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GENERALIZATION OF GAUSSIAN ESTIMATES AND INTERPOLATION OF THE SPECTRUM IN L^p

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Abstract. Recently, it has been revealed that the semigroups satisfying Gaussian estimates inherit some of the nice properties enjoyed by the Gaussian semigroup itself. Arendt [1] gives a result in this direction, by proving the invariance of the spectrum of the generators of consistent C_0-semigroups with Gaussian estimates. In this paper, we generalize this result to the semigroups estimated by the one generated by the fractional power $-(I-\Delta)^\alpha$ $(1/2 < \alpha \le 1)$.

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§1. Introduction

Let $\Omega \subset \mathbf{R}^N$ be an open set, and suppose that a C_0-semigroup $T_p = \{T_p(t)\}_{t\ge 0}$ on $L^p(\Omega)$ with generator A_p is given for each $1 \le p < \infty$. Assume further that T_p's are consistent in the sense that $T_p(t) = T_q(t)$ on $L^p(\Omega) \cap L^q(\Omega)$ for all $t \ge 0$. Then it is natural to expect that the spectrum $\sigma(A_p)$ of A_p is independent of p. Unfortunately, this is not always true. Some simple but subtle examples are given in Arendt [1, Section 3]. However, concerning a Schrödinger operator as the generator of consistent semigroups, the following result was proved by Hempel and Voigt: *Let V be a real-valued measurable function such that V^+ is admissible and $V^- \in \hat{K}_N$ with $c_N(V^-) < 1$. Then $\sigma(H_{p,V})$ is independent of $p \in [1,\infty)$.* Here, $H_{p,V} = -\Delta + V$ denotes a Schrödinger operator acting in $L^p(\mathbf{R}^N)$ with suitable domain that generates consistent C_0-semigroups on $L^p(\mathbf{R}^N)$ ([5, Theorem]). On the other hand, Arendt [1] proved the following result: *Assume that T_2 satisfies an upper Gaussian estimate. Then $\rho_\infty(A_p)$*

is independent of $p \in [1, \infty)$. *If, in addition,* A_2 *is self-adjoint, then* $\sigma(A_p)$ *is independent of* $p \in [1, \infty)$. Here, $\rho(A_p) := \mathbf{C} \backslash \sigma(A_p)$ and $\rho_\infty(A_p)$ is the connected component of $\rho(A_p)$ which contains a right half-plane $\{\lambda \in \mathbf{C} : \mathrm{Re}\lambda > w\}$ for some $w \in \mathbf{R}$ ([1, Theorem 4.2, Corollary 4.3]). It is well known that if V^+ is admissible and $V^- \in \hat{K}_N$ with $c_N(V^-) = 0$ then the form sum of $-\Delta$ and V generates a (positive) C_0-semigroup on $L^2(\mathbf{R}^N)$ satisfying an upper Gaussian estimate. Therefore, the result of Arendt partly contains [5, Theorem].

One of the key to the proof of Arendt [1] is the estimate of the integral kernel of $T_2(t)$ by a (modified) heat kernel. So, one is naturally led to the question whether an estimate by a certain well-behaved kernel also guarantees the p-independence of $\sigma(A_p)$. The purpose of this paper is to partly answer this question affirmatively by showing that the estimate by the integral kernel of $e^{-t(I-\Delta)^\alpha}$ for $1/2 < \alpha \leq 1$ does imply the p-independence of $\sigma(A_p)$ (Theorem 2.7, Corollary 2.8). In the case where $\alpha = 1$, the assertions of Arendt is equivalent to that of ours (*cf.* Remark 2.2 below). As to the reason why we consider $(I - \Delta)^\alpha$ instead of $(-\Delta)^\alpha$, see the remark following the statement of Theorem 2.7.

The paper consists of two parts. In section 2, we prove the main theorem. In section 3, we apply the results in section 2 to some examples.

In this paper, ξ denotes a vector in $\mathbf{C}^N$ and $\xi_1, \cdots, \xi_N \in \mathbf{C}$ denote the components of ξ: $\xi = (\xi_1, \cdots, \xi_N)$. $\xi \in \mathbf{C}^N$ is also written as $\xi = \eta + i\zeta$ $(\eta, \zeta \in \mathbf{R}^N)$. $\xi^2 \in \mathbf{C}$ is the number defined by $\sum_{j=1}^N \xi_j^2$, and $|\xi|^2$ denotes the length of the vector ξ, defined by $|\xi|^2 = \sum_{j=1}^N |\xi_j|^2 = |\eta|^2 + |\zeta|^2$. In the case of $\xi \in \mathbf{R}^N$, ξ^2 coincides with $|\xi|^2$. We also use the symbol Re ξ_j or Im ξ_j to denote the real and imaginary part of ξ_j, respectively (Re $\xi_j = \eta_j$, Im $\xi_j = \zeta_j$).

In the following, constants "C" may vary from place to place.

§2. The main results

Throughout this paper, let $\Omega \subset \mathbf{R}^N$ denote an open set and let $T = \{T(t)\}_{t \geq 0}$ [resp. $S = \{S(t)\}_{t \geq 0}$] be a C_0-semigroup on $L^2(\Omega)$ [resp. $L^2(\mathbf{R}^N)$] with generator A [resp. B]. Assume that $S(t)$ $(t \geq 0)$ is positive, i.e., $S(t)f \geq 0$ for $f \geq 0$ (Nagel (ed.) [9] is a standard reference book for the theory of positive semigroups). We identify $L^2(\Omega)$ with a subspace of $L^2(\mathbf{R}^N)$ by considering the elements of $L^2(\Omega)$ to have value 0 on $\mathbf{R}^N \backslash \Omega$.

To make things clear, we introduce the notion of essential domination of semigroups as a generalization of that of domination (see [9, p. 269]) and upper Gaussian estimate ([1]).

Definition 2.1. We say that $T = \{T(t)\}_{t\geq 0}$ is essentially dominated by $S = \{S(t)\}_{t\geq 0}$ if there exist $M > 0$, $\omega \in \mathbf{R}$ and $b > 0$ such that

$$|T(t)f| \leq Me^{\omega t}S(bt)|f| \tag{2.1}$$

holds for all $t \geq 0$ and $f \in L^2(\Omega)$.

Remark 2.2. (1) In the case where $S(t) = e^{t\Delta}$, T is essentially dominated by S iff T satisfies an upper Gaussian estimate (see [1, Definition 4.1], for definition). For the proof, see [1, p. 1160].
(2) For every $\varepsilon > 0$, T is essentially dominated by $S(t) = e^{tB}$ iff T is essentially dominated by $e^{-t(\varepsilon - B)}$.

Next we collect the basic facts concerning the semigroup generated by the fractional power $(I-\Delta)^\alpha$. Here, Δ denotes the usual Laplacian in $L^2(\mathbf{R}^N)$ with domain $H^2(\mathbf{R}^N)$. For every real number α, $(I-\Delta)^\alpha$ is a positive definite self-adjoint operator in $L^2(\mathbf{R}^N)$, and hence $S_\alpha(t) := e^{-t(I-\Delta)^\alpha}$ is a C_0-semigroup on $L^2(\mathbf{R}^N)$. Especially in the case of $0 < \alpha \leq 1$, $S_\alpha(t)$ possesses the following nice properties.

