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NOTES ON DIFFERENTIAL IDEALS OF LASKERIAN RINGS

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Abstract. Let R be a ring and d a derivation of R. We consider the following three conditions: (a) every quasi-prime d-ideal of R is prime, (b) any weak associated prime of every d-ideal of R is a d-ideal and (c) every d-prime d-ideal of R is prime. In this paper we show that if R is a Laskerian ring, then the two conditions (a) and (b) are equivalent. Furthermore we show that if R is a strongly Laskerian ring, then any d-prime d-ideal of R is quasi-prime, and then the three conditions (a), (b) and (c) are equivalent.

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§1. Introduction

All rings in this paper are assumed to be commutative with a unit element. Let R be a ring. A *derivation* d of a ring R is an additive endomorphism $d : R \to R$ such that $d(ab) = d(a)b + ad(b)$ for every $a, b \in R$. Let d be a derivation of R. An ideal I of R is called a *d-ideal* if $d(I) \subset I$. A proper d-ideal Q of R is called a *d-prime* d-ideal if for d-ideals I and J of R the relation $IJ \subset Q$ implies either $I \subset Q$ or $J \subset Q$. A proper d-ideal Q of R is called a *quasi-prime* d-ideal if there is a multiplicative subset S of R such that Q is maximal among d-ideals disjoint from S. Some of the properties of the d-prime d-ideals and the quasi-prime d-ideals are given in [3], [8], [9], [11], [12], [14].

Let I be an ideal of R. A prime ideal P of R is called a *minimal prime divisor* of I if P is minimal among the prime ideals containing I. A prime ideal P of R is called a *weak associated prime* of I if there exists $x \in R$ such that P is a minimal prime divisor of $I : (x)$; We denote by $\mathrm{Ass}_f(R/I)$ the set of weak associated primes of I (cf.[1, IV, §1,Exercise 17]). It is known that if R is Noetherian, then the weak associated primes of I coincides with the usual associated primes of I. If I can be expressed as an intersection of a finite number of primary ideals, we say that I has a *primary decomposition.* A ring R is called *Laskerian* if every ideal of R has a primary decomposition.

A Laskerian ring R is called *strongly Laskerian* if each primary ideal of R contains a power of its radical (cf.[1, IV, §2, Exercise 23, 28]).

Let R be a ring and d a derivation of R. A. Nowicki ([11], [12]) obtained the following results under the assumption that the ring R is Noetherian:

(1) *An ideal Q of R is a d-prime d-ideal if and only if Q is a quasi-prime d-ideal.*

(2) *The following three conditions are equivalent:*

(a) *every quasi-prime ideal of R is a prime ideal,*
(b) *any weak associated prime of every d-ideal of R is a d-ideal,*
(c) *every d-prime d-ideal of R is a prime ideal.*

The aim of this paper is to try to weaken the condition "Noetherian" of the ring R in A. Nowicki's results above. The results which we obtained are as follows:

If R is a Laskerian ring, then the conditions (a) and (b) above are equivalent (see Theorem 4.1). Furthermore, if R is a strongly Laskerian ring, the results (1) and (2) above hold (see Theorem 4.2). If the ring R is not strongly Laskerian, then the result (1) is not necessarily true, and then the conditions (a) and (c) of (2) are not equivalent in general, even if R is Laskerian (see Example 4.3).

§2. Preliminaries

Throughout this paper, let R be a ring, d a derivation of R and $\mathbb{Z}$ the rational integers. In this section we record several lemmas for convenience, which are known.

Lemma 2.1 ([4, Proposition (1.4)]). *Let I be a d-ideal of R and P a minimal prime divisor of I. Then the primary component Q $(= sat_P(I))$ of I belonging to P is also d-ideal.*

Lemma 2.2 ([9, Exercise 3, p.63]). *Any quasi-prime d-ideal of R is primary. If R contains the rational numbers, then every quasi-prime d-ideal of R is prime.*

For an ideal I of R, we denote by $I_\#$ the biggest d-ideal contained in I. Note that $I_\# = \{x \in I \mid d^n(x) \in I, \text{ for all } n \geq 1\}$.

Lemma 2.3 ([8, Proposition 2.2]). *For a d-ideal Q of R, the following three conditions are equivalent:*

(1) Q is quasi-prime.
(2) Q is primary and $Q = (\sqrt{Q})_\#$.
(3) There is a prime ideal P in R such that $Q = P_\#$.

