SECOND METHOD OF LYAPUNOV AND EXISTENCE OF INTEGRAL MANIFOLDS FOR IMPULSIVE DIFFERENTIAL EQUATIONS

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Abstract. In the present paper sufficient conditions of the existence of integral manifolds for impulsive differential equations are obtained. The investigations are carried on by means of piecewise continuous functions which are analogues of Lyapunov's functions.

1. INTRODUCTION

Impulsive differential equations represent a natural apparatus for mathematical simulations of real processes and phenomena studied in biology, physics, control theory etc. On the other hand the mathematical theory of impulsive differential equations is much richer than the corresponding theory of equations without impulses [1–5].

Since the solutions of the impulsive differential equations are piecewise continuous functions it is necessary to introduce certain analogous of Lyapunov's functions which have discontinuities of the first kind.

By means of such functions the extension of Lyapunov's second method to impulsive differential equations is much more effective [1], [4–5].

In the present paper the problem of the existence of integral manifold for systems of impulsive differential equations is considered. Piecewise continuous Lyapunov's functions are used in the investigations. It is proved that the existence of such functions with certain properties is a sufficient conditions for existence of integral manifolds.

2. PRELIMINARY NOTES AND DEFINITIONS

Let R^n be an n-dimensional Euclidean space with norm $\|\cdot\|$, scalar product $\langle\cdot,\cdot\rangle$ and let $I = [0,\infty)$.

With $PC^\kappa(J, R^n)$, where $J \subseteq I, \kappa = 1, 2, \ldots$, we denote the space which is constructed from all piecewise continuous functions such that:

1. If by $\{t_i \in J, i = 1, 2, \ldots\}$ we denote the set of all points t_i at which the function $x \in PC^\kappa(J, R^n)$ is discontinuous, and $x(t_i - 0) = x(t_i)$ is finite. The set $\{t_i \in J, i = 1, 2, \ldots\}$ have no finite accumulation point.
2. If $t \in J \backslash \{t_i \in J, i = 1, 2, \ldots\}$, then x is of class C^κ.

Let $\Omega \subset R^n$, $f : I \times \Omega \to R^n$, and $\Phi_i : \Omega \to R^n$, $i = 1, 2, \ldots$.
Introduce the following conditions:

H1. $f \in C^1(I \times \Omega, R^n)$.

H2. $\phi_i \in C^1(\Omega, R^n)$, $i = 1, 2, \ldots$.

H3. If $x \in \Omega$, then $x + \Phi_i(x) \in \Omega$, $F_i(x) = x + \Phi_i(x)$ where $F_i : \Omega \to \Omega$ are invertible in Ω, and $F_i^{-1}(x) \in \Omega$ for $i = 1, 2, \ldots$, and $x \in \Omega$.

H4. The impulsive moments $\{t_i\}_{i=1}^\infty$ forms a strictly increasing sequence such that $\lim_{i \to \infty} t_i = \infty$.

Let the conditions H1–H4 are satisfied. We consider the system of impulsive differential equations with fixed moments of time $\{t_i\}_{i=1}^\infty$

$$x' = f(t, x), \quad t \neq t_i, \tag{1}$$

$$\Delta x(t) = \Phi_i(x(t)), \quad t = t_i, \quad i = 1, 2, \ldots, \tag{2}$$

where $x' = \dfrac{dx}{dt}, \Delta x(t_i) = x(t_i + 0) - x(t_i - 0)$.

We shall denote that from [1], [2] for any $(t_0, x_0) \in I \times \Omega$ the solution of the system (1), (2) with initial condition $x(t_0) = x_0$ is any function $x(t; t_0, x_0)$ for which:

1. $x(t; t_0, x_0) \in PC^2(J, R^n)$ and for any $i = 1, 2, \ldots$, $x(t_i + 0; t_0, x_0) = x(t_i; t_0, x_0) + \Phi_i(x(t_i; t_0, x_0))$.
2. For any $t \in J \backslash \{t_i \in J, i = 1, 2, \ldots\}$, (1) holds.