Proposition 2.3. *Let $0 < \alpha \leq 1$. Then the following hold.*
(1) $S_\alpha(t) \geq 0$ $(t \geq 0)$.
(2) *For each $t > 0$, there exists $0 \leq K_\alpha(t, \cdot) \in L^1(\mathbf{R}^N)$ such that*

$$S_\alpha(t)f = K_\alpha(t, \cdot) * f \quad (f \in L^2(\mathbf{R}^N)), \tag{2.2}$$

where $u * v$ is the convolution of u and v.
(3) *For each $1 \leq p < \infty$, $t > 0$, a bounded linear operator $S_{\alpha,p}(t)$ on $L^p(\mathbf{R}^N)$ is defined by the following formula:*

$$S_{\alpha,p}(t)f := K_\alpha(t, \cdot) * f \quad (f \in L^p(\mathbf{R}^N)). \tag{2.3}$$

Then $S_{\alpha,p}$ are consistent positive C_0-semigroups of contractions on $L^p(\mathbf{R}^N)$ such that $S_{\alpha,2} = S_\alpha$. (Note that $K_\alpha(\cdot, \cdot)$ is independent of $p \in [1, \infty)$.)

Proof. Since the assertions are well known for $\alpha = 1$, we assume $0 < \alpha < 1$.
(1) It is well known that the following equality holds for $\lambda \geq 0$ (cf. [11, p. 37]):

$$[\lambda + (I-\Delta)^\alpha]^{-1} = \frac{\sin \pi\alpha}{\pi} \int_0^\infty \frac{\mu^\alpha(\mu + 1 - \Delta)^{-1}}{\mu^{2\alpha} + 2\lambda\mu^\alpha \cos \pi\alpha + \lambda^2} d\mu.$$

Because of the positivity of $e^{t\Delta} \geq 0$, $(\mu + 1 - \Delta)^{-1} \geq 0$ for $\mu \geq 0$, and so $0 \leq S_\alpha(t)$ $(t \geq 0)$.

(2) Since

$$\mathcal{F}\left[e^{-t(I-\Delta)^\alpha} f\right](\xi) = e^{-t(1+\xi^2)^\alpha}\hat{f}(\xi) \quad (f \in \mathcal{S}(\mathbf{R}_x^N))$$

and $e^{-t(1+\xi^2)^\alpha} \in \mathcal{S}(\mathbf{R}_\xi^N)$ for each $t > 0$,

$$\left(e^{-t(I-\Delta)^\alpha} f\right)(x) = \left(\mathcal{F}^{-1}\left[e^{-t(1+\xi^2)^\alpha}\right] * f\right)(x) \quad (f \in \mathcal{S}(\mathbf{R}_x^N)).$$

Here, $\mathcal{F}[u](\xi) := \int_{\mathbf{R}^N} e^{-ix\xi} u(x)\, dx$ is the Fourier transform of $u \in \mathcal{S}(\mathbf{R}^N)$ and $\mathcal{S}(\mathbf{R}^N)$ is the Schwartz space. Put

$$\text{(2.4)} \quad K_\alpha(t,x) := \mathcal{F}^{-1}\left[e^{-t(1+\xi^2)^\alpha}\right](x) = \frac{1}{(2\pi)^N}\int_{\mathbf{R}^N} e^{ix\xi} e^{-t(1+\xi^2)^\alpha} d\xi.$$

Then, (1) implies $0 \le K_\alpha(t,\cdot) \in \mathcal{S}(\mathbf{R}^N) \subset L^1(\mathbf{R}^N)$ and

$$\text{(2.5)} \qquad \|K_\alpha(t,\cdot)\|_1 = \int_{\mathbf{R}^N} K_\alpha(t,x)dx = \mathcal{F}\left[K_\alpha(t,\cdot)\right](0) = e^{-t} < 1.$$

Moreover, (2.2) holds since $K_\alpha(t,\cdot) \in L^1(\mathbf{R}^N)$ and $\mathcal{S}(\mathbf{R}^N)$ is a dense subset of $L^2(\mathbf{R}^N)$.
(3) The assertion readily follows from (1), (2.2) and (2.5) (cf. [3, Theorem 1.4.1]). □

The following two estimates for $K_\alpha(\cdot,\cdot)$ defined in (2.4) are crucial to our main result.

Proposition 2.4. *Let $0 < \alpha \le 1$ and $0 < \delta < 1$. Then there exists a constant $C_{\alpha,\delta} > 0$ such that*

$$\text{(2.6)} \qquad 0 \le K_\alpha(t,x) \le C_{\alpha,\delta}\frac{e^{-\delta|x|}}{t^{N/2\alpha}} \quad (0 < {}^\forall t < \infty).$$

Proposition 2.5. *Let $\frac{1}{2} < \alpha \le 1$ and $0 < \delta < 1$. Then there exists a constant $C_{\alpha,\delta} > 0$ such that*

$$\text{(2.7)} \qquad 0 \le K_\alpha(t,x) \le C_{\alpha,\delta}\frac{e^{-\delta|x|}}{|x|^{N+1}} t^{1/2\alpha} \quad (0 < {}^\forall t \le 1).$$

For the proof of these two estimates, we need the following lemma.

Lemma 2.6. *Let $0 < \alpha \le 1$, $0 < \delta < 1$, and let $\xi = (\xi_1, \cdots, \xi_N) = \eta + i\zeta \in \mathbf{C}^N$ (η, $\zeta \in \mathbf{R}^N$) with $|\zeta| \le \delta$. Then we obtain*

(1) $\quad 1 \le \dfrac{1+|\xi|^2}{1+|\eta|^2} \le 2,$

(2) $\quad |\arg(1+\xi^2)| \le \theta_0 := \arctan(\delta/\sqrt{1-\delta^2}) \in (0, \pi/2)$ *and*

(3) $\quad \mathrm{Re}\,\left[(1+\xi^2)^\alpha\right] \ge \cos\alpha\theta_0 \cdot |\eta|^{2\alpha}.$

Proof. (1) This is clear.
(2) Let $X := \mathrm{Re}\ (1+\xi^2)$ and $Y := \mathrm{Im}\ (1+\xi^2)$. Then we have

$$X = 1 - |\zeta|^2 + |\eta|^2 \ge 1 - |\zeta|^2 + \left(\frac{Y}{2|\zeta|}\right)^2 \ge 1 - \delta^2 + \frac{Y^2}{4\delta^2}.$$