Lemma 2.4 ([12, Proposition 2.1]). *Any quasi-prime d-ideal of R is d-prime.*

Lemma 2.5 ([8, Proposition 2.1]). *The following four conditions are equivalent:*

(1) Every quasi-prime d-ideal of R is prime.
(2) Any minimal prime divisor of every d-ideal of R is a d-ideal.
(3) The radical of any d-ideal of R is a d-ideal.
(4) For any prime ideal P of R, the ideal $P_{\#}$ is prime.

§3. d-prime d-ideals and quasi-prime d-ideals

In this section we study some conditions under which a d-prime d-ideal is primary. Furthermore we discuss a relation among quasi-prime ideals and prime ideals, and we give some examples.

For a ring R to be Laskerian, it is necessary and sufficient that it satisfies the following two conditions:

(LA_1) For every ideal I of R and every prime ideal P of R, the saturation of I with respect to P in R is of the form $I : (a)$ for some $a \notin P$.

(LA_2) For every ideal I of R, every decreasing sequence $(\mathrm{sat}_{S_n}(I))$ (where (S_n) is any decreasing sequence of multiplicative subset of R) is stationary. (cf.[1, IV, §2, Exercise 23]).

Proposition 3.1. *Let R be a ring and d a derivation of R. If R satisfies the conditions (LA_1) above or char(R) $\neq 0$, then evry d-prime d-ideal of R is primary. In particular, if the ring R is Laskerian, then every d-prime d-ideal of R is primary.*

Proof. First we sssume that R satisfies the condition (LA_1). Let I be a d-prime d-ideal of R and P a minimal prime divisor of I. Then the primary component Q of I belonging to P is d-ideal by Lemma 1.1. Since R satisfies the condition (LA_1), the saturation $sat_P(I)(= Q)$ of I is of the form $I : (x)$ for some $x \notin P$. It follows that $(x) \subset I : Q$ and so $[x] \subset I : Q$, where $[x]$ is the smallest d-ideal containing x. Hence $[x]Q \subset I$. Since $[x] \not\subset I$, we have that $Q \subset I$, and hence $Q = I$. Therefore I is primary.

Next we sssume that $\mathrm{char}(R) = n(\neq 0)$. Let I be a d-prime d-ideal. Suppose that $xy \in I$ and $x \notin \sqrt{I}$. Then we have $x^n y \in I$ and hence $y \in I : (x^n)$. Since $I : (x^n)$ is a d-ideal, we have $[y] \subset I : (x^n)$, where $[y]$ is the smallest d-ideal containing y. Therefore $(x^n)[y] \subset I$. On the other hand, since $(x^n) \not\subset I$, we have that $[y] \subset I$, and hence $y \in I$. Therefore I is primary.

Proposition 3.2. *Let R be a ring and d a derivation of R. If a d-prime d-ideal I of R has a primary decomposition, then I is primary.*

Proof. Let P be a minimal prime divisor of I and Q the primary component of I belonging to P. Since I has a primary decomposition, it is clear that $Q = I : (x)$ for some $x \notin P$. Therefore by the same way as the proof of the first case of Proposition 3.1, we have that I is primary.

Proposition 3.3. *Let R be a ring of characteristic 0 and d a derivation of R. Let I be a d-prime d-ideal of R. If $I \cap \mathbb{Z} \neq (0)$, where $\mathbb{Z}$ is the rational integers, then I is primary.*

Proof. Put $I \cap \mathbb{Z} = (n)(n \neq 0)$. Then the residue ring R/I is of characteristic n. Let $\bar{d}$ be the derivation of R/I defined by $\bar{d}(x+I) = d(x)+I$ $(x \in R)$. Since I is a d-prime d-ideal of R, (0) is a $\bar{d}$-prime $\bar{d}$-ideal of R/I. By Proposition 3.1, (0) is a primary ideal of R/I, and thus I is a primary ideal of R.

Remark. We do not know whether a d-prime d-ideal is a primary ideal in general.

Proposition 3.4. *Let R be a ring of characteristic 0 and d a derivation of R. Let Q be a quasi-prime d-ideal of R. If $Q \cap \mathbb{Z} = (0)$, then Q is prime.*

Proof. Let R' be the quotient ring $S^{-1}R$ with respect to $S = \mathbb{Z} - \{0\}(\subset R)$ and d' the derivation of R' induced by d. Put $P = \sqrt{Q}$. Then Q is P-primary and $Q = P_{\#} \subset P$. Put $Q' = QR'$ and $P' = PR'$. Then Q' is a d'-ideal and P'- primary. Furthermore we have $(P')_{\#} = (P_{\#})R'$. Thus $(P')_{\#} = QR' = Q'$ and therefore Q' is a quasi-prime d'-ideal of R'. Since R' contains the rational integers, Q' is a prime ideal by Lemma 2.2. It follows that $Q' = P'$ and so we have $Q = P$. Consequently Q is prime.

