameter $\theta(i_1, i_2, \ldots, i_m)$

having k nonzero arguments with respect to k factors is called the k-factor interaction and is expressed as $\theta(U^1U^2)$ by indicating the sets of arguments U^1 and U^2.

Let T be a fraction of 3^m factorial design with m factors composed of n assemblies $(j_1^{(\alpha)}, j_2^{(\alpha)}, \dots, j_m^{(\alpha)})$; $j_p^{(\alpha)} = 0,\ 1$ or 2, $p = 1, 2, \dots, m$, $\alpha = 1, 2, \dots, n$; and suppose $\boldsymbol{y}(T)$ be the corresponding vector of observations, i.e.,

$$T = \begin{bmatrix} j_1^{(1)}, j_2^{(1)}, \dots, j_m^{(1)} \\ \vdots \\ j_1^{(\alpha)}, j_2^{(\alpha)}, \dots, j_m^{(\alpha)} \\ \vdots \\ j_1^{(n)}, j_2^{(n)}, \dots, j_m^{(n)} \end{bmatrix} \text{ and } \boldsymbol{y}(T) = \begin{bmatrix} y(j_1^{(1)}, j_2^{(1)}, \dots, j_m^{(1)}) \\ \vdots \\ y(j_1^{(\alpha)}, j_2^{(\alpha)}, \dots, j_m^{(\alpha)}) \\ \vdots \\ y(j_1^{(n)}, j_2^{(n)}, \dots, j_m^{(n)}) \end{bmatrix}. \tag{1.5}$$

The vector of observations of the design T is expressed as

$$\boldsymbol{y}(T) = E(T)\Theta + \boldsymbol{e}(T) \tag{1.6}$$

in terms of $E(T)$, Θ, and $\boldsymbol{e}(T)$, where Θ is the parameter vector obtained by rearranging $\Theta(\boldsymbol{Z})$ in a natural order of the number of factors and levels concerned, $E(T)$ is the design matrix whose element in the row and the column correspond respectively to αth observation and the effect $\theta(i_1, i_2, \dots, i_m)$ is given by

$$e(\alpha; (i_1, i_2, \dots, i_m)) = d_{j_1^{(\alpha)} i_1} d_{j_2^{(\alpha)} i_2} \dots d_{j_m^{(\alpha)} i_m} \tag{1.7}$$

and $\boldsymbol{e}(T)$ is the error vector with the usual assumption that the components are distributed uncorrelatedly with $(0, \sigma^2)$.

Since $d_{j0} = 1$ for every j,

$$\begin{aligned} &e(\alpha; \phi) = 1, \text{ for every } \alpha, \\ &e(\alpha; p^{i_p}) = d_{j_p^{(\alpha)} i_p}, \text{ for } p \in \Omega \text{ and } i_p = 1, 2, \\ &e(\alpha; p^{i_p} q^{i_q}) = d_{j_p^{(\alpha)} i_p} d_{j_q^{(\alpha)} i_q}, \text{ for } p \neq q \in \Omega \text{ and } i_p, i_q = 1, 2, \end{aligned} \tag{1.8}$$

and in general,

$$e(\alpha; U^1U^2) = \prod_{p \in U^1} d_{j_p^{(\alpha)} 1} \prod_{q \in U^2} d_{j_q^{(\alpha)} 2}.$$

The column vector of the design matrix $E(T)$ corresponding to the factorial effect $\theta(i_1, i_2, \dots, i_m)$ is expressed as:

$$\begin{aligned} \boldsymbol{d}(i_1, i_2, \dots, i_m) = (&d_{j_1^{(1)} i_1} d_{j_2^{(1)} i_2} \cdots d_{j_m^{(1)} i_m}, \dots, \\ &d_{j_1^{(\alpha)} i_1} d_{j_2^{(\alpha)} i_2} \cdots d_{j_m^{(\alpha)} i_m}, \dots, d_{j_1^{(n)} i_1} d_{j_2^{(n)} i_2} \cdots d_{j_m^{(n)} i_m})'. \end{aligned} \tag{1.9}$$

Since $d_{j0} = 1$ for every j, the above expression may be simplified to a vector of the products of d_{ji}'s with nonzero i's only.

In particular,

$$\begin{aligned} &\boldsymbol{d}(\phi) = (1, 1, \ldots, 1)', \text{ and} \\ &\boldsymbol{d}(p^{i_p}) = (d_{j_p^{(1)} i_p}, d_{j_p^{(2)} i_p}, \ldots, d_{j_p^{(\alpha)} i_p}, \ldots, d_{j_p^{(n)} i_p})' \end{aligned} \tag{1.10}$$

for $p \in \Omega$ and $i_p = 1, 2$.