Therefore, we have $|\arg(1+\xi^2)| \le \theta_0 := \arctan(\delta/\sqrt{1-\delta^2}) \in (0,\pi/2)$ since $0 < \delta < 1$.
(3) Using (2), we have

$$\begin{aligned} \mathrm{Re}\ \left[(1+\xi^2)^\alpha\right] &= |(1+\xi^2)|^\alpha \cos[\alpha \arg(1+\xi^2)] \\ &\ge \left[\mathrm{Re}\ (1+\xi^2)\right]^\alpha \cos\alpha\theta_0 \ge \cos\alpha\theta_0 \cdot |\eta|^{2\alpha}. \end{aligned}$$

So, the proof is complete. □

Proof of Proposition 2.4. Hereafter, ξ denotes a vector in $\mathbf{C}^N$ and $\xi_j \in \mathbf{C}$ ($j = 1, \cdots, N$) denotes the component of ξ. We shall obtain the desired estimate by shifting the integration area $\mathbf{R}^N$ in the formula (2.4) to complex region. Let us now fix $0 < \alpha \le 1$ and $0 < \delta < 1$. Then for each fixed $x \in \mathbf{R}^N$ and $t > 0$, the function $F(\xi, x, t) : \xi \in \mathbf{C}^N \longmapsto e^{ix\xi}e^{-t(1+\xi^2)^\alpha} \in \mathbf{C}$ is a well-defined analytic function in a neighborhood of $\Omega_\delta := \{\xi \in \mathbf{C}^N : \xi = \eta + i\zeta,\ \eta, \zeta \in \mathbf{R}^N \text{ with } |\zeta| \le \delta\}$. First, note that for each $\zeta \in \mathbf{R}^N$ with $|\zeta| \le \delta$, $F(\eta + i\zeta, x, t)$ is absolutely integrable with respect to η on $\mathbf{R}^N$. Hence we can calculate $\displaystyle\int_{\mathbf{R}^N} F(\eta + i\zeta, x, t)d\eta$ as an successive integration with respect to $\eta_1, \cdots, \eta_N$ ($\eta = (\eta_1, \cdots, \eta_N)$) in any order. Now fix a $\zeta^0 = (\zeta_1^0, \cdots, \zeta_N^0) \in \mathbf{R}^N$ with $|\zeta^0| \le \delta$. Set $D_{\zeta_1^0, R} \subset \mathbf{C}$ as

$$\{\xi_1 \in \mathbf{C} : 0 \le \mathrm{Im}\ \xi_1 \le \zeta_1^0,\ |\mathrm{Re}\ \xi_1| \le R\}$$

or

$$\{\xi_1 \in \mathbf{C} : 0 \ge \mathrm{Im}\ \xi_1 \ge \zeta_1^0,\ |\mathrm{Re}\ \xi_1| \le R\}$$

according as $\zeta_1^0 \ge 0$ or $\zeta_1^0 < 0$. For each fixed $\xi_2, \cdots, \xi_N \in \mathbf{R}$ (even for ξ_j with $|\mathrm{Im}\ \xi_j| \le |\zeta_j^0|$ $(2 \le j \le N)$), $F(\xi, x, t)$ is an analytic function of

ξ_1 in a neighborhood of $D_{\zeta_1^0,R}$. Moreover, Lemma 2.6 implies $|F(\xi,x,t)| \le e^{|\zeta_1^0||x|}e^{-t\cos\alpha\theta_0\cdot|\eta|^2}$ for $\xi_1 \in D_{\zeta_1^0,R}$ with $|\mathrm{Re}\ \xi_1| = R$. Therefore, the path on the "vertical" edges of $\partial D_{\zeta_1^0,R}$ of the integral $0 = \oint_{\partial D_{\zeta_1^0,R}} F(\xi,x,t)d\xi_1$ tends to 0 as $R \to 0$, and hence

$$\int_{\mathbf{R}} F(\xi,x,t)d\xi_1 = \int_{\mathbf{R}+i\zeta_1^0} F(\xi,x,t)d\xi_1 = \int_{\mathbf{R}} F((\eta_1+i\zeta_1^0,\xi_2,\cdots,\xi_N),x,t)d\eta_1.$$

This means that we can shift the integration path for $\xi_1 \in \mathbf{R}$ in (2.4) to the path $\mathbf{R}+i\zeta_1^0$ in the complex region:

$$K_\alpha(t,x) = \frac{1}{(2\pi)^N}\int_{\mathbf{R}^{N-1}} d\xi_2\cdots d\xi_N \left(\int_{\mathbf{R}+i\zeta_1^0} F(\xi,x,t)d\xi_1\right).$$

Change of the order of integration yields

$$K_\alpha(t,x) = \frac{1}{(2\pi)^N}\int_{\mathbf{R}+i\zeta_1^0} d\xi_1 \int_{\mathbf{R}^{N-2}} d\xi_3\cdots d\xi_N \left(\int_{\mathbf{R}} F(\xi,x,t)d\xi_2\right).$$

By applying a similar argument as above, we see that

$$\int_{\mathbf{R}} F(\xi,x,t)d\xi_2 = \int_{\mathbf{R}+i\zeta_2^0} F(\xi,x,t)d\xi_2$$

for all $\xi \in \mathbf{R}+i\zeta_1^0$, $\xi_j \in \mathbf{R}$ $(j \ne 1,2)$. Hence we obtain

$$K_\alpha(t,x) = \frac{1}{(2\pi)^N}\int_{\mathbf{R}+i\zeta_1^0} d\xi_1 \int_{\mathbf{R}^{N-2}} d\xi_3\cdots d\xi_N \left(\int_{\mathbf{R}+i\zeta_2^0} F(\xi,x,t)d\xi_2\right).$$

We can further rewrite $K_\alpha(t,x)$ as

$$K_\alpha(t,x) = \frac{1}{(2\pi)^N}\int_{\mathbf{R}+i\zeta_1^0} d\xi_1 \int_{\mathbf{R}+i\zeta_2^0} d\xi_2 \int_{\mathbf{R}^{N-3}} d\xi_4\cdots d\xi_N \left(\int_{\mathbf{R}} F(\xi,x,t)d\xi_3\right),$$

and shift the integration path for ξ_3 from $\mathbf{R}$ to $\mathbf{R}+i\zeta_3^0$. Continuing a process like this, we finally reach the equality

$$(2.8)\qquad \begin{aligned} K_\alpha(t,x) &= \frac{1}{(2\pi)^N}\int_{\mathbf{R}+i\zeta_1^0} d\xi_1 \cdots \int_{\mathbf{R}+i\zeta_N^0} d\xi_N F(\xi,x,t) \\ &= \frac{1}{(2\pi)^N}\int_{\mathbf{R}^N} F(\eta+i\zeta^0,x,t)d\eta \\ &= \frac{e^{-x\zeta^0}}{(2\pi)^N}\int_{\mathbf{R}^N} e^{ix\eta}e^{-t(1+\xi^2)^\alpha}d\eta \quad (\xi = \eta+i\zeta^0). \end{aligned}$$