In case of $\mathrm{char}(R) \neq 0$, a quasi-prime d-ideal of R is not necessarily prime as in the following example.

Example 3.5. Let k be a field of characteristic $p > 0$ and $R = k[X]$ a polynomial ring over k. Let d be a k-derivation of R such that $d(X) = 1$. Put $P = (X)$ and $Q = (X^p)$. Then Q is a P-primary ideal of R and by a simple calculation we have $Q = P_{\#}$. Thus Q is a quasi-prime d-ideal by Lemma 2.3, but Q is not a prime ideal.

In case of $\mathrm{char}(R) = 0$, let Q be a quasi-prime d-ideal of R such that $Q \cap \mathbb{Z} \neq (0)$. Then Q is not necessarily prime as shown in the following example.

Example 3.6. Let $R = \mathbb{Z}[X]$ be a polynomial ring over the rational integers $\mathbb{Z}$ and d a derivation of R such that $d(X) = 1$. Then $Q := (X^2, 2)$ is a d-ideal of R. Put $P = (X, 2)$. Then Q is a P-primary ideal. It is clear that $Q = P_{\#}$. Thus Q is a quasi-prime d-ideal by Lemma 2.3, but Q is not a prime ideal.

§4. Main results

We are now ready to prove the main results.

Theorem 4.1. *Let R be a Laskerian ring and d a derivation of R. The following two conditions are equivalent:*

(a) Every quasi-prime d-ideal of R is prime.

(b) Any weak associated prime of every d-ideal of R is a d-ideal.

Proof. (a) $\Longrightarrow$ (b). Let I be a d-ideal of R. First, we consider the case $\text{char}(R) \neq 0$ or char $(R) = 0$ and $I \cap \mathbb{Z} \neq (0)$. Then I can be written as an irredundant intersection of a finite number of primary d-ideals $Q_i (i = 1, ..., n)$ by [5, Theorem 2 and Proposition 6]. Furthermore we have that $Ass_f(R/I) = \{\sqrt{Q_1}, ..., \sqrt{Q_n}\}$. By the hypothesis (a) and Lemma 1.5, all $\sqrt{Q_i}$ are d-ideals. Thus every weak associated prime of I is a d-ideal.

Next, suppose that $\text{char}(R) = 0$ and $I \cap \mathbb{Z} = (0)$. Let $I = Q_1 \cap \cdots \cap Q_n$ be an irredundant primary decomposition such that $P_i \cap \mathbb{Z} = (0)(i = 1, \ldots, t)$ and $P_i \cap \mathbb{Z} \neq (0)(i = t+1, \ldots, n)$, where $P_i = \sqrt{Q_i}$ $(i = 1, \ldots, n)$. Note that $Ass_f(R/I) = \{P_1, ..., P_n\}$. By [5, Theorem 1], $P_i(i = 1, \ldots, t)$ are d-ideals. Put $I_1 = Q_1 \cap \cdots \cap Q_t$ and $I_2 = Q_{t+1} \cap \cdots \cap Q_n$. Then $I_2 \cap \mathbb{Z} = (q)$ for some non-zero integer q. Put $I'_2 = qR + I$. Then I'_2 is a d-ideal and $I \subset I'_2 \subset I_2$. Thus we have $I = I_1 \cap I'_2$, and I'_2 can be written as an intersection $Q'_1 \cap \cdots \cap Q'_m$ of primary d-ideals $Q'_i (i = 1, .., m)$ by [5, Proposition 6]. Therefore we have that $I = Q_1 \cap \cdots \cap Q_t \cap Q'_1 \cap \cdots \cap Q'_m$. By the same reason as the first step, each $\sqrt{Q'_i}$ is a d-ideal. For any $i(t+1 \leq i \leq n)$, $P_i = \sqrt{Q'_j}$ for some $j(1 \leq j \leq m)$. Thus $P_i(1 \leq i \leq n)$ are d-ideals.

(b) $\Longrightarrow$ (a). Let I be a d-ideal of R and P a minimal prime divisor of I. Then clearly P is a weak associated prime of I. Thus P is a d-ideal. Therefore, the assertion follows from Lemma 1.5.