In general,

$$\begin{aligned} \boldsymbol{d}(U^1U^2) = \Big(&\prod_{p \in U^1} d_{j_p^{(1)}1} \prod_{q \in U^2} d_{j_q^{(1)}2}, \prod_{p \in U^1} d_{j_p^{(2)}1} \prod_{q \in U^2} d_{j_q^{(2)}2} \\ &, \ldots, \prod_{p \in U^1} d_{j_p^{(\alpha)}1} \prod_{q \in U^2} d_{j_q^{(\alpha)}2}, \ldots, \prod_{p \in U^1} d_{j_p^{(n)}1} \prod_{q \in U^2} d_{j_q^{(n)}2} \Big)'. \end{aligned} \tag{1.11}$$

Definition 1. For a fractional 3^m factorial design T, the vector $\boldsymbol{d}(i_1, i_2, \ldots, i_m)$ or $\boldsymbol{d}(U^1U^2)$ is called the *loading vector* of a factorial effect $\theta(i_1, i_2, \ldots, i_m)$ or $\theta(U^1U^2)$.

Using loading vectors of $2m$ main effects given in (1.10), every loading vector can be obtained by the Schur products ($*$) of related loading vectors for main effects as is given by the formula (1.11). For a simplest example, $\boldsymbol{d}(p^{i_p} q^{i_q}) = \boldsymbol{d}(p^{i_p}) * \boldsymbol{d}(q^{i_q})$.

Let $S_p[\boldsymbol{x}]$ be the *spur* of a vector $\boldsymbol{x}$ being defined by the sum of its components.

Definition 2. The spur $S_p[\boldsymbol{d}(U^1U^2)]$ of the loading vector of a factorial effect $\theta(U^1U^2)$, denoted by $\gamma(U^1U^2)$, is called the *loading coefficient* of $\theta(U^1U^2)$ of the design T.

In particular,

$$\begin{aligned} &\gamma(\phi) = n, \\ &\gamma(p^{i_p}) = \sum_{\alpha=1}^{n} d_{j_p^{(\alpha)} i_p} \quad \text{for } p \in \Omega \text{ and } i_p = 1, 2, \\ &\gamma(p^{i_p} q^{i_q}) = \sum_{\alpha=1}^{n} d_{j_p^{(\alpha)} i_p} d_{j_q^{(\alpha)} i_q} \quad \text{for } p \neq q \in \Omega \text{ and } i_p, i_q = 1, 2, \end{aligned} \tag{1.12}$$

and, in general,

$$\gamma(U^1U^2) = \sum_{\alpha=1}^{n} \prod_{p \in U^1} d_{j_p^{(\alpha)}1} \prod_{q \in U^2} d_{j_q^{(\alpha)}2}. \tag{1.13}$$

The normal equation for estimating Θ is given by

$$M(T)\Theta = E'(T)\boldsymbol{y}(T) \tag{1.14}$$

where $M(T) = E'(T)E(T)$ is the information matrix of a design T.

The element $\varepsilon(i_1 i_2 \ldots i_m, k_1 k_2 \ldots k_m)$ of the information matrix in the $\theta(i_1, i_2, \ldots, i_m)$ row and the $\theta(k_1, k_2, \ldots, k_m)$ column is given by

$$\begin{aligned} &\varepsilon(i_1 i_2 \ldots i_m, k_1 k_2 \ldots k_m) \\ &= \sum_{\alpha=1}^{n} d_{j_1^{(\alpha)} i_1} d_{j_2^{(\alpha)} i_2} \cdots d_{j_m^{(\alpha)} i_m} d_{j_1^{(\alpha)} k_1} d_{j_2^{(\alpha)} k_2} \cdots d_{j_m^{(\alpha)} k_m} \\ &= \sum_{\alpha=1}^{n} \prod_{p \in \Omega} d_{j_p^{(\alpha)} i_p} d_{j_p^{(\alpha)} k_p}. \end{aligned} \tag{1.15}$$

The following is a modification of the lemma due to Kuwada [3].