From the estimate in Lemma 2.6, we obtain a constant C such that

$$\left|\int_{\mathbf{R}^N} e^{ix\eta}e^{-t(1+\xi^2)^\alpha}d\eta\right| \le \int_{\mathbf{R}^N} e^{-t\cos\alpha\theta_0\cdot|\eta|^{2\alpha}}d\eta = \frac{C}{t^{N/2\alpha}}$$

for every ζ^0 with $|\zeta^0| \le \delta$. Thus we get

$$K_\alpha(t,x) \le \inf_{|\zeta^0|\le\delta} e^{-x\zeta^0}\frac{C}{t^{N/2\alpha}} = C\frac{e^{-\delta|x|}}{t^{N/2\alpha}}.$$

So, the proof is complete □

Proof of Proposition 2.5. Let $\frac{1}{2} < \alpha \le 1$, $0 < \delta < 1$ and $j = 1, \cdots, N$. Then we obtain from (2.8) and the integration by parts,

$$x_j^{N+1}K_\alpha(t,x) = \frac{i^{N+1}}{(2\pi)^N}e^{-x\zeta}\int_{\mathbf{R}^N} e^{ix\eta}\frac{\partial^{N+1}}{\partial\eta_j^{N+1}}e^{-t(1+\xi^2)^\alpha}d\eta, \tag{2.9}$$

where $\xi = \eta + i\zeta$ with η, $\zeta \in \mathbf{R}^N$ and $|\zeta| \le \delta$. By induction with respect to n, it is easy to show that the following equation holds for every infinitely differentiable function f of η, ζ ($\xi = \eta + i\zeta$):

$$\frac{\partial^n}{\partial\eta_j^n}e^{-tf(\xi)} = \sum_{(k_l)_l\in D_n} C_{(k_l)_l}t^{\sum_{l=1}^n k_l}\prod_{l=1}^n\left[\frac{\partial^l}{\partial\eta_j^l}f(\xi)\right]^{k_l}e^{-tf(\xi)}, \tag{2.10}$$

where $D_n := \{(k_l)_l \in \mathbf{Z}_+^n : \sum_{l=1}^n lk_l = n\}$ and $C_{(k_l)_l}$ denotes a constant depending on $(k_l)_l \in D_n$. On the other hand, by the analyticity of $(1+\xi^2)^\alpha$ and induction with respect to l, we obtain

$$\frac{\partial^l}{\partial\eta_j^l}(1+\xi^2)^\alpha = \frac{\partial^l}{\partial\xi_j^l}(1+\xi^2)^\alpha = (1+\xi^2)^{\alpha-l}p_l(\xi), \tag{2.11}$$

where $p_l(\xi)$ is a polynomial of ξ of degree l. For a $(k_l)_l \in D_{N+1}$, put $\sum_{l=1}^{N+1} k_l = m$. Note that $m \in \{1, \cdots, N+1\}$. With the aid of Lemma 2.6 (1), we obtain the following estimate:

$$\begin{aligned}\left|\prod_{l=1}^{N+1}\left[(1+\xi^2)^{\alpha-l}p_l(\xi)\right]^{k_l}\right| &= \left|\frac{\prod_{l=1}^{N+1}p_l(\xi)^{k_l}}{\prod_{l=1}^{N+1}[(1+\xi^2)^{l-\alpha}]^{k_l}}\right| \\ &\le C\left|\frac{\prod_{l=1}^{N+1}(1+|\xi|^2)^{\frac{lk_l}{2}}}{(1+\xi^2)^{N+1-\alpha m}}\right| \\ &\le C\frac{(1+|\xi|^2)^{\frac{N+1}{2}}}{[\text{Re }(1+\xi^2)]^{N+1-\alpha m}} \\ &\le C/(1+|\eta|^2)^{\frac{N+1}{2}-\alpha m}.\end{aligned} \tag{2.12}$$

Hence, (2.9), (2.10), (2.11) and (2.12) imply

$$\begin{aligned}
&|x_j|^{N+1}K_\alpha(t,x)\\
\leq\ & \frac{1}{(2\pi)^N}\inf_{|\zeta|\leq\delta} e^{-x\zeta}\int_{\mathbf{R}^N}\left|\frac{\partial^{N+1}}{\partial\eta_j^{N+1}}e^{-t(1+\xi^2)^\alpha}\right|d\eta\\
\leq\ & Ce^{-\delta|x|}\sum_{m=1}^{N+1}t^m\int_{\mathbf{R}^N}\frac{1}{(1+|\eta|^2)^{(N+1)/2-\alpha m}}e^{-t\cos\alpha\theta_0\cdot|\eta|^{2\alpha}}d\eta\\
=\ & Ce^{-\delta|x|}\sum_{m=1}^{N+1}t^m\left(\int_0^1+\int_1^\infty\right)\frac{r^{N-1}}{(1+r^2)^{(N+1)/2-\alpha m}}e^{-t\cos\alpha\theta_0\cdot r^{2\alpha}}dr\\
\leq\ & Ce^{-\delta|x|}\sum_{m=1}^{N+1}t^m\left(\int_0^1\frac{r^{N-1}}{(1+r^2)^{(N+1)/2-\alpha m}}dr\right.\\
&\left.+\int_1^\infty\frac{r^{N-1}}{r^{N+1-2\alpha m}}\left(\frac{r^2}{1+r^2}\right)^{\frac{N+1}{2}-\alpha m}e^{-t\cos\alpha\theta_0\cdot r^{2\alpha}}dr\right)\\
\leq\ & Ce^{-\delta|x|}\sum_{m=1}^{N+1}\left(t^m+t^m\int_1^\infty r^{2\alpha m-2}e^{-t\cos\alpha\theta_0\cdot r^{2\alpha}}dr\right)\\
\leq\ & Ce^{-\delta|x|}\sum_{m=1}^{N+1}\left(t^m+t^{1/2\alpha}\int_0^\infty r^{2\alpha m-2}e^{-r^{2\alpha}}dr\right),
\end{aligned}$$

where the last integral is finite because $2\alpha m-2>-1$. It follows from $\frac{1}{2\alpha}<1\leq m$ that $t^m\leq t^{1/2\alpha}$ for $0<t\leq 1$, so

$$|x_j|^{N+1}K_\alpha(t,x)\leq Ce^{-\delta|x|}\left(\sum_{m=1}^{N+1}t^m+t^{1/2\alpha}\right)\leq Ce^{-\delta|x|}t^{1/2\alpha} \tag{2.13}$$

holds for $0<t\leq 1$. Therefore,

$$K_\alpha(t,x)\leq C\frac{e^{-\delta|x|}}{|x|^{N+1}}t^{1/2\alpha}\quad(0<{}^\forall t\leq 1),$$

since the constant C of the right hand side of (2.13) depends only on α, δ. □

Now our main results reads as follows:

Theorem 2.7. *Assume that a C_0-semigroup T on $L^2(\Omega)$ ($\Omega\subset\mathbf{R}^N$) is essentially dominated by $S_\alpha=\{e^{-t(I-\Delta)^\alpha}\}_{t\geq0}$ (on $L^2(\mathbf{R}^N)$) for some $\alpha\in(\frac{1}{2},1]$. Then there exist consistent C_0-semigroups T_p on $L^p(\Omega)$ ($1\leq p<\infty$) such that $T_2=T$ and $\rho_\infty(A_p)$ is independent of $p\in[1,\infty)$, where A_p is the generator of T_p.*

