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THE SPECTRAL GEOMETRY OF SOME ALMOST HERMITIAN MANIFOLDS

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Abstract. Let (M_i, g_i) be a certain almost Hermitian $2n$-manifold M_i with a Hermitian metric g_i for $i = 1, 2$, which is more general than an almost L manifold (a Kählerian manifold is known to be a special almost L manifold). Let $Spec^p(M_i, g_i)$ denote the spectrum of the real Laplacian on p-forms on M_i. The purpose of this paper is to show that for some special values of p and n, if $Spec^p(M_1, g_1) = Spec^p(M_2, g_2)$, then (M_1, g_1) is of constant holomorphic sectional curvature H_1 if and only if (M_2, g_2) is of constant holomorphic sectional curvature H_2, and $H_2 = H_1$. The corresponding results on almost L manifolds were obtained by C. C. Hsiung and C. X. Wu (The spectral geometry of almost L manifolds, Bull. Inst. Math. Acad. Sinica, 23 (1995), 229–241).

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1. Introduction

Let (M, g) be an m-dimensional compact Riemannian manifold M with a Riemannian metric g. Throughout this paper all manifolds are supposed to be C^∞ and connected. The set of the eigenvalues associated with all the p-eigenforms, $0 \le p \le m$, with respect to real Laplacian Δ and the metric g of M is called the spectrum of Δ on p-forms on M, which will be denoted by $Spec^p(M, g)$.Thus

$$Spec^p(M, g) = \{0 \ge \lambda_{1,p} \ge \lambda_{2,p} \ge \cdots > -\infty\}, \tag{1.1}$$

where each eigenvalue $\lambda_{i,p}$, $i = 1, 2, \dots,$ is repeated as many times as its multiplicity; which is finite, and the spectrum $Spec^p(M, g)$ is discrete since Δ is an elliptic operator.

It is well known that there are various examples ([13], [7] and the references there) of a pair of nonisometric manifolds with the same spectrum. Thus the

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spectra do not determine a manifold up to an isometry. However, the relationship between the geometry of a Riemannian or Kählerian manifold and its spectra has been extensively studied. Let (M, g) and (M', g') be compact Riemannian (respectively, Kählerian) manifolds M, M' with Riemannian (respectively, Hermitian) metrics g and g' and $Spec^p(M, g) = Spec^p(M', g')$ for a fixed p. Various authors [1], [4], [6], [14], …, [17] have shown that for some spacial values of p and m, (M, g) is of constant sectional (respectively, holomorphic sectional) curvature H if and only if (M', g') is of constant sectional (respectively holomorphic sectional) curvature H' and $H = H'$.

Recently Hsiung and Wu [8] have generalized the results on Kählerian manifolds to almost L manifolds of which Kählerian manifolds are special ones. The purpose of this paper is to further generalize the results of Hsiung and Wu to more general almost Hermitian manifolds.

In §2 there is a classification of all almost complex structures on a Riemannian manifold, together with inclusion relations among all classes, by means of the Riemann curvature tensor and the tensor of an almost complex structure.

In §3 we define various almost Hermitian structures and manifolds, and give necessary and sufficient or just necessary conditions for some classes of almost complex manifolds defined in §2 with an Hermitian structure to have constant holomorphic sectional curvature at each point.

§4 contains some fundamenzal formulas for a Riemannian structure and the well-known Minakshisundaram-Pleijel-Gaffney's formula for the spectra of a Riemannian manifold.

§5 deals with the spectral geometry of some almost Hermitian manifolds which are more general than L manifolds.

2. Almost complex structures

Let M be a Riemannian $2n$-manifold, and let $g_{ij}, R_{hijk}, R_{ij}, R$, and $J_i{}^j$ denote, respectively, the Riemannian metric tensor, the Riemann curvature tensor, the Ricci curvature tensor, the scalar curvature and the tensor of an almost complex structure J of M. Let (g^{ij}) be the inverse matrix of the matrix (g_{ij}). Throughout this paper all Latin indices take values $1, \cdots, 2n$ unless stated otherwise. We shall follow the usual tensor convention that indices of tensors can be raised and lowered by using g^{ij} and g_{ij} respectively, and that repeated indices imply summation. Moreover, if we multiply, for example, the components a_{ij} of a tensor of type (0,2) by the components b^{jk} of a tensor of type(2,0), it will be always understood that j is to be summed.