Remark. By the same way as the proof of Theorem 4.1, we get the following result:

Let R be a Laskerian ring of characteristic 0, d a derivation of R and I a proper d-ideal of R. Let $\mathbb{Z}$ be the rational integers. Then I can be represented as an irredundant intersection $Q_1 \cap \cdots \cap Q_t \cap \cdots \cap Q_n (0 \leq t \leq n)$ of primary ideals Q_i of R such that: (1) $P_i \cap \mathbb{Z} = (0)(i = 1, \ldots, t), P_j \cap \mathbb{Z} \neq (0)(j = t+1, \ldots, n)$ (where $P_i = \sqrt{Q_i}$). (2) $P_i(i = 1, \ldots, t), Q_j(j = t+1, \ldots, n)$ are d-ideals. Obviously, (i) if the ring R contains the rational numbers, then the number t equal to n, and (ii) if $I \cap \mathbb{Z} \neq (0)$, then the number t equal to 0.

When R is a Noetherian ring, the following Theorem 4.2 was proved by A. Nowicki in [11] and [12].

Theorem 4.2. *Let R be a strongly Laskerian ring and d a derivation of R, then the following statements hold.*

(1) For a d-ideal Q, Q is d-prime if and only if Q is quasi-prime.

(2) The following three conditions are equivalent:

(a) Every quasi-prime d-ideal of R is prime.

(b) Any weak associated prime of every d-ideal of R is a d-ideal.

(c) Every d-prime d-ideal of R is prime.

Proof. (1) In virtue of Lemma 1.4, it suffices to show that if a d-ideal Q is d-prime, then Q is quasi-prime. Put $\sqrt{Q} = P$. Then P is prime by Proposition

3.1. Furthermore we have that $Q \subset P_\# \subset P$. Since Q is P-primary, $P^n \subset Q$ for some $n \geq 1$, and hence $(P_\#)^n \subset Q$. Since Q is d-prime, we have that $P_\# \subset Q$ and therefore $Q = P_\#$. Thus Q is quasi-prime by Lemma 1.3.

(2) The equivalence of (a) and (b) follows from Theorem 4.1 and the equivalence of (a) and (c) follows from (1).

Remarks. (1) Let R be a ring and d a derivation of R. If every quasi-prime d-ideal of R is prime, then R is called a d-MP ring (cf. [11]), or a special differential ring (cf. [8]).

(2) In Example 4.3 below we show that there is a Laskerian ring R which is not a strongly Laskerian and there is a derivation d of R such that R has a d-prime d-ideal which is neither prime nor quasi-prime. Therefore Example 4.3 shows that if R is not strongly Laskerian, Theorem 4.2 is not necessarily true even if R is Laskerian.

Example 4.3 (cf. [2, Example 2.1]). Let $T = k[X_1, X_2, ...]$ be a polynomial ring over the field $k(:= \mathbb{Z}/(p))$ of prime characteristic p. For the ideal $A = (X_1^p, X_2^p, ...)$, put $R = T/A = k[x_1, x_2, ...]$, where $x_n = X_n + A$. Then R is a local ring with the maximal ideal $M = (x_1, x_2, ...)$. Let d be a derivation of R such that $d(x_n) = x_{n+1}$ for every $n \geq 1$.

In this situation, the following properties hold.

(1) (0) is a d-prime d-ideal of R, but it is not prime.

(2) M is the only one quasi-prime d-ideal of R.

(3) Every quasi-prime d-ideal of R is prime.

(4) R is a Laskerian ring.

(5) R is not a strongly Laskerian ring.

Proof. (1) Assume that I and J are d-ideals of R such that $IJ = (0)$. If $I \neq (0)$ and $J \neq (0)$, then $I \ni x_1^{p-1} \cdots x_n^{p-1}$ and $J \ni x_{n+1}^{p-1} \cdots x_{n+m}^{p-1}$ for some $n \geq 1$ and $m \geq 1$ (see the proof of Lemma 2.3 (p. 291) of [2]). Hence we have $IJ \ni x_1^{p-1} \cdots x_{n+m}^{p-1} \neq 0$, which is a contradiction. Consequently, (0) is a d-prime d-ideal.

(2) Since $\mathrm{Spec}(R) = \{M\}$ and M is a d-ideal, M is the only one quasi-prime d-ideal of R.

(3) This is a immediate consequence of (2).

(4) Let I be any ideal of R. Then $\sqrt{I} = M$, and so I is primary. Hence R is a Laskerian ring.

(5) Note that (0) is a M-primary ideal of R and $\{x_1, x_2, \dots\}$ is a p-basis of R over R^p. For every $n \geq 1$, M^n contains $x_1 x_2 \cdots x_n \neq 0$, and hence we have $M^n \neq (0)$. Therefore R is not a strongly Laskerian ring.

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