With $J^+ = J^+(t_0, x_0), (J^- = J^-(t_0, x_0))$ we shall denote the maximal interval of the form $(t_0, \omega), ((\omega, t_0))$ in which $x(t; t_0, x_0)$ is defined.

With $\theta_+(t_0, x_0), \theta_-(t_0, x_0)$, and $\theta(t_0, x_0)$ we shall denote the integral orbit of the solution $x(t; t_0, x_0)$ for $t \in J^+, t \in J^-$, and $t \in J$ respectively.

DEFINITION 1. We shall say that a manifold M in the extended phase space is:

a) an *r-integral manifold*, if $(t_0, x_0) \in M$ it follows that $\theta_+(t_0, x_0) \subset M$.
b) an *l-integral manifold*, if $(t_0, x_0) \in M$ it follows that $\theta_-(t_0, x_0) \subset M$.
c) an *integral manifold*, if M is an r-integral manifold and an l-integral manifold.

In this paper we give sufficient conditions for the existence of integral manifolds of the system (1), (2).

Consider the sets

$$G_i = \{(t, x) \in I \times \Omega, t_{i-1} < t < t_i\}, i = 1, 2, \ldots,$$

$$G = \bigcup_{i=1}^{\infty} G_i.$$

DEFINITION 2. The function $L : I \times \Omega \to R$, $(t, x) \to L(t, x)$ is called a function of type Lyapunov with kernel manifold M for the system of impulsive differential equations (1), (2) if the following conditions hold

    1. $L(t, x) \geq 0$ for any $(t, x) \in I \times \Omega$, and $L(t, x) = 0$ only when $(t, x) \in M$.
    2. For any $i = 1, 2, \ldots, x_0 \in \Omega$ there exist finite limits

$$L(t_i - 0, x_0) = \lim_{\substack{(t,x) \to (t_i, x_0) \\ t < t_i}} L(t, x) \quad L(t_i + 0, x_0) = \lim_{\substack{(t,x) \to (t_i, x_0) \\ t > t_i}} L(t, x),$$

    and the equality $L(t_i - 0, x_0) = L(t_i, x_0)$ holds.
    3. $L \in C^1(G, R)$.

Let $L(t, x)$ be a function of Lyapunov with kernel manifold M for the system (1), (2). Then in G we define the function

$$\dot{L} = \left\langle \frac{\partial L(t, x)}{\partial x}, f(t, x) \right\rangle + \frac{\partial L(t, x)}{\partial t}.$$

Obviously $\dfrac{d}{dt} L(t, x(t; t_0, x_0)) = \dot{L}(t, x(t; t_0, x_0))$ for $(t, x(t; t_0, x_0)) \in G$.

In the further considerations we shall use the class K of all functions $a : I \to I$ that are continuous and strictly increasing, and such that $a(0) = 0$.

3. MAIN RESULTS

Theorem 1. *Let the following conditions are satisfied:*
    1. *The conditions H1–H4 hold.*
    2. *There exists a function $L(t, x)$ of Lyapunov with kernel manifold M for the system (1), (2) such that:*

$$\dot{L}(t, x) \leq 0 \qquad for \quad (t, x) \in G, \tag{3}$$

$(resp. \quad \dot{L}(t, x) \geq 0 \qquad for \quad (t, x) \in G),$

$$L(t_i + 0, x + \Phi_i(x)) \leq L(t_i, x) \qquad for \quad i = 1, 2, \ldots, x \in \Omega, \tag{4}$$

$(resp. \quad L(t_i + 0, x + \Phi_i(x)) \geq L(t_i, x) \qquad for \quad i = 1, 2, \ldots, x \in \Omega).$

Then M is an r-integral manifold (resp. an l-integral manifold) of the system (1), (2).

