SUT Journal of Mathematics
Vol. 32, No. 2 (1996), 157–161

NEW IDEA FOR PROOF OF ANALYTICITY OF SOLUTIONS TO ANALYTIC NONLINEAR ELLIPTIC EQUATIONS

Keiichi Kato

(Received September 18, 1996)

Abstract. In this article, we give a new idea to prove analyticity of solutions to analytic nonlinear elliptic equations.

AMS 1991 *Mathematics Subject Classification.* Primary 35J60, Secondary 35J15.

Key words and phrases. Analyticity, nonlinear, elliptic partial differential equation.

1. Introduction

In this article, we give a new idea to prove analyticity of solutions to analytic nonlinear elliptic equations. To illustrate our idea, we only treat the simple equation,

$$\triangle u = \lambda u^2 \qquad \text{in} \quad \Omega, \tag{1}$$

where $\triangle = \sum_{j=1}^{n} \partial^2/\partial x_j^2$, Ω is a domain in $\mathbf{R}^n$ and λ is a constant in $\mathbf{R}$. In next section, we prove the following theorem by using our method.

Theorem 1. *Suppose that u is in $C^\infty(\Omega)$ and that u satisfies the equation (1). Then u is real analytic in Ω.*

Many proofs of analyticity of solutions to analytic nonlinear elliptic equations have been given by many mathematicians. There are two families of methods to prove analyticity. One is the method to estimate higher order derivatives of solutions([2], [3], [4], [5], [13], [15]). And another is the method to extend the variables of the corresponding integral equations to complex values([9], [11], [12], [14]). Our method belongs to the former. But the author believes that our proof is new and simple.

In these papers([2], [3], [4], [5], [13], [15]), the Sobolev norm of M times derivative of a solution in a domain B contained in Ω is estimated by the Sobolev norm of $(M-1)$ times derivative of a solution in $B_\delta = \{x; \mathrm{dist}(x, B) < \delta\}$. To estimate the Sobolev norm of any times derivative of a solution by that of a solution itself, we must prepare countably many domains $B_0, B_1, \dots, B_M, \dots$

with $B_{M+1} = (B_M)_{\delta_M}$ and must check convergence with respect to δ_M carefully. The method in this article needs only two domains B and B' with $\bar{B} \subset B' \subset \Omega$ because we use a cut-off function $r(x)$ to the power M for M-th derivative of a solution.

We briefly exhibit our method. Our method is to multiply a cut-off function $r(x)$ to the power $|\alpha|$ to the α-th derivative of a solution u and estimate its Sobolev norm. The point of our method is not multiplication of a cut-off function $r(x)$ itself to the α-th derivative of a solution but multiplication of a cut-off function $r(x)$ to the power $|\alpha|$ to the α-th derivative of a solution. As we see in the following, the term $r(x)^{|\alpha|}\partial^\alpha u$ is adapted to nonlinear term. So it is easy to estimate $\|r(x)^{|\alpha|}\partial^\alpha u\|$ with $|\alpha| = M+1$ by $\|r(x)^{|\alpha|}\partial^\alpha u\|$ with $|\alpha| \le M$.

As an application of our method, we refer two papers, [8], [6]. In [8], we consider the evolution equation,

$$iu_t + \triangle u = f(u), \quad u(0,x) = \phi(x).$$

Applying our method, we show that if the initial data ϕ satisfies $\|(x\cdot\nabla)^l\phi\|_{H^m} \le CA^l(l!)^2$ for all $l \in \mathbf{N}$ with $m > n/2$, the solution u is real analytic in x for $t > 0$. In [6], we show by our method how analytic singularities for semilinear wave equations $\Box u = f(u)$ propagate.