By using the following identities for the relationship between $J_i{}^j$ and R_{hijk}, Hsiung and Xiong [9] have defined the following four classes of almost complex structures on the manifold M:

$$R_{hijk} = J_h{}^r J_i{}^s R_{rsjk}, \tag{2.1}$$

$$R_{hijk} = J_h{}^r J_i{}^s R_{rsjk} + J_h{}^r J_j{}^s R_{risk} + J_h{}^r J_k{}^s R_{rijs}, \tag{2.2}$$

$$R_{hijk} = J_h{}^p J_i{}^q J_j{}^r J_k{}^s R_{pqrs}, \tag{2.3}$$

$$J_{i_1}{}^r J_{i_2}{}^s R_{rsi_3k} + J_{i_2}{}^r J_{i_3}{}^s R_{rsi_1k} + J_{i_3}{}^r J_{i_1}{}^s R_{rsi_2k} = 0. \tag{2.4}$$

Let $\mathfrak{L}$ and $\mathfrak{K}$ denote respectively the classes of almost complex structures (or manifolds) and the Kählerian structure (or manifolds). Let $\mathfrak{L}_1$,$\mathfrak{L}_2$,$\mathfrak{L}_3$ and $\mathfrak{C}$ denote the classes of almost complex structures (or manifolds) satisfying (2.1),$\cdots$, (2.4) respectively. Hsiung and Xiong [9] have showed the following inclusion relations.

$$\mathfrak{K} \subset \mathfrak{L}_1 \begin{array}{c} \mathfrak{L}_2 \\ \\ \mathfrak{C} \end{array} \mathfrak{L}_3 \subset \mathfrak{L}. \tag{2.5}$$

Thus for $i = 1, 2, 3$, as i decreases the structures (or manifolds) in $\mathfrak{L}_i$ resemble Kählerian structures (or manifolds) more closely.

For simplicity, throughout this paper if an almost complex manifolds M admits a certain special almost complex structure, then M is also called by the same name as the structure's.

If $J_i{}^j$ and g_{ij} satisfy

$$g_{ij} J_h{}^i J_k{}^j = g_{hk}, \tag{2.6}$$

then the almost complex structure J is called an almost Hermitian structure, and g_{ij} is called an Hermitian metric. For simplicity, throughout this paper, unless stated otherwise, by an almost Hermitian manifold M we shall always mean a manifold with an almost Hermitian structure J and an Hermitian metric g_{ij}. Friedland and Hsiung [5] called an almost Hermitian structure J an almost L structure if it satisfies

$$[\nabla_j, \nabla_k] J_i{}^h \equiv (\nabla_j \nabla_k - \nabla_k \nabla_j) J_i{}^h = 0, \tag{2.7}$$

where ∇ denotes the covariant derivation with respect to g_{ij}. Obviously, Kählerian structures are almost L structures since an almost Hermitian structure $J_i{}^j$ is Kählerian if

$$\nabla_i J_j{}^k = 0 \qquad \text{for all } i,\ j,\ k. \tag{2.8}$$

For simplicity we shall denote an almost Hermitian $\mathfrak{L}_i$ structure by AH_i for $i = 1, 2, 3$, and a Kählerian structure, an almost Hermitian $\mathfrak{C}$ structure and an almost Hermitian structure respectively by K, AHC, and AH. From (2.5) we thus obtain the following inclusion relations among almost Hermitian structures:

$$K \subset AH_1 \begin{array}{c} AH_2 \\ \\ AHC \end{array} AH_3 \subset AH. \tag{2.9}$$

We [10] also have defined an AH_1' manifold to be an almost Hermitian manifold satisfying

$$R_{hijk} = -J_h{}^r J_i{}^s R_{rsjk}. \tag{2.10}$$

Since the difference between (2.1) and (2.10) is only a sign, $AH_1' \subset AHC \subset AH_3$ and the intersection of the two classes AH_1 and AH_1' is the class of locally Euclidean spaces, that is, the classe of spaces with $R_{hijk} = 0$.

Now we introduce one new classe of AH_4 manifolds which are almost Hermitian manifolds satisfying

$$2R_{hijk} = J_h{}^r J_i{}^s R_{rsjk} + J_h{}^r J_j{}^s R_{rkis} + J_h{}^r J_k{}^s R_{rjsi}. \tag{2.11}$$

3. Almost Hermitian structures

In this section M is a Riemannian manifold as in §2. If there exists on M a tensor $J_i{}^j$ of type (1,1) satisfying

$$J_i{}^j J_j{}^k = -\delta_i^k, \tag{3.1}$$

where δ_i^k are the Kronecker deltas defined by

$$\delta_i^k = \begin{cases} 1, & i = k, \\ 0, & i \neq k, \end{cases} \tag{3.2}$$

then $J_i{}^j$ is said to define an almost complex structure on M, and M is called an almost complex manifold.