Proof. We shall prove Theorem 1 for r-integral manifold. For l-integral manifold the proof is analogous. Suppose that M is not an r-integral manifold. Then there exists $t' > t_0$ such that, if $(t_0, x_0) \in M$ then $(t, x(t; t_0, x_0)) \in M$ for $t_0 \geq t \geq t'$ and $(t, x(t; t_0, x_0)) \notin M$ for $t > t'$. Then $L(t', x') = 0$, where $x' = x(t'; t_0, x_0)$. Moreover the function $x(t) = x(t; t_0, x_0)$ is piecewise continuous with a finite number of points of discontinuity in the interval $[t_0, t']$ and the following two cases are possible.

   a) If $t' = t_i, i = j, j+1, \cdots$, then $(t', x(t'+0; t_0, x_0)) = (t'+0, x(t'; t_0, x_0) + \Phi_i(x'))$, $(t' + 0, x(t' + 0; t_0, x_0)) \notin M$ and from Definition 2 it follows that $L(t' + 0, x(t' + 0; t_0, x_0)) > 0$.

   Consequently $L(t' + 0, x(t' + 0; t_0, x_0)) > L(t', x') = 0$ which is a cotradiction by (4).

   b) If $t' \neq t_i, i = j, j + 1, \cdots$, ,then there exists t'' such that $t'' > t$ and $(t'', x(t''; t', x')) \notin M$. From (3) and (4) it follows that the funcion $L(t, x(t))$ is not increasing in (t_0, ∞).

   From Definition 2 it follows that $L(t'', x(t''; t', x')) > 0$ so $L(t'', x(t''; t', x')) > L(t', x')$ for $t'' > t'$ which is a contradiction to the fact that the function $L(t, x(t))$ is not increasing in (t_0, ∞).

   From a) and b) it follows that M is an r-integral manifold.

Theorem 2. *Let the following conditions are satisfied:*

   1. *The conditions H1–H4 hold.*
   2. *There exists a function $L(t, x)$ of Lyapunov with kernel manifold M for the system (1),(2) and the function $c \in K$ such that:*

$$\dot{L}(t, x) \leq -c\|x\| \qquad for \quad (t, x) \in G,$$
$$(resp \quad \dot{L}(t, x) \geq c\|x\| \qquad for \quad (t, x) \in G),$$
$$L(t_i + 0, x + \Phi_i(x)) \leq L(t_i, x) \qquad for \quad i = 1, 2, \ldots, x \in \Omega,$$
$$(resp \quad L(t_i + 0, x + \Phi_i(x)) \geq L(t_i, x) \qquad for \quad i = 1, 2, \ldots, x \in \Omega).$$

Then M is an r-integral manifold (resp an l-integral manifold) of the system (1), (2).

Proof. The proof of Theorem 2 is analogous to the proof of Theorem 1.

Theorem 3. *Let the following conditions are satisfied:*

   1. *The conditions H1–H4 hold.*
   2. *There exist a functions $L(t, x)$ and $V(t, x)$ of Lyapunov with kernel*

manifold M for the system (1),(2) such that:

$$\dot{L}(t,x) \le 0 \qquad \text{for} \quad (t,x) \in G,$$
$$\dot{V}(t,x) \ge 0 \qquad \text{for} \quad (t,x) \in G,$$
$$L(t_i + 0, x + \Phi_i(x)) \le L(t_i, x), \qquad \text{for} \quad i = 1, 2, \dots, x \in \Omega,$$
$$V(t_i + 0, x + \Phi_i(x)) \ge L(t_i, x), \qquad \text{for} \quad i = 1, 2, \dots, x \in \Omega.$$

Then M is an integral manifold of (1),(2).

Proof. The proof of Theorem 3 follows from Theorem 1 and Theorem 2.

Example. We consider the system of impulsive defferential equations

$$\begin{cases} \dfrac{dy}{dt} = -y - t^2\sqrt{y}z^2, & \dfrac{dz}{dt} = t^2 y^{-2}(z-2), \quad t \ne i, \\[2mm] \Delta y = -\dfrac{1}{2}, & \Delta z = 0, \qquad t = i, \ i = 1, 2, \dots, \end{cases} \tag{5}$$

where $t \in I, \ y \in I, \ z \in I$.