Lemma 1. *Every product $d_{ji}d_{jk}$ of the elements of the matrix D satisfy the following irrespective of the value $j = 0, 1, 2$, i.e.,*

$$\begin{aligned} &d_{j0}d_{j0} = d_{j0} = 1,\ d_{j0}d_{j1} = d_{j1}d_{j0} = d_{j1},\ d_{j0}d_{j2} = d_{j2}d_{j0} = d_{j2}, \\ &d_{j1}d_{j1} = 1 + \sqrt{\frac{1}{2}}d_{j2},\ d_{j2}d_{j2} = 1 - \sqrt{\frac{1}{2}}d_{j2},\ d_{j1}d_{j2} = \sqrt{\frac{1}{2}}d_{j1}. \end{aligned} \tag{1.16}$$

Let $K^{xy} = (U^x \cap V^y) \cup (U^y \cap V^x)$ with cardinality $|K^{xy}| = k_{xy}$ for every pair of $x \le y = 0, 1$, and 2. Then, we have:

Theorem 2. *The $(\theta(U^1U^2), \theta(V^1V^2))$ element of the information matrix $M(T)$ of a fractional 3^m factorial design T is given by*

$$\begin{aligned} \varepsilon(U^1U^2, V^1V^2) = &\sum_{\alpha=1}^{n} \prod_{p \in K^{01}} d_{j_p^{(\alpha)} 1} \prod_{q \in K^{02}} d_{j_q^{(\alpha)} 2} \\ &\cdot \prod_{r \in K^{11}} \Big(1 + \sqrt{\frac{1}{2}} d_{j_r^{(\alpha)} 2}\Big) \cdot \prod_{s \in K^{22}} \Big(1 - \sqrt{\frac{1}{2}} d_{j_s^{(\alpha)} 2}\Big) \cdot \prod_{t \in K^{12}} \Big(\sqrt{\frac{1}{2}} d_{j_t^{(\alpha)} 1}\Big). \end{aligned} \tag{1.17}$$

Definition 3. The first row $\Gamma(T)$ of the information matrix $M(T)$ which is composed of all loading coefficients $\gamma(U^1U^2)$'s arranged in a natural order of $\theta(U^1U^2)$ is called the *characteristic vector* of $M(T)$ or the design T.

Theorem 3. *The information matrix $M(T)$ of the design T is completely determined by its characteristic vector $\Gamma(T)$.*

Proof. The formula (1.17) shows that every component of $M(T)$ is a linear combination of the terms each composed of the sum of the products of at most m $d_{j_p^{(\alpha)}i}$'s with respect to α, *i.e.*, the loading coefficients.

In particular,

$\varepsilon(\phi,\phi)=n.$

$\varepsilon(\phi,p^{i_p})=\gamma(p^{i_p})$ for $p\in\Omega$ and $i_p=1,2.$

$\varepsilon(\phi,p^{i_p}q^{i_q})=\gamma(p^{i_p}q^{i_q})$ for $p\neq q\in\Omega$ and $i_p,i_q=1,2.$

$\varepsilon(\phi,U^1U^2)=\gamma(U^1U^2).$

$\varepsilon(p^1,p^1)=n+\sqrt{\frac{1}{2}}\gamma(p^2),\ \varepsilon(p^1,p^2)=\sqrt{\frac{1}{2}}\gamma(p^1),$ and

$\varepsilon(p^2,p^2)=n-\sqrt{\frac{1}{2}}\gamma(p^2),$ for $p\in\Omega.$

$\varepsilon(p^{i_p},q^{i_q})=\gamma(p^{i_p}q^{i_q})$ for $p\neq q\in\Omega$ and $i_p,i_q=1,2.$

$\varepsilon(p^1,p^1q^{i_q})=\gamma(q^{i_q})+\sqrt{\frac{1}{2}}\gamma(p^2q^{i_q}),\ \varepsilon(p^1,p^2q^{i_q})=\sqrt{\frac{1}{2}}\gamma(p^1q^{i_q}),$ and

$\varepsilon(p^2,p^2q^{i_q})=\gamma(q^{i_q})-\sqrt{\frac{1}{2}}\gamma(p^2q^{i_q}),$ for $p\neq q\in\Omega$ and $i_q=1,2.$

$\varepsilon(p^{i_p},q^{i_q}r^{i_r})=\gamma(p^{i_p}q^{i_q}r^{i_r})$ for $q\neq r,p\neq q,r\in\Omega$ and $i_p,\ i_q,\ i_r=1,2.$

$\varepsilon(p^1q^1,p^1q^1)=n+\sqrt{\frac{1}{2}}\Big(\gamma(p^2)+\gamma(q^2)\Big)+\frac{1}{2}\gamma(p^2q^2),$

$\varepsilon(p^1q^1,p^1q^2)=\sqrt{\frac{1}{2}}\gamma(q^1)+\frac{1}{2}\gamma(p^2q^1),$