If an almost complex $J_i{}^j$ is almost Hermitian, then as a consequence of (3.1) and (2.6) the tensor J_{ij} of type (0,2) defined by

$$J_{ij} = g_{jk} J_i{}^k \tag{3.3}$$

is skew-symmetric. Thus

$$J_{ij} J_k{}^j = g_{ik}, \quad J^{ji} J_j{}^k = g^{ik}, \tag{3.4}$$

and for any tangent vector v^i of M,

$$g_{ij} J_k{}^i v^k v^j = 0, \tag{3.5}$$

which shows that v^i is orthogonal to its transform $J_j{}^i v^j$. Furthermore, on an almost Hermitian manifold M, there is a differential form

$$\omega = J_{ij} dx^i \wedge dx^j, \tag{3.6}$$

where $x^1, \dots, x^{2n}$ are local coordinates on M, and the wedge $\wedge$ denotes the exterior product. If the differential form ω is closed, that is, if

$$d\omega = 0, \tag{3.7}$$

then $J_i{}^j$ is called an almost Kählerian structure. From (3.6) and (3.7) it follows that an almost Kählerian structure satisfies

$$J_{hij} \equiv \nabla_h J_{ij} + \nabla_i J_{jh} + \nabla_j J_{hi} = 0. \tag{3.8}$$

The tensor J_{hij} is skew-symmetric in all indices.

Lemma 3.1. *An almost Hermitian $2n$-manifold M with an Hermitian metric g_{ij} and an almost Hermitian structure $J_i{}^j$ is Kählerian if it satisfies*

$$Q \equiv R + \frac{1}{2} J^{ij} J^{kl} R_{ijkl} = 0, \tag{3.9}$$

$$R_{hijk} = \frac{1}{4} H G_{hijk}, \tag{3.10}$$

where H is a nonzero constant, and

$$G_{hijk} = g_{hk} g_{ij} - g_{hj} g_{ik} + J_{hk} J_{ij} - J_{hj} J_{ik} - 2J_{hi} J_{jk}. \tag{3.11}$$

Proof. Substituting (3.10) and (3.11) and four similar equations in Bianchi identity (4.2) gives

$$\begin{aligned} \nabla_l (J_{hk} J_{ij} - J_{hj} J_{ik} - 2J_{hi} J_{jk}) + \nabla_j (J_{hl} J_{ik} - J_{hk} J_{il} - 2J_{hi} J_{kl}) \\ + \nabla_k (J_{hj} J_{il} - J_{hl} J_{ij} - 2J_{hi} J_{lj}) = 0. \end{aligned} \tag{3.12}$$

Expanding (3.12) by differentiating covariantly, multiplying the resulting equation by J^{ik} and using $J^{ij} \nabla_h J_{ij} = 0$ we obtain (3.8) which shows that M is almost Kählerian. Hence M is Kählerian since it is known that an almost Kählerian manifold with condition (3.9) is Kählerian (see for instance, [5, p.261]). □

Let M be an almost Hermitian manifold with an almost Hermitian structure $J_i{}^j$ satisfying (2.6). Then the two-dimensional plane determined by an arbitrary tangent vector u^i of M and the tangent vector $J_j{}^i u^j$ at a point p of M is called a holomorphic plane (u, Ju), and the sectional curvature with respect to the plane (u, Ju) is called the holomorphic sectional curvature $H(u, Ju)$ of M at p. If the holomorphic sectional curvature at a point p is independent of the holomorphic plane through p, then M is said to have constant holomorphic sectional curvature at p.