2. Proof of Theorem 1

First we introduce some notation and prepare several propositions. Let Ω be a domain in $\mathbf{R}^n$ and m be a real number. We denote an usual Sobolev space of order m with respect to $L^2(\Omega)$ by $H^m(\Omega)$ and let $H_0^m(\Omega)$be the completion of $C_0^\infty(\Omega)$ with the norm of $H^m(\Omega)$. For a multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$, we denote $x^\alpha = x_1^{\alpha_1}\cdots x_n^{\alpha_n}$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha! = \alpha_1!\cdots\alpha_n!$ and $\partial_x^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\cdots\partial_n^{\alpha_n}$ with $\partial_j = \partial/\partial x_j$ $(j = 1, 2, \dots, n)$. For multi-indices α and β, we write $\alpha \le \beta$ if $\alpha_j \le \beta_j$ for $1 \le \forall j \le n$ and define $\alpha+\beta = (\alpha_1+\beta_1, \dots, \alpha_n+\beta_n)$ and $\binom{\alpha}{\beta} = \alpha!/(\beta!(\alpha-\beta)!)$.

Proposition 2.1. *Let α be a multi-index and k be an integer with $0 \le k \le |\alpha|$. We have*

$$\sum_{\substack{|\beta|=k \\ \beta\le\alpha}} \binom{\alpha}{\beta} = \binom{|\alpha|}{k}.$$

Proof. Comparing the coefficients of t^k in both sides of

$$(1+t)^{\alpha_1}\cdots(1+t)^{\alpha_n} = (1+t)^{|\alpha|},$$

we have the proposition. $\Box$

Proposition 2.2. *Let Ω be a domain in $\mathbf{R}^n$ with smooth boundary. If f and g are in $H^m(\Omega)$ with $m > n/2$, then we have*

$$\|fg\|_{H^m(\Omega)} \leq C_1 \|f\|_{H^m(\Omega)} \|g\|_{H^m(\Omega)}, \tag{2}$$

where C_1 is a constant which does not depend on f and g.

Proof. See for example Adams[1]. □

Proposition 2.3. *Let Ω be a domain in $\mathbf{R}^n$. We have*

$$\|\partial_x^\alpha v\|_{H^m(\Omega)} \leq \|\triangle v\|_{H^m(\Omega)}, \tag{3}$$

for all $v \in H_0^{m+2}(\Omega)$ and multi-indices α with $|\alpha| = 2$.

Proof. It suffices to prove (2.1) for $v \in C_0^\infty(\Omega)$. From Plancherel's theorem, we have

$$\begin{aligned}
\|\partial_x^\alpha v\|_{H^m(\Omega)} &= \|\partial_x^\alpha v\|_{H^m(\mathbf{R}^n)} \\
&= \|(1+|\xi|^2)^{m/2} \xi^\alpha \hat{v}(\xi)\|_{L^2(\mathbf{R}^n)} \\
&\leq \|(1+|\xi|^2)^{m/2} |\xi|^2 \hat{v}(\xi)\|_{L^2(\mathbf{R}^n)} \\
&= \|\triangle v\|_{H^m(\mathbf{R}^n)} \\
&= \|\triangle v\|_{H^m(\Omega)}.
\end{aligned}$$

Proof of Theorem 1. It suffices to prove that u is real analytic in every open ball B in Ω with $\bar{B} \subset \Omega$. We take an open ball B' with $\bar{B} \subset B'$ and $\bar{B}' \subset \Omega$. We take and fix a real valued function $r(x)$ in $C_0^\infty(B')$ such that $0 \leq r(x) \leq 1$ and $r(x) \equiv 1$ in a neighborhood of B. To prove that u is real analytic in B, we show that there exist positive constants C and A such that

$$\|r(x)^{|\alpha|} \partial_x^\alpha u\|_{H^m(B')} \leq C A^{|\alpha|} |\alpha|!, \tag{4}$$