$\varepsilon(p^1q^1,p^2q^2)=\frac{1}{2}\gamma(p^1q^1),$

$\varepsilon(p^1q^2,p^1q^2)=n+\sqrt{\frac{1}{2}}\Big(\gamma(p^2)-\gamma(q^2)\Big)-\frac{1}{2}\gamma(p^2q^2),$

$\varepsilon(p^1q^2,p^2q^1)=\frac{1}{2}\gamma(p^1q^1),$

$\varepsilon(p^1q^2,p^2q^2)=\sqrt{\frac{1}{2}}\gamma(p^1)-\frac{1}{2}\gamma(p^1q^2),$ and

$\varepsilon(p^2q^2,p^2q^2)=n-\sqrt{\frac{1}{2}}\Big(\gamma(p^2)+\gamma(q^2)\Big)+\frac{1}{2}\gamma(p^2q^2),$ for $p\neq q\in\Omega.$

$\varepsilon(p^1q^{i_q},p^1r^{i_r})=\gamma(q^{i_q}r^{i_r})+\sqrt{\frac{1}{2}}\gamma(p^2q^{i_q}r^{i_r}),$

$\varepsilon(p^1q^{i_q},p^2r^{i_r})=\sqrt{\frac{1}{2}}\gamma(p^1q^{i_q}r^{i_r}),$ and

$\varepsilon(p^2q^{i_q},p^2r^{i_r})=\gamma(q^{i_q}r^{i_r})-\sqrt{\frac{1}{2}}\gamma(p^2q^{i_q}r^{i_r}),$

for $p\neq q,r\neq p,q\in\Omega$ and $i_q,i_r=1,2.$

$\varepsilon(p^{i_p}q^{i_q}, r^{i_r}s^{i_s}) = \gamma(p^{i_p}q^{i_q}r^{i_r}s^{i_s})$ for $p \neq q,\ r \neq p, q,\ s \neq p, q, r \in \Omega$ and $i_p, i_q, i_r, i_s = 1, 2.$

§2. Normal equation of a fractional 3^m factorial design

The first member of the normal equation (1.14) is given by

$$n\theta(\phi) + \sum_{k=1}^{m} \sum_{|V^1 \cup V^2| = k} \gamma(V^1V^2)\theta(V^1V^2) = \boldsymbol{d}'(\phi)\boldsymbol{y}(T), \tag{2.1}$$

$$\text{or } = \sum_{\alpha=1}^{n} y(j_1^{(\alpha)}, j_2^{(\alpha)}, \ldots, j_m^{(\alpha)}).$$

In some sense, the left hand member of equation (2.1) may be called the *defining formula*. This is an extension of the so-called *defining relation* introduced by Box and Hunter [1,2] in the case of 2^m factorial designs.

Those members corresponding to the main effects $\theta(p^1)$ and $\theta(p^2)$, for $p \in \Omega$, are given by

$$\begin{aligned} &\gamma(p^1)\theta(\phi) + \Big(n + \sqrt{\frac{1}{2}}\gamma(p^2)\Big)\theta(p^1) + \sqrt{\frac{1}{2}}\gamma(p^1)\theta(p^2) \\ &+ \sum_{q \neq p}\Big(\gamma(p^1q^1)\theta(q^1) + \gamma(p^1q^2)\theta(q^2)\Big) \\ &+ \sum_{k=2}^{m} \sum_{|V^1 \cup V^2| = k} \varepsilon(p^1, V^1V^2)\theta(V^1V^2) = \boldsymbol{d}'(p^1)\boldsymbol{y}(T), \\ &\text{or } = \sum_{\alpha=1}^{n} d_{j_p^{(\alpha)}1} y(j_1^{(\alpha)}, j_2^{(\alpha)}, \ldots, j_m^{(\alpha)}) \text{ for } \theta(p^1), \text{ and} \end{aligned} \tag{2.2}$$

$$\begin{aligned} &\gamma(p^2)\theta(\phi) + \sqrt{\frac{1}{2}}\gamma(p^1)\theta(p^1) + \Big(n - \sqrt{\frac{1}{2}}\gamma(p^2)\Big)\theta(p^2) \\ &+ \sum_{q \neq p}\Big(\gamma(p^2q^1)\theta(q^1) + \gamma(p^2q^2)\theta(q^2)\Big) \\ &+ \sum_{k=2}^{m} \sum_{|V^1 \cup V^2| = k} \varepsilon(p^2, V^1V^2)\theta(V^1V^2) = \boldsymbol{d}'(p^2)\boldsymbol{y}(T), \\ &\text{or } = \sum_{\alpha=1}^{n} d_{j_p^{(\alpha)}2} y(j_1^{(\alpha)}, j_2^{(\alpha)}, \ldots, j_m^{(\alpha)}) \text{ for } \theta(p^2). \end{aligned} \tag{2.3}$$