Now we consider the manifold

$$M = \{(t,y,z) \in I^3 : z = 2, t > 0, y > 0\} \tag{6}$$

and the functions

$$V(t,y,z) = \left(\frac{3}{4}\right)^i \exp\left\{-\left(\frac{t}{y}\right)^2\right\}(z-2)^2, \quad i-1 < t < i,$$
$$W(t,y,z) = (z-2)^2.$$

Then

$$\dot{V}(t,y,z) = \left(\frac{3}{4}\right)^i \left(-2ty^{-2}\exp\left\{-\left(\frac{t}{y}\right)^2\right\}(z-2)^2\right)$$
$$+ 2\left(\frac{3}{4}\right)^i \exp\left\{-\left(\frac{t}{y}\right)^2\right\}(z-2)^2 t^2 y^{-2}$$
$$+ \left(\frac{3}{4}\right)^i 2t^2 y^{-3}\exp\left\{-\left(\frac{t}{y}\right)^2\right\}(z-2)^2\left(-y - t^2\sqrt{y}z^2\right)$$
$$= -2\left(\frac{3}{4}\right)^i ty^{-2}\exp\left\{-\left(\frac{t}{y}\right)^2\right\}(z-2)^2\left(1 + t^3 y^{-\frac{1}{2}}z^2\right) \le 0, \tag{7}$$
$$i < t < i+1, \ i = 1, 2, \dots, \ y > 0, \ z > 0$$

On the other hand

$$\dot{W}(t,y,z) = 2(z-2)^2 t^2 y^{-2} \geq 0, \quad i < t < i+1, \ i = 1, 2, \ldots, \ y > 0, \ z > 0, \tag{8}$$

$$V\left(i+0, y-\frac{1}{2}, z\right) \leq V(i,y,z), \quad y > 0, \ z > 0, \ i = 1, 2, \ldots, \tag{9}$$

$$W\left(i+0, y-\frac{1}{2}, z\right) = W(i,y,z), \quad y > 0, \ z > 0, \ i = 1, 2, \ldots, \tag{10}$$

From (7), (8), (9) and (10) it follows that the conditions of Theorem 3 are satisfied. Therefore (6) is an integral manifold of the system (5).

Now we consider the function

$$W(t,s,x) = \begin{cases} L(t,x), t > s, (t,x) \in I \times \Omega, (s,x) \in I \times \Omega, \\ V(t,x), t < s, (t,x) \in I \times \Omega, (s,x) \in I \times \Omega, \\ \max\{L(t,x), V(t,x)\}, (t,x) \in I \times \Omega, t = s, \\ \max\{L(t+0, x+\Phi_i(x)), V(t+0, x+\Phi_i(x))\}, \\ (t,x) \in I \times \Omega, t = s = t_i, i = 1, 2, \ldots, x \in \Omega \end{cases} \tag{11}$$

where $L(t,x)$ and $V(t,x)$ are defined in Theorem 3.

Theorem 4. *Let the condition H1–H4 hold. Then a manifold M in the extended phase space of (1), (2) is an integral manifold (1), (2) if there exist a function $W(t,s,x)$ in the form (11) such that:*

$$\dot{W}(t,s,x) \leq 0 \quad for \quad t > s, (t,x) \in G, (s,x) \in G,$$

$$W(t_i+0, s+0, x+\Phi_i(x)) \leq W(t_i, s, x) \quad for \quad t_i > s, x \in \Omega,$$

$$0 \leq \dot{W}(t,s,x) \quad for \quad t < s, (t,x) \in G, (s,x) \in G,$$

$$W(t_i, s, x) \leq W(t_i+0, s+0, x+\Phi_i(x)) \quad for \quad t_i < s, x \in \Omega.$$

Proof. The proof of Theorem 4 follows from (11) and Theorem 3.

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