Concerning constant holomorphic sectional curvature we have the following theorems:

Theorem 3.1[5]**.** *A necessary and sufficient condition for an almost L 2n-manifold M to be of constant holomorphic sectional curvature H at each point is that the Riemann curvature tensor* R_{hijk} *with respect to the metric* g_{ij} *satisfy* (3.10). *Furthermore, the Ricci tensor and scalar curvature of such a manifold are given respectively by*

$$R_{ij} = \frac{n+1}{2} H g_{ij}, \tag{3.13}$$

$$R = n(n+1)H. \tag{3.14}$$

As a consequence of (3.13), *M is an Einstein manifold.*

Theorem 3.2[9]**.** *A necessary and sufficient condition for an AHC 2n-manifold M to be of constant holomorphic sectional curvature H at each point is that the Riemann curvature tensor* R_{hijk} *with respect to the metric* g_{ij} *satisfy*

$$R_{hijk} = J_h{}^r J_j{}^s R_{skir} - \frac{1}{2} H (g_{kj} g_{hi} + g_{ki} g_{hj} - J_{kj} J_{hi} - J_{ki} J_{hj}). \tag{3.15}$$

Furthermore, the Ricci tensor and scalar curvature of such a manifold are given respectively by (3.13) *and* (3.14). *As a consequence of* (3.13), *M is an Einstein manifold.*

Theorem 3.3[10]**.** *A necessary condition for an* AH_2 *2n-manifold M to be of constant holomorphic sectional curvature H at each point is that the Riemann curvature tensor* R_{hijk} *with respect to the metric* g_{ij} *satisfy*

$$R_{hijk} = \frac{1}{2} (R_{pqkj} J_h{}^p J_i{}^q + R_{pkqi} J_h{}^p J_j{}^q + R_{pjiq} J_h{}^p J_k{}^q) + \frac{1}{2} H G_{hijk}. \tag{3.16}$$

Furthermore, the Ricci tensor and scalar curvature of such a manifold are given respectively by

$$R_{ij} = 3 R_{piqr} J^{pq} J_j{}^r + 2(n+1) H g_{ij}, \tag{3.17}$$

$$R = 3 R_{psrq} J^{pq} J^{rs} + 4n(n+1)H. \tag{3.18}$$

4. Spectra of Riemannian manifolds

Let (M, g) be a Riemannian manifold of dimension $m \geq 2$ with a Riemannian mertic $g = (g_{ij})$. We shall use all the notation with $2n = m$, in §2, and also the following identities:

$$R_{hijk} + R_{hjki} + R_{hkij} = 0, \tag{4.1}$$

and

$$\nabla_l R_{hijk} + \nabla_j R_{hikl} + \nabla_k R_{hilj} = 0. \tag{4.2}$$

(4.2) is called the Bianchi identity. In 1919 Einstein suggested the following equation, called the Einstein equation,

$$T_{ij} = R_{ij} - \frac{R}{m} g_{ij}, \tag{4.3}$$

where R is the scalar curvature:

$$R = g^{ij} R_{ij}, \tag{4.4}$$

and T_{ij} is called the stress-energy tensor. When $T_{ij} = 0$, (M, g) is called an Einstein manifold and g_{ij} an Einstein mertic.

Assume that M is compact. To study $Spec^p(M, g)$ given by (1.1) we need the following Minakshisundaram-Pleijel-Gaffney's formula:

$$\sum_{i=0}^{\infty} e^{\lambda_{i,p} t} \underset{t \downarrow 0}{\sim} \frac{1}{4\pi t^{\frac{m}{2}}} \sum_{i=0}^{\infty} a_{i,p} t^i, \tag{4.5}$$

where

$$a_{0,p} = \binom{m}{p} \int_M dM, \tag{4.6}$$

$$a_{1,p} = \left[\frac{1}{6} \binom{m}{p} - \binom{m-2}{p-1} \right] \int_M R \, dM, \tag{4.7}$$

$$a_{2,p} = \int_M [c_1(m,p) R^2 + c_2(m,p) |R_{ij}|^2 + c_3(m,p) |R_{hijk}|^2] dM, \tag{4.8}$$

and

$$c_1(m,p) = \frac{1}{72} \binom{m}{p} - \frac{1}{6} \binom{m-2}{p-1} + \frac{1}{2} \binom{m-4}{p-2}, \tag{4.9}$$

$$c_2(m,p) = -\frac{1}{180} \binom{m}{p} + \frac{1}{2} \binom{m-2}{p-1} - 2 \binom{m-4}{p-2}, \tag{4.10}$$

$$c_3(m,p) = \frac{1}{180} \binom{m}{p} - \frac{1}{12} \binom{m-2}{p-1} + \frac{1}{2} \binom{m-4}{p-2}, \tag{4.11}$$

dM being the volume element of M, $\binom{m}{p}$ a binomial coefficient, and $|R_{ij}|^2$ and $|R_{hijk}|^2$ the square of the lengths of the Ricci and Riemann curvature tensors respectively given by

$$|R_{ij}|^2 = R^{ij}R_{ij}, \quad |R_{hijk}|^2 = R^{hijk}R_{hijk}. \tag{4.12}$$

The coefficients $a_{0,p}$,$a_{1,p}$ and $a_{2,p}$ have been calculated for $p = 0$ by many authors (see [1], [12]), and determined for all p by V.K.Patodi [14].