The member corresponding to $\theta(p^{i_p}q^{i_q})$, for $p \neq q \in \Omega$, is given by

$$\gamma(p^{i_p}q^{i_q})\theta(\phi) + \sum_{r} \sum_{i_r = 1,2} \varepsilon(p^{i_p}q^{i_q}, r^{i_r})\theta(r^{i_r}) \tag{2.4}$$

$$+\sum_{r\neq s}\sum_{i_r,i_s=1,2}\varepsilon(p^{i_p}q^{i_q},r^{i_r}s^{i_s})\theta(r^{i_r}s^{i_s})$$
$$+\sum_{k=3}^{m}\sum_{|V^1\cup V^2|=k}\varepsilon(p^{i_p}q^{i_q},V^1V^2)\theta(V^1V^2)=\boldsymbol{d}'(p^{i_p}q^{i_q})\boldsymbol{y}(T).$$

In general, the member corresponding to $\theta(U^1U^2)$, which may be called *principal equation* of estimating the parameter, is given by

$$\sum_{k=0}^{m}\sum_{|V^1\cup V^2|=k}\varepsilon(U^1U^2,V^1V^2)\theta(V^1V^2)=\boldsymbol{d}'(U^1U^2)\boldsymbol{y}(T), \tag{2.5}$$
$$\text{or } =\sum_{\alpha=1}^{n}\prod_{p\in U^1}d_{j_p^{(\alpha)}1}\prod_{q\in U^2}d_{j_q^{(\alpha)}2}y(j_1^{(\alpha)},j_2^{(\alpha)},\dots,j_m^{(\alpha)}).$$

Those left hand members of (2.2), (2.3), (2.4) and, in general, (2.5) may be called the derived formulas of the design.

§3. Designs derived from three-symbol orthogonal arrays of strength 2

Let T in (1.5) be a design derived from a three-symbol orthogonal array of strength 2, m constraints and index λ, denoted by 3-OA$(2,m,\lambda)$, having $n=9\lambda$ runs. Then, since $\sum_{\alpha=1}^{n}d_{j_p^{(\alpha)}i_p}=0$ and also $\sum_{\alpha=1}^{n}d_{j_p^{(\alpha)}i_p}d_{j_q^{(\alpha)}i_q}=0$ hold true for all $p\neq q\in\Omega$ and $i_p,i_q=1,2$, all loading coefficients $\gamma(p^{i_p})$ of $2m$ main effects and those $\gamma(p^{i_p}q^{i_q})$ of $2m(m-1)$ two-factor interactions vanish simultaneously.

In such a circumstance, we have the following:

$\varepsilon(\phi,\phi)=n,\ \varepsilon(\phi,p^{i_p})=\varepsilon(\phi,p^{i_p}q^{i_q})=0,$ for $p\neq q\in\Omega$ and $i_p,i_q=1,2.$

$\varepsilon(p^1,p^1)=\varepsilon(p^2,p^2)=n$ and $\varepsilon(p^1,p^2)=\varepsilon(p^2,p^1)=0,$ for $p\in\Omega.$

$\varepsilon(p^{i_p},q^{i_q})=0$ for $p\neq q\in\Omega$ and $i_p,i_q=1,2.$

$\varepsilon(p^1,p^1q^{i_q})=\varepsilon(p^1,p^2q^{i_q})=\varepsilon(p^2,p^1q^{i_q})=\varepsilon(p^2,p^2q^{i_q})=0$
for $p\neq q\in\Omega$ and $i_q=1,2.$

$\varepsilon(p^{i_p},q^{i_q}r^{i_r})=\gamma(p^{i_p}q^{i_q}r^{i_r})$ for $q\neq r,p\neq q,r\in\Omega$ and $i_p,i_q,i_r=1,2.$

$\varepsilon(p^1q^1,p^1q^1)=\varepsilon(p^1q^2,p^1q^2)=\varepsilon(p^2q^2,p^2q^2)=n,$ and

$\varepsilon(p^1q^1,p^1q^2)=\varepsilon(p^1q^1,p^2q^1)=\varepsilon(p^1q^1,p^2q^2)=\varepsilon(p^1q^2,p^1q^1)=\varepsilon(p^1q^2,p^2q^1)$
$=\varepsilon(p^1q^2,p^2q^2)=\varepsilon(p^2q^2,p^1q^1)=\varepsilon(p^2q^2,p^1q^2)=\varepsilon(p^2q^2,p^2q^1)=0,$
for $p\neq q\in\Omega.$