Corollary 2.8. *Assume that the generator A of T is self-adjoint and that T is essentially dominated by S_α for some $\alpha \in (\frac{1}{2}, 1]$. Then there exist consistent C_0-semigroups T_p on $L^p(\Omega)$ $(1 \le p < \infty)$ such that $T_2 = T$ and $\sigma(A_p)$ is independent of $p \in [1, \infty)$, where A_p is the generator of T_p.*

Remark 2.9. By an inspection of the proof of our main theorem, we can easily see that the essential domination by $e^{-t(\varepsilon-\Delta)^\alpha}$ for an $\varepsilon > 0$ and $1/2 < \alpha \le 1$ implies the same conclusion as in our main theorem. So, it may be conjectured that the assumption of essential domination by $e^{-t(I-\Delta)^\alpha}$ can be relaxed to that by $e^{-t(-\Delta)^\alpha}$. Note that $0 \le e^{-t(\varepsilon-\Delta)^\alpha} \le e^{-t(-\Delta)^\alpha}$ holds for all $t \ge 0$. At present, the authors are not able to prove or disprove this conjecture. However, we would like to note that the Fourier multiplier theory yields the p-independence of the spectrum of $(-\Delta)^\alpha$ in $L^p(\mathbf{R}^N)$ with a suitable domain ([10, p. 96]).

In the following, we state a full proof of our main theorem for the reader's convenience. However, the authors would like to emphasize that we owe the method of the proof to Arendt [1]. Our own contribution mainly lies in the estimate of the integral kernel of $S_\alpha(t)$, which makes Arendt's method work.

Proof of Theorem 2.7 From Proposition 2.3 and (2.1), it follows that there exist consistent C_0-semigroups T_p on $L^p(\Omega)$ $(1 \le p < \infty)$ such that $T = T_2$ and

$$|T_p(t)f| \le Me^{\omega t} S_{\alpha,p}(bt)|f| \quad (f \in L^p(\Omega),\ t \ge 0) \tag{2.14}$$

(cf. [1, p. 1160]).

To continue the proof, we need the idea of Arendt [1] to use the following auxiliary spaces.

For each vector $\varepsilon, x \in \mathbf{R}^N$, we let $\varepsilon x := \sum_{j=1}^N \varepsilon_j x_j$. Let $L^p := L^p(\Omega)$, $L^p_\varepsilon := L^p(\Omega, e^{-p\varepsilon x}dx)$, for $1 \le p < \infty$, $\varepsilon \in \mathbf{R}^N$. Then $(U_{\varepsilon,p}f)(x) := e^{-\varepsilon x}f(x)$ defines an isometric isomorphism of L^p_ε onto L^p. Hence $\widetilde{T}_{\varepsilon,p}(t) := U_{\varepsilon,p}^{-1} T_p(t) U_{\varepsilon,p}$ defines a C_0-semigroup $\widetilde{T}_{\varepsilon,p}$ on L^p_ε.

Since $S_{\alpha,p}(t)$ is an integral operator for $t > 0$ (see Proposition 2.3), it follows from (2.14) and a well-known fact (cf. [1, Proposition 6.2]) that for $t > 0$ there exists a measurable function $K(t, \cdot, \cdot)$ on $\Omega \times \Omega$ such that

$$(T_p(t)f)(x) = \int_\Omega K(t, x, y) f(y) dy$$

holds for all $f \in L^p$. We write such a relation of $T_p(t)$ and the integral kernel $K(t, \cdot, \cdot)$ as $T_p(t) \overset{dx}{\sim} K(t, \cdot, \cdot)$. Note that $K(\cdot, \cdot, \cdot)$ is independent of $p \in [1, \infty)$. Consequently, $\widetilde{T}_{\varepsilon,p}(t) \overset{dx}{\sim} K_\varepsilon(t, \cdot, \cdot)$ where

$$K_\varepsilon(t, x, y) = e^{\varepsilon(x-y)} K(t, x, y). \tag{2.15}$$

We use the following notation. Let $1 \leq p, q, r \leq \infty$, $Q \in \mathcal{L}(L^p)$. Then we set

$$\|Q\|_{q,r} := \sup\left\{\|Qf\|_r : f \in L^p \cap L^q,\ \|f\|_q \leq 1\right\}.$$

Concerning $\widetilde{S}_{\alpha,\varepsilon,p}$, we obtain the following lemma.

Lemma 2.10. *Let $0 < \varepsilon_0 < 1$, $\varepsilon \in \mathbf{R}^N$ with $|\varepsilon| \leq \varepsilon_0$ and $1 \leq p < \infty$. Then there exists a C_0-semigroup $S_{\alpha,\varepsilon,p}$ on L^p satisfying*

$$S_{\alpha,\varepsilon,p}(t)f = \widetilde{S}_{\alpha,\varepsilon,p}(t)f \quad (f \in L^p \cap L^p_\varepsilon) \tag{2.16}$$

and

$$\sup_{\substack{|\varepsilon|\leq\varepsilon_0\\ 0\leq t\leq 1}} \|S_{\alpha,\varepsilon,p}(t)\|_{p,p} \leq C < \infty. \tag{2.17}$$

Moreover, for $q_1, q_2 \in [1,\infty]$ with $q_1 \leq q_2$ and $t > 0$

$$\sup_{|\varepsilon|\leq\varepsilon_0} \|S_{\alpha,\varepsilon,p}(t)\|_{q_1,q_2} \leq C_t < \infty \tag{2.18}$$

holds for all $1 \leq p < \infty$ (C_t is a constant depending on $t > 0$).

Proof. Let $0 < \varepsilon_0 < 1$, $\varepsilon \in \mathbf{R}^N$ with $|\varepsilon| \leq \varepsilon_0$, $1 \leq p < \infty$, and $\varepsilon_0 < \delta < 1$. Then for $t > 0$ and $f \in L^p \cap L^p_\varepsilon$,

$$|\widetilde{S}_{\alpha,\varepsilon,p}(t)f| \leq \widetilde{S}_{\alpha,\varepsilon,p}(t)|f| = (e^{\varepsilon\cdot}K_\alpha(t,\cdot)) * |f|. \tag{2.19}$$