for all multi-indices α, where $m = [n/2] + 1$. We prove (4) by induction with respect to $|\alpha|$. For simplicity we assume that A is larger than or equal to 1. The inequality (4) is valid for $|\alpha| \leq 1$ if C is large enough. We fix a constant C so that (4) is valid for $|\alpha| \leq 1$. Assuming that (4) is valid for $|\alpha| \leq N (\geq 1)$, we show that (4) is valid for $|\alpha| = N + 1$ by taking a constant A sufficiently large. In the following, we write $\|\cdot\| = \|\cdot\|_{H^m(B')}$ for abbreviation. Let α and β be multi-indices with $|\alpha| = N - 1$ and $|\beta| = 2$. From Proposition 2.3, we have

$$\begin{aligned}
\|r^{N+1} \partial_x^{\alpha+\beta} u\| &\leq \|\partial_x^\beta r^{N+1} \partial_x^\alpha u\| + \|[\partial_x^\beta, r^{N+1}] \partial_x^\alpha u\| \\
&\leq \|\triangle r^{N+1} \partial_x^\alpha u\| + \|[\partial_x^\beta, r^{N+1}] \partial_x^\alpha u\| \\
&\leq \|r^{N+1} \partial_x^\alpha \triangle u\| + \|[\triangle, r^{N+1}] \partial_x^\alpha u\| + \|[\partial_x^\beta, r^{N+1}] \partial_x^\alpha u\| \\
&= I_1 + I_2 + I_3,
\end{aligned}$$

where $I_1 = \|r^{N+1}\partial_x^\alpha \lambda u^2\|$, $I_2 = \|[\triangle, r^{N+1}]\partial_x^\alpha u\|$ and $I_3 = \|[\partial_x^\beta, r^{N+1}]\partial_x^\alpha u\|$. We estimate each $I_j (j = 1, 2, 3)$ by $(1/3)CA^{N+1}(N+1)!$.

First we estimate I_3. We put $\partial_x^\beta = \partial_j\partial_k$. Since the commutator $[\partial_x^\beta, r^{N+1}]$ is equal to

$$(N+1)[r^N(\partial_j r)\partial_k + r^N(\partial_k r)\partial_j] + (N+1)r^N(\partial_x^\beta r) + (N+1)Nr^{N-1}(\partial_j r)(\partial_k r),$$

we have from Proposition 2.2 and the assumption of induction,

$$\begin{aligned} I_3 &\leq (N+1)C_1[\|\partial_j r\|\|r^N\partial_k\partial_x^\alpha u\| + \|\partial_k r\|\|r^N\partial_j\partial_x^\alpha u\|] \\ &\quad + (N+1)C_1\|r(\partial^\beta r)\|\|r^{N-1}\partial_x^\alpha u\| + (N+1)NC_1\|(\partial_j r)(\partial_k r)\|\|r^{N-1}\partial_x^\alpha u\| \\ &\leq 4C_1C_2CA^N(N+1)!, \end{aligned}$$

where $C_2 = \max_{1\leq j,k\leq n}(\|\partial_j r\|, \|r(\partial_j\partial_k r)\|, \|(\partial_j r)(\partial_k r)\|, \|r^2\|)$. If A is larger than or equal to $12C_1C_2$, we have $I_3 \leq (1/3)CA^{N+1}(N+1)!$.

Next we estimate I_2. By the same estimate as in the estimate of I_3, we have $I_2 \leq 4nC_1C_2A^N(N+1)!$. If A is larger than or equal to $12nC_1C_2$, we have $I_2 \leq (1/3)A^{N+1}(N+1)!$.

Thirdly we estimate I_1. By Leibniz's rule, Proposition 2.2 and the assumption of induction, we have

$$\begin{aligned} I_1 &\leq |\lambda|C_1^2 \sum_{\gamma\leq\alpha}\binom{\alpha}{\gamma}\|r^2\|\|r^{|\gamma|}\partial_x^\gamma u\|\|r^{|\alpha-\gamma|}\partial_x^{\alpha-\gamma}u\| \\ &\leq |\lambda|C_1^2C_2\sum_{\gamma\leq\alpha}\binom{\alpha}{\gamma}CA^{|\gamma|}|\gamma|!CA^{|\alpha-\gamma|}|\alpha-\gamma|! \\ &\leq |\lambda|C_1^2C_2C^2A^{|\alpha|}|\alpha|!\sum_{k=0}^{|\alpha|}\sum_{\substack{|\gamma|=k\\ \gamma\leq\alpha}}\binom{|\alpha|}{|\gamma|}^{-1}\binom{\alpha}{\gamma}. \end{aligned}$$