$\varepsilon(p^1q^{i_q},p^1r^{i_r})=\sqrt{\frac{1}{2}}\gamma(p^2q^{i_q}r^{i_r}),$

$$\varepsilon(p^2q^{i_q}, p^2r^{i_r}) = -\sqrt{\frac{1}{2}}\gamma(p^2q^{i_q}r^{i_r}), \text{ and}$$
$$\varepsilon(p^1q^{i_q}, p^2r^{i_r}) = \sqrt{\frac{1}{2}}\gamma(p^1q^{i_q}r^{i_r}),$$
for $p \neq q, r \neq p, q \in \Omega$ and $i_q, i_r = 1, 2$.
$$\varepsilon(p^{i_p}q^{i_q}, r^{i_r}s^{i_s}) = \gamma(p^{i_p}q^{i_q}r^{i_r}s^{i_s})$$
for $p \neq q, r \neq p, q, s \neq r, p, q \in \Omega$ and $i_p, i_q, i_r, i_s = 1, 2$.

In this orthogonal case, the principal member of the normal equation for the general mean $\theta(\phi)$ is given by

$$n\theta(\phi) + \sum_{k=3}^{m} \sum_{|V^1 \cup V^2|=k} \gamma(V^1V^2)\theta(V^1V^2) = \boldsymbol{d}'(\phi)\boldsymbol{y}(T), \tag{3.1}$$
$$\text{or } = \sum_{\alpha=1}^{n} y(j_1^{(\alpha)}, j_2^{(\alpha)}, \ldots, j_m^{(\alpha)}).$$

The principal member of the normal equation for the main effect $\theta(p^{i_p})$ is given by

$$n\theta(p^{i_p}) + \sum_{\{q,r\}(q,r\neq p)} \sum_{i_q,i_r=1,2} \gamma(p^{i_p}q^{i_q}r^{i_r})\theta(q^{i_q}r^{i_r}) \tag{3.2}$$
$$+ \sum_{k=3}^{m} \sum_{|V^1 \cup V^2|=k} \varepsilon(p^{i_p}, V^1V^2)\theta(V^1V^2) = \boldsymbol{d}'(p^{i_p})\boldsymbol{y}(T),$$
$$\text{or } = \sum_{\alpha=1}^{n} d_{j_p^{(\alpha)}1}\, y(j_1^{(\alpha)}, j_2^{(\alpha)}, \ldots, j_m^{(\alpha)}),$$
for $p \in \Omega$, $i_p = 1, 2$.

The principal member corresponding to the two-factor interaction $\theta(p^{i_p}q^{i_q})$ is given by

$$\sum_{r(\neq p,q)} \sum_{i_r=1,2} \gamma(p^{i_p}q^{i_q}r^{i_r})\theta(r^{i_r}) + n\theta(p^{i_p}q^{i_q}) \tag{3.3}$$
$$+ \sum_{\{r,s\}(\neq\{p,q\})} \sum_{i_r,i_s=1,2} \varepsilon(p^{i_p}q^{i_q}, r^{i_r}s^{i_s})\theta(r^{i_r}s^{i_s})$$
$$+ \sum_{k=3}^{m} \sum_{|V^1 \cup V^2|=k} \varepsilon(p^{i_p}q^{i_q}, V^1V^2)\theta(V^1V^2) = \boldsymbol{d}'(p^{i_p}q^{i_q})\boldsymbol{y}(T),$$
$$\text{or } = \sum_{\alpha=1}^{n} d_{j_p^{(\alpha)}i_p} d_{j_q^{(\alpha)}i_q}\, y(j_1^{(\alpha)}, j_2^{(\alpha)}, \ldots, j_m^{(\alpha)}),$$
for $p \neq q \in \Omega$, $i_p, i_q = 1, 2$.