From Proposition 2.4 and Proposition 2.5, we have

$$\begin{aligned}
&\sup_{\substack{|\varepsilon|\leq\varepsilon_0\\ 0<t\leq 1}} \|e^{\varepsilon\cdot}K_\alpha(t,\cdot)\|_1 \\
= &\sup_{\substack{|\varepsilon|\leq\varepsilon_0\\ 0<t\leq 1}} \left(\int_{|x|\geq t^{1/2\alpha}} + \int_{|x|\leq t^{1/2\alpha}}\right) e^{\varepsilon x}K_\alpha(t,x)dx \\
\leq &\ C \sup_{\substack{|\varepsilon|\leq\varepsilon_0\\ 0<t\leq 1}} \left(\int_{|x|\geq t^{1/2\alpha}} e^{\varepsilon x}\frac{e^{-\delta|x|}}{|x|^{N+1}}t^{1/2\alpha}dx + \int_{|x|\leq t^{1/2\alpha}} e^{\varepsilon x}\frac{e^{-\delta|x|}}{t^{N/2\alpha}}dx\right) \\
\leq &\ C \sup_{\substack{|\varepsilon|\leq\varepsilon_0\\ 0<t\leq 1}} \left(t^{1/2\alpha}\int_{t^{1/2\alpha}}^{\infty} \frac{1}{r^{N+1}}r^{N-1}dr + t^{-N/2\alpha}\int_0^{t^{1/2\alpha}} r^{N-1}dr\right) \\
= &\ C \sup_{\substack{|\varepsilon|\leq\varepsilon_0\\ 0<t\leq 1}} \left(t^{1/2\alpha}\left[-\frac{1}{r}\right]_{t^{1/2\alpha}}^{\infty} + t^{-N/2\alpha}\left[\frac{r^N}{N}\right]_0^{t^{1/2\alpha}}\right) \\
= &\ C \sup_{\substack{|\varepsilon|\leq\varepsilon_0\\ 0<t\leq 1}} \left(t^{1/2\alpha}\cdot t^{-1/2\alpha} + t^{-N/2\alpha}\cdot\frac{t^{N/2\alpha}}{N}\right) = \frac{N+1}{N}C < \infty.
\end{aligned}$$

Thus, by (2.19), there exists a semigroup $S_{\alpha,\varepsilon,p}$ on L^p such that (2.16) and (2.17) hold for all $1 \le p < \infty$. It remains to show that $S_{\alpha,\varepsilon,p}(t)f \to f$ $(t \downarrow 0)$ in L^p for all $f \in L^p$. By (2.17), it suffices to consider functions with compact support. Let $f \in L^p$ such that $f(x) = 0$ for $|x| \ge r$, where $r > 0$. Then

$$\limsup_{t\downarrow 0} \|S_{\alpha,\varepsilon,p}(t)f - f\|_p^p \le \limsup_{t\downarrow 0} \int_{|x|\ge r+1} [(e^{\varepsilon\cdot} K_\alpha(t,\cdot)) * |f|(x)]^p \, dx$$

(cf. [1, p. 1165]). If $|x| \ge r+1$ then

$$\begin{aligned}
\int_{\mathbf{R}^N} \frac{e^{-(\delta-\varepsilon_0)|x-y|}}{|x-y|^{N+1}} |f(y)| dy &= \int_{|y|\le r} \frac{e^{-(\delta-\varepsilon_0)|x-y|}}{|x-y|^{N+1}} |f(y)| dy \\
&\le \int_{|x-y|\ge 1} \frac{e^{-(\delta-\varepsilon_0)|x-y|}}{|x-y|^{N+1}} |f(y)| dy \\
&\le \left(e^{-(\delta-\varepsilon_0)|\cdot|} * |f|\right)(x),
\end{aligned}$$

so

$$\begin{aligned}
&\int_{|x|\ge r+1} \left(\int_{\mathbf{R}^N} \frac{e^{-(\delta-\varepsilon_0)|x-y|}}{|x-y|^{N+1}} |f(y)| dy \right)^p dx \\
\le\ & \int_{\mathbf{R}^N} \left[\left(e^{-(\delta-\varepsilon_0)|\cdot|} * |f|\right)(x) \right]^p dx \\
\le\ & \|f\|_p^p \int_{\mathbf{R}^N} e^{-(\delta-\varepsilon_0)|x|} dx < \infty.
\end{aligned}$$

Thus, by Proposition 2.5,

$$\begin{aligned}
&\limsup_{t\downarrow 0} \int_{|x|\ge r+1} [(e^{\varepsilon\cdot} K_\alpha(t,\cdot)) * |f|(x)]^p \, dx \\
\le\ & \lim_{t\downarrow 0} C t^{p/2\alpha} \int_{|x|\ge r+1} \left(\int_{\mathbf{R}^N} \frac{e^{-(\delta-\varepsilon_0)|x-y|}}{|x-y|^{N+1}} |f(y)| dy \right)^p dx = 0.
\end{aligned}$$

This implies $\lim_{t\downarrow 0} \|S_{\alpha,\varepsilon,p}(t)f - f\|_p^p = 0$.

Moreover, let $q_1, q_2 \in [1,\infty]$ with $q_1 \le q_2$ and $t > 0$. Then by Proposition 2.4,

$$|S_{\alpha,\varepsilon,p}(t)f| \le (e^{\varepsilon\cdot} K_\alpha(t,\cdot)) * |f| \le \frac{C}{t^{N/2\alpha}} e^{-(\delta-\varepsilon_0)|\cdot|} * |f| \quad (f \in L^p \cap L^{q_1}).$$

Since $e^{-(\delta-\varepsilon)|\cdot|} \in L^r$ $\left(\frac{1}{r} = 1 + \frac{1}{q_2} - \frac{1}{q_1}\right)$, Young's inequality implies that

$$\sup_{|\varepsilon|\le\varepsilon_0} \|S_{\alpha,\varepsilon,p}(t)\|_{q_1,q_2} \le \frac{C}{t^{N/2\alpha}} \left\| e^{-(\delta-\varepsilon_0)|\cdot|} \right\|_r < \infty$$

holds for all $1 \le p < \infty$. □

By virtue of Lemma 2.10, we obtain

Proposition 2.11. *Let $0 < \varepsilon_0 < 1$, $\varepsilon \in \mathbf{R}^N$ with $|\varepsilon| \le \varepsilon_0$ and $1 \le p < \infty$. Then the following assertions hold:*
(1) *There exists a C_0-semigroup $T_{\varepsilon,p}$ on L^p such that*

$$T_{\varepsilon,p}(t)f = \widetilde{T}_{\varepsilon,p}(t)f \quad (f \in L^p \cap L^p_\varepsilon,\ t \ge 0). \tag{2.20}$$

(2) *There exist $M_1 > 0$ and $\omega_1 \in \mathbf{R}$ such that*

$$\|T_{\varepsilon,p}(t)\|_{p,p} \le M_1 e^{\omega_1 t} \quad (t \ge 0)$$

for all $|\varepsilon| \le \varepsilon_0$, $1 \le p < \infty$.
(3) *For $\lambda > \omega_1$*

$$\lim_{|\varepsilon|\downarrow 0} \|R(\lambda, A_{\varepsilon,p}) - R(\lambda, A_p)\|_{p,p} = 0,$$

where $A_{\varepsilon,p}$ denotes the generator of $T_{\varepsilon,p}$ on L^p $(1 \le p < \infty)$.