By Proposition 2.1, we have

$$\begin{aligned} I_1 &\leq |\lambda|C^2C_1^2C_2A^{|\alpha|}|\alpha|!\sum_{k=0}^{|\alpha|}1 \\ &\leq |\lambda|C^2C_1^2C_2A^{N-1}N!. \end{aligned}$$

If A is larger than or equal to $3|\lambda|CC_1^2C_2$, we have $I_1 \leq (1/3)CA^{N+1}(N+1)!$.

We consequently have

$$\|r^{N+1}\partial_x^{\alpha+\beta}u\| \leq CA^{N+1}(N+1)!,$$

if A is larger than or equal to $\max(1, 12nC_1C_2, 3|\lambda|CC_1^2C_2)$. This completes the proof. □

Remark. We can prove analyticity of solutions to analytic fully nonlinear elliptic equations. We give the proof of analyticity of solutions to analytic fully nonlinear elliptic equations of second order in forthcoming paper([7]).

Acknowledgment

The author would like to thank the referee for his useful comments and suggestions.

References

1. R. Adams, *Sobolev spaces*, Acad. Press, New York.
2. S. Bernstein, *Sur la nature analytique des solutions des équations aux dérivées partièlles du second ordre*, Math. Annal. **59** (1904), 20–76.
3. ———, *Demonstration du théorème de M. Hilbert sur la nature analytique des solutions des équations du type elliptique sans l'emploi des séries normales*, Math. Zeit. **28** (1928), 330–348.
4. A. Friedman, *On the regularity of the solutions of nonlinear elliptic and parabolic systems of partial differential equations*, J. of Math. and Mech. **7** (1958), 43–59.
5. M. Gevrey, *Demonstration du théorème de Picard-Bernstein par la method des contours successifs; prolongement analytique*, Bull. des Sci. Math. **50** (1926), 113–126.
6. K. Kato, *Interaction of analytic singularities for semilinear wave equations*, Preprint..
7. ———, *New proof of analyticity of solutions to analytic nonlinear elliptic equations of second order*, in preparation.
8. K. Kato and K. Taniguchi, *Regularizing effect for nonlinear Schrödinger equations*, to appear in Osaka J. of Math..
9. H. Lewy, *Neuer Beweis des analytischen Charakters Lösungen elliptischer Differentialgleichungen*, Math. Annal. **101** (1929), 609–619..
10. K. Masuda, *HisenkeiDaengataHouteisiki (Nonlinear elliptic partial differential equations)*, Iwanami shoten, Tokyo, 1977.
11. C. B. Morrey, *On the analyticity of the solutions of analytic nonlinear systems of partial differential equations, part I*, Amer. J. of Math. **80** (1958), 198–218.
12. ———, *On the analyticity of the solutions of analytic nonlinear systems of partial differential equations, part II*, Amer. J. of Math. **80** (1958), 219–237.
13. C. B. Morrey and L. Nirenberg, *On the analyticity of the solutions of linear elliptic systems of partial differential equations*, Comm. Pure Appl. Math. **10** (1957), 271–290.
14. I. Petrowski, *Sur l'analyticité des solutions des systèmes d'équations difféntielles*, Mat. Sbor- nik **5** (1939), 3–70.
15. T. Radó, *Das Hilbertsche Theorem über den analytischen Charakter der Lösungen de partiellen differentialgleichungen zweiter Ordnung*, Math. Zeit. **25** (1926), 514–589.

Keiichi Kato
Department of Mathematics, Science University of Tokyo
Wakamiya 26, Tokyo 162, Japan
E-mail: `kato@ma.kagu.sut.ac.jp`