If three-factor or more interactions are assumed to be negligible, those equations (3.1), (3.2) and (3.3) may be simplified as follows:

(3.4) $n\theta(\phi) = \boldsymbol{d}'(\phi)\boldsymbol{y}(T)$,

(3.5) $n\theta(p^{i_p}) + \sum_{\{q,r\}(q,r\neq p)} \sum_{i_q,i_r=1,2} \gamma(p^{i_p}q^{i_q}r^{i_r})\theta(q^{i_q}r^{i_r}) = \boldsymbol{d}'(p^{i_p})\boldsymbol{y}(T)$,

for $\theta(p^{i_p}),\ p \in \Omega,\ i_p = 1,2,$ and

(3.6) $\sum_{r(\neq p,q)} \sum_{i_r=1,2} \gamma(p^{i_p}q^{i_q}r^{i_r})\theta(r^{i_r}) + n\theta(p^{i_p}q^{i_q})$

$+ \sum_{\{r,s\}(\neq\{p,q\})} \sum_{i_r,i_s=1,2} \varepsilon(p^{i_p}q^{i_q}, r^{i_r}s^{i_s})\theta(r^{i_r}s^{i_s}) = \boldsymbol{d}'(p^{i_p}q^{i_q})\boldsymbol{y}(T)$,

for $\theta(p^{i_p}q^{i_q}),\ p \neq q \in \Omega,\ i_p, i_q = 1,2.$

It can be seen that in estimating the main effects $\theta(p^{i_p})$'s and the two-factor interactions $\theta(p^{i_p}q^{i_q})$'s using principal equations (3.5) and (3.6), the estimates may be more or less confounded by several effects, i.e., the estimate of a main effect may be partially confounded by at most $4\times\binom{m-1}{2}$ two-factor interactions and that of a two-factor interaction may be partially confounded by $2\times(m-2)$ main effects and $2m(m-1)-4$ two-factor interactions.

With respect to the confounding coefficient of a two-factor interaction to the main effect to be estimated and that to the two-factor interaction to be estimated, the following theorem shows that,

$$|\gamma(p^{i_p}q^{i_q}r^{i_r})|/n \leq 1, \text{ and } |\varepsilon(p^{i_p}q^{i_q}, r^{i_r}s^{i_s})|/n \leq 1,$$

hold true, respectively.

Theorem 4. *The absolute value of the coefficient* $\gamma(p^{i_p}q^{i_q}r^{i_r})$ *of* $\theta(q^{i_q}r^{i_r})$ *in equation* (3.5) *is bounded by* n. *The absolute values of the coefficients* $\gamma(p^{i_p}q^{i_q}r^{i_r})$ *of* $\theta(r^{i_r})$ *and* $\varepsilon(p^{i_p}q^{i_q}, r^{i_r}s^{i_s})$ *of* $\theta(r^{i_r}s^{i_s})$ *in* (3.6) *are also bounded by* n, *respectively.*

Proof. In proving the theorem, it is sufficient to show the following:

$$(\gamma(p^{i_p}q^{i_q}r^{i_r}))^2 = (\sum_{\alpha=1}^{n} d_{j_p^{(\alpha)}i_p} d_{j_q^{(\alpha)}i_q} d_{j_r^{(\alpha)}i_r})^2$$
$$\leq (\sum_{\alpha=1}^{n} d^2_{j_p^{(\alpha)}i_p})(\sum_{\alpha=1}^{n} d^2_{j_q^{(\alpha)}i_q} d^2_{j_r^{(\alpha)}i_r}) = n^2, \text{ and}$$
$$(\varepsilon(p^{i_p}q^{i_q}, r^{i_r}s^{i_s}))^2 = (\sum_{\alpha=1}^{n} d_{j_p^{(\alpha)}i_p} d_{j_q^{(\alpha)}i_q} d_{j_r^{(\alpha)}i_r} d_{j_s^{(\alpha)}i_s})^2$$
$$\leq (\sum_{\alpha=1}^{n} d^2_{j_p^{(\alpha)}i_p} d^2_{j_q^{(\alpha)}i_q})(\sum_{\alpha=1}^{n} d^2_{j_r^{(\alpha)}i_r} d^2_{j_s^{(\alpha)}i_s}) = n^2.$$

§4. 18-run orthogonal 3^4 factorial designs

An orthogonal n-run 3^4 factorial design can be provided by a three-symbol orthogonal array, 3-OA(t, m, λ), of size n, m constraints, strength $t = 2$ and index λ, where $n = 9\lambda$.

The class of three-symbol orthogonal arrays of strength t having $m = t + 2$ and index $\lambda = 2$ (3-OA$(t, m = t + 2, \lambda = 2)$) has been investigated by Yamamoto, Fujii and Mitsuoka [4]. They have shown that the number of all possible 3-OA(2,4,2)'s amounts to 31,356 and these arrays are classified into 12 cosets with respect to the group of the symbol (level) and column (factor) permutations. In the case of 3-OA(3,5,2)'s the number amounts to 62,944 and these arrays are classified into 4 cosets. In their subsequent paper [5], the class of three-symbol orthogonal arrays of strength 2 and index 2 having maximal or saturated ($m = 7$) constraints have been investigated and it has been shown that there are three nonisomorphic classes of 3-OA$(t=2, m=7, \lambda=2)$.