Proof. Let $0 < \varepsilon_0 < 1$, $\varepsilon \in \mathbf{R}^N$ with $|\varepsilon| \le \varepsilon_0$, $1 \le p < \infty$ and $\varepsilon_0 < \delta < 1$.
(1) It follows from (2.14) that

$$|\widetilde{T}_{\varepsilon,p}(t)f| \le M e^{\omega t} \widetilde{S}_{\alpha,\varepsilon,p}(bt)|f| \quad (t \ge 0,\ f \in L^p \cap L^p_\varepsilon). \tag{2.21}$$

Thus, by (2.16) and (2.17), there exists a semigroup $T_{\varepsilon,p}$ on L^p such that (2.20) holds. In order to prove (1) it remains to show that $T_{\varepsilon,p}(t)f \to f$ $(t \downarrow 0)$ in L^p for all $f \in L^p$. By (2.17) and (2.21), it suffices to consider functions with compact support. Let $f \in L^p$ such that $f(x) = 0$ for $|x| \ge r$, where $r > 0$. Then, by a calculation similar to [1, p. 1165] and the strong continuity of $S_{\varepsilon,p}$

$$\limsup_{t\downarrow 0} \|T_{\varepsilon,p}(t)f - f\|_p^p \le \lim_{t\downarrow 0} M^p \|e^{\omega t} S_{\varepsilon,p}(bt)|f| - |f|\|_p^p = 0.$$

(2) By (2.16), (2.17) and (2.21),

$$M_1 := \sup_{0\le t\le 1} \|T_{\varepsilon,p}(t)\|_{p,p} < \infty$$

holds for all $|\varepsilon| \le \varepsilon_0$, $1 \le p < \infty$. Thus, by the semigroup property, (2) follows $(\omega_1 := \log M_1)$.
(3) First, we show that for $0 < a < 1$

$$\lim_{|\varepsilon|\downarrow 0} \sup_{a\le t\le 1/a} \|T_{\varepsilon,p}(t) - T_p(t)\|_{p,p} = 0. \tag{2.22}$$

In fact, since $T_{\varepsilon,p}(t) - T_p(t) \overset{dx}{\sim} K(t,x,y)(e^{\varepsilon(x-y)} - 1)$ (see (2.15)) and, by Proposition 2.4,

$$|K(t,x,y)(e^{\varepsilon(x-y)} - 1)| \le C e^{\omega t} \frac{e^{-\delta|x-y|}}{(bt)^{N/2\alpha}} |e^{\varepsilon(x-y)} - 1|,$$

we have for $a \le t \le 1/a$, $|(T_{\varepsilon,p}(t) - T_p(t))f| \le C g_\varepsilon * |f|$, where $g_\varepsilon(x) := e^{-\delta|x|}|e^{\varepsilon x} - 1|$. Since $|g_\varepsilon(x)| \le e^{-(\delta-\varepsilon_0)|x|} + e^{-\delta|x|} \in L^1$ and $g_\varepsilon(x) \to 0$ a.e. x as $|\varepsilon| \downarrow 0$, (2.22) follows by the dominated convergence theorem.

Now let $\lambda > \omega_1$. Then for $0 < a < 1$

$$\begin{aligned}\limsup_{|\varepsilon|\downarrow 0} \|R(\lambda, A_{\varepsilon,p}) - R(\lambda, A_p)\|_{p,p} &\le \limsup_{|\varepsilon|\downarrow 0} \int_0^\infty e^{-\lambda t}\|T_{\varepsilon,p}(t) - T_p(t)\|_{p,p} dt \\ &\le 2M_1 \left(\int_0^a + \int_{1/a}^\infty\right) e^{-(\lambda-\omega_1)t} dt \\ &\le 2M_1 \left(a + \frac{1}{\lambda - \omega_1} e^{-(\lambda-\omega_1)/a}\right).\end{aligned}$$

Since $0 < a < 1$ is arbitrary, the assertion follows. □

It is clear from the construction that the semigroups $T_{\varepsilon,p}$ on L^p and $\widetilde{T}_{\varepsilon,p}$ on L^p_ε are consistent. Consequently, $R(\lambda, A_{\varepsilon,p})$ and $R(\lambda, \widetilde{A}_{\varepsilon,p})$ are consistent for $\lambda \in \mathbf{C}$ if Re λ is sufficiently large. Thus, we obtain from [1, Proposition 2.2] the following assertion: *$R(\lambda, A_{\varepsilon,p})$ and $R(\lambda, \widetilde{A}_{\varepsilon,p})$ are consistent for all $\lambda \in [\rho(A_{\varepsilon,p}) \cap \rho(\widetilde{A}_{\varepsilon,p})]_\infty$.* Here, for open subset O of $\mathbf{C}$, we let O_∞ be the connected component of O which contains a right half-plane $\{\lambda \in \mathbf{C} : \text{Re } \lambda > w\}$ for some $w \in \mathbf{R}$.

Note that by construction $\rho(\widetilde{A}_{\varepsilon,p}) = \rho(A_p)$ and

$$R(\lambda, \widetilde{A}_{\varepsilon,p})f = e^{\varepsilon x}\left(R(\lambda, A_p)(e^{-\varepsilon \cdot} f)\right) \quad (f \in L^p_\varepsilon)$$

for all $\lambda \in \rho(A_p)$. Therefore, if $f, e^{-\varepsilon \cdot} f \in L^p$ and $\lambda \in [\rho(A_{\varepsilon,p}) \cap \rho(A_p)]_\infty$, then the following holds:

$$R(\lambda, A_{\varepsilon,p})f = e^{\varepsilon x}\left(R(\lambda, A_p)(e^{-\varepsilon \cdot} f)\right). \tag{2.23}$$

Now we can continue the proof of Theorem 2.7.

Proof of Theorem 2.7 (continued). (cf. [1, Proof of Theorem 4.4]) Let $1 \le p, q < \infty$, $\mu \in \rho_\infty(A_p)$. First, we have to show that $\mu \in \rho(A_q)$. By [1, Proposition 2.3], it suffices to show that $\|R(\mu, A_p)\|_{q,q} < \infty$. Since

$$R(\mu, A_p) = \int_0^1 e^{-\mu t} T_p(t) dt + e^{-\mu} T_p(1) R(\mu, A_p),$$

it is enough to show that

$$\|T_p(1) R(\mu, A_p)\|_{q,q} < \infty. \tag{2.24}$$

Let K be the image of a continuous path in $\rho(A_p)$ connecting μ with a point in $\{\lambda \in \mathbf{C} : \mathrm{Re}\,\lambda > \omega_1\}$. By Proposition 2.11 (3) and [1, Proposition 6.6], there exists $0 < \varepsilon_0 < 1$ such that $K \subset \rho(A_{\varepsilon,p})$ for all $\varepsilon \in \mathbf{R}^N$ with $|\varepsilon| \leq \varepsilon_0$. Consequently, $\mu \in [\rho(A_{\varepsilon,p}) \cap \rho(A_p)]_\infty$. It follows from (2.16), (2.18) and (2.21) that

$$\sup_{|\varepsilon|\leq\varepsilon_0} \|T_{\varepsilon,p}(\tfrac{1}{2})\|_{1,p} < \infty \quad \text{and} \quad \sup_{|\varepsilon|\leq\varepsilon_0} \|T_{\varepsilon,p}(\tfrac{1}{2})\|_{p,\infty} < \infty.$$

Since $T_{\varepsilon,p}(1)R(\mu, A_{\varepsilon,p}) = T_{\varepsilon,p}(\frac{1}{2})R(\mu, A_{\varepsilon,p})T_{\varepsilon,p}(\frac{1}{2})$, it follows that

$$C_0 := \sup_{|\varepsilon|\leq\varepsilon_0} \|T_{\varepsilon,p}(1)R(\mu, A_{\varepsilon,p})\|_{1,\infty} < \infty. \tag{2.25}$$

Therefore, the operator $T_{\varepsilon,p}(1)R(\mu, A_{\varepsilon,p})$ is given by a kernel $K_\varepsilon(\cdot,\cdot)$ such that $|K_\varepsilon(x,y)| \leq C_0$ (a.e. $x, y \in \Omega$) (cf. [1, Proposition 6.1]). In particular, $K_0(\cdot,\cdot) \overset{dx}{\sim} T_p(1)R(\mu, A_p)$. (2.23) implies that

$$\begin{aligned} T_{\varepsilon,p}(1)R(\mu, A_{\varepsilon,p})f &= e^{\varepsilon x}T_p(1)\left[e^{-\varepsilon y}\left(e^{\varepsilon y}R(\mu, A_p)(e^{-\varepsilon \cdot}f)\right)\right] \\ &= e^{\varepsilon x}\left[T_p(1)R(\mu, A_p)\right](e^{-\varepsilon \cdot}f) \end{aligned}$$

whenever $f, e^{-\varepsilon \cdot}f \in L^p$, hence

$$K_\varepsilon(x,y) = e^{\varepsilon(x-y)}K_0(x,y) \quad (\text{a.e. } x, y \in \Omega).$$

So, by (2.25), $|K_0(x,y)| \leq C_0 e^{-\varepsilon_0|x-y|}$ (a.e. $x, y \in \Omega$), which yields (2.24). Thus we obtain $\rho_\infty(A_p) \subset \rho(A_q)$, hence $\rho_\infty(A_p) \subset \rho_\infty(A_q)$. □

§3. Applications

In this section, we apply the results in the section 2 to some examples. Let $\Omega \subset \mathbf{R}^N$, T, S, A and B be as in first paragraph of section 2, and let $S_\alpha(t) = e^{-t(I-\Delta)^\alpha}$ for $0 < \alpha \leq 1$.

(I) *The fractional power of the Schrödinger operators*: First, we show the following proposition.

Proposition 3.1. *If $T(t) = e^{tA}$ is dominated by $S(t) = e^{tB}$ in the sense that $|T(t)f| \leq S(t)|f|$ for $f \in L^2(\Omega)$, and suppose that $S(t)$ is uniformly bounded, then $e^{-t(-A)^\alpha}$ is dominated by $e^{-t(-B)^\alpha}$.*

Proof. Let $\lambda > 0$ and $0 \le f \in L^2(\Omega)$. Then

$$\begin{aligned}
\left|[\lambda + (-A)^\alpha]^{-1} f\right| &= \left|\frac{\sin \pi\alpha}{\pi} \int_0^\infty \frac{\mu^\alpha(\mu - A)^{-1} f}{\mu^{2\alpha} + 2\lambda\mu^\alpha \cos \pi\alpha + \lambda^2} d\mu\right| \\
&\le \frac{\sin \pi\alpha}{\pi} \int_0^\infty \frac{\mu^\alpha|(\mu - A)^{-1} f|}{\mu^{2\alpha} + 2\lambda\mu^\alpha \cos \pi\alpha + \lambda^2} d\mu \\
&\le \frac{\sin \pi\alpha}{\pi} \int_0^\infty \frac{\mu^\alpha(\mu - B)^{-1} f}{\mu^{2\alpha} + 2\lambda\mu^\alpha \cos \pi\alpha + \lambda^2} d\mu \\
&= [\lambda + (-B)^\alpha]^{-1} f,
\end{aligned}$$

hence we conclude the proof (see [9, Proposition 4.1]). □

Proposition 3.1 implies that if e^{tA} is dominated by $e^{t\Delta}$, then $e^{-t(I-A)^\alpha}$ is essentially dominated by $e^{-t(I-\Delta)^\alpha}$. Therefore, we can obtain many C_0-semigroups essentially dominated by $e^{-t(I-\Delta)^\alpha}$. For example, if $-A$ is the form sum of either the negative Dirichlet Laplacian $-\Delta_D$ or the negative Neumann Laplacian $-\Delta_N$ (in the latter case Ω is assumed to be bounded and to have the extension property) and $0 \le V \in L^1_{\text{loc}}$, then A is self-adjoint and generates a positive C_0-semigroup T on L^2 which is dominated by $e^{t\Delta}$. Proposition 3.1 implies that $e^{-t(I-A)^\alpha}$ is dominated by S_α. Therefore, Corollary 2.8 implies that there exist consistent positive C_0-semigroups on L^p generated by $-{(I-A)^\alpha}_p$ $(1 \le p < \infty)$ such that $-{(I-A)^\alpha}_2 = -(I-A)^\alpha$ and $\sigma(-{(I-A)^\alpha}_p)$ is independent of $p \in [1, \infty)$. Especially, $\sigma(-{(I-\Delta_D)^\alpha}_p)$ and $\sigma(-{(I-\Delta_N)^\alpha}_p)$ is independent of $p \in [1, \infty)$.

(II) *The generator of absorption semigroups*: For the theory of absorption semigroups, see [12], [13] and [7].

Let $0 < \alpha \le 1$, B_α the generator of S_α and $V^{(n)} := V \wedge n$. Then s-$\lim_{n\to\infty} e^{t(B_\alpha - V^{(n)})}$ exists for all $t \ge 0$ if $0 \le V$ is measurable. We denote this limit by $S_{\alpha,V}(t)$. If, in addition, V is $S_\alpha(\cdot)$-admissible (see [12], for definition), then $S_{\alpha,V} := \{S_{\alpha,V}(t)\}_{t\ge 0}$ is a C_0-semigroup on L^2. It is clear that

$$0 \le S_{\alpha,V}(t) \le S_\alpha(t) \quad (t \ge 0)$$

and the generator $B_{\alpha,V}$ of $S_{\alpha,V}$ is self-adjoint. Therefore, if $\alpha \in (\frac{1}{2}, 1]$ then there exist consistent positive C_0-semigroups $S_{\alpha,p,V}$ on L^p with generator $B_{\alpha,p,V}$ $(1 \le p < \infty)$ such that $B_{\alpha,2,V} = B_{\alpha,V}$ and $\sigma(B_{\alpha,p,V})$ is independent of $p \in [1, \infty)$ (Corollary 2.8).

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