Representative arrays of the 3 cosets of 3-OA$(t=2, m=7, \lambda=2)$ (labeled as [A], [B], [C]) and those of 12 cosets of 3-OA$(t=2, m=4, \lambda=2)$ (labeled as (1), (2), ..., (12)) will be referred to here in Table 1.

Table 1. Representatives of the three isomorphic classes of saturated 3-OA($t=2, m=7, \lambda=2$) and twelve classes of 3-OA($t=2, m=4, \lambda=2$)

[A]	[B]	[C]
0021000	0021000	0021000
0022111	0022111	0022111
0110002	0110002	0110002
0112221	0112221	0112221
0200112	0200120	0200120
0201220	0201212	0201212
1010120	1010112	1010210
1011212	1011220	1011122
1101011	1101011	1101011
1102100	1102100	1102100
1220021	1220021	1220021
1222202	1222202	1222202
2000201	2000201	2000201
2002022	2002022	2002022
2120210	2120210	2120112
2121122	2121122	2121220
2211101	2211101	2211101
2212010	2212010	2212010

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
0022	0022	0022	0022	0022	0021	0021	0021	0021	0021	0021	0012
0022	0022	0022	0022	0022	0022	0022	0022	0022	0022	0022	0021
0111	0111	0110	0110	0101	0110	0110	0110	0110	0100	0102	0102
0111	0111	0111	0111	0110	0112	0112	0112	0112	0110	0110	0120
0200	0200	0200	0200	0200	0200	0200	0200	0200	0202	0200	0200
0200	0200	0201	0201	0211	0201	0201	0201	0201	0211	0211	0211
1010	1001	1001	1000	1001	1010	1000	1000	1000	1000	1002	1001
1010	1010	1011	1011	1010	1011	1010	1010	1011	1011	1010	1010
1102	1102	1102	1102	1112	1101	1101	1102	1102	1101	1101	1111
1102	1120	1120	1121	1121	1102	1121	1121	1121	1122	1121	1122
1221	1212	1212	1212	1202	1220	1212	1211	1212	1212	1212	1202
1221	1221	1220	1220	1220	1222	1222	1222	1220	1220	1220	1220
2001	2001	2000	2001	2000	2000	2002	2002	2002	2002	2000	2000
2001	2010	2010	2010	2011	2002	2011	2011	2010	2010	2011	2022
2120	2102	2102	2102	2102	2120	2102	2101	2101	2112	2112	2101
2120	2120	2121	2120	2120	2121	2120	2120	2120	2121	2120	2110
2212	2212	2212	2212	2212	2211	2211	2212	2211	2201	2201	2212
2212	2221	2221	2221	2221	2212	2220	2220	2222	2220	2222	2221

After editing the results appearing in Yamamoto, Fujii and Mitsuoka [5], the possibility of embedding those 12 classes of 3-OA($t = 2, m = 4, \lambda = 2$) into those 3 classes of saturated 3-OA($t = 2, m = 7, \lambda = 2$) having maximal constraints can be summarized in the following Table 2. This table, of course, shows the possibility of deriving the former from the latter.

Table 2. Possibility of deriving 3-OA$(t=2, m=4, \lambda=2)$ from saturated 3-OA$(t=2, m=7, \lambda=2)$

		3-OA$(t=2, m=4, \lambda=2)$											
		(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
3-OA$(t=2,$	[A]	×	×	×	×	×	○	○	×	○	×	×	○
$m=7,$	[B]	×	×	×	×	×	○	×	○	○	×	○	○
$\lambda=2)$	[C]	×	×	×	×	×	○	×	×	×	○	○	○

(○: possible, ×: impossible)

With respect to each of the 18-run orthogonal 3^4 factorial designs provided by the 12 representative arrays (1), (2), ..., (12), the loading vectors, the characteristic vector and the information matrix are calculated. Those 12 information matrices $M(2, T)$, under the assumption that three or more factor interactions are negligible, will be given in Table 3.

The number of two-factor interactions actually confounded with (though partially) the main effect to be estimated by the principal equation is at most 12 in this case. The number, however, varies from a main effect to another and from a design to another. The largest is 12 which can be seen in the design (1) and some of others and the smallest is 4 which can be seen in the design (7).

The average of the confounding coefficients among 12 two-factor interactions having the possibility of confounding also varies from a main effect to another and a design to another(see Table 3).

These considerations along Table 3 show that the design (11) and also the design (7) seem to be recommendable for the practical use.

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