Remarks. **1.** Let (M, g) and (M', g') be compact Riemannian manifolds. If $Spec^p(M, g) = Spec^p(M', g')$ for some p, then from (4.5) and (4.6) we have

(i) $m = \dim M = \dim M' = m'$,

(ii) $Vol.M = Vol.M'$.

2. For a geometric quantity A on (M, g), we shall denote the corresponding quantity on (M', g') by A'.

5. $AH_{2,4}$ manifolds

The purpose of this section is to study the spectral geometry of the almost Hermitian manifold M satisfying (2.2) and (2.11), which we call an $AH_{2,4}$-manifold. At first we need

Lemma 5.1. *An AH_1-manifold is an $AH_{2,4}$-manifold.*

Proof. Substituting (2.1) for the second and the third terms on the right-hand side of (2.11) and using (4.1) we can easily see that the right-hand side of (2.11) becomes automatically the left-hand side of (2.11). So an AH_1-manifold is an AH_4-manifold and therefore an $AH_{2,4}$-manifold, since from (2.9) an AH_1-manifold is also an AH_2-manifold. □

Theorem 5.1. *A necessary and sufficient condition for an $AH_{2,4}$ $2n$-manifold M to be of constant holomorphic sectional curvature H at each point is that the Riemann curvature tensor R_{hijk} with respect to the metric g_{ij} satisfy* (3.10). *Furthermore, the Ricci tensor and scalar curvature of such a manifold are given respectively by* (3.13) *and* (3.14). *As a consequence of* (3.13), *M is an Einstein manifold.*

Proof. Suppose that M is of constant holomorphic sectional curvature H at each point. Since M is an AH_2 manifold, (3.16) holds. Since M is also an AH_4 manifold, (2.11) also holds. Substituting (2.11) in the right-hand side of (3.16) gives condition (3.10) immediately.

For the proof of the sufficiency of condition (3.10) one may see [5] as Theorem 3.1 also has this condition. □

Now we want to derive some relations among the tensors $J_i{}^j$, R_{hijk} and R_{ij} of an AH_4 $2n$-manifold M. At first, from (4.1) follows

$$J^{kr}R_{rkis} = -J^{kr}R_{risk} - J^{rk}R_{ksri} = 2J^{kr}R_{riks}. \tag{5.4}$$

Multiplying (2.11) by g^{hk} and using (3.4), (5.4), we obtain

$$3R_{ij} = J^{kr}J_i{}^s R_{rsjk} + 2J^{kr}J_j{}^s R_{riks}. \tag{5.5}$$

Similarly, multiplication of (2.11) by g^{ij} and use of (5.4) give

$$2R_{hk} + J_h{}^r J_k{}^s R_{rs} = 3J_h{}^r J^{is} R_{ikrs}. \tag{5.6}$$

Substituting the right-hand side of (5.6) for each term on the right-hand side of (5.5) yields readily

$$R_{ij} = J_i{}^r J_j{}^s R_{rs}, \tag{5.7}$$

which together with (5.6) implies that

$$R_{hk} = J_h{}^r J^{is} R_{ikrs} = J_k{}^r J^{is} R_{ihrs}, \tag{5.8}$$

where the last equality is due to the symmetry of h and k. Multiplying (5.8) by g^{hk} gives

$$J^{is}J^{hr}R_{ihrs} = R. \tag{5.9}$$

Multiplying (5.4) by $J_j{}^i$ and using (5.8) we obtain

$$J^{kr}J_j{}^i R_{krsi} = -2R_{js}. \tag{5.10}$$

Multiplication of (5.10) by g^{is} gives

$$J^{kr}J^{si}R_{krsi} = -2R. \tag{5.11}$$

Multiplying (5.7) by $J_k{}^i$ yields

$$J_k{}^i R_{ij} = -J_j{}^s R_{ks}, \tag{5.12}$$

which together with (3.5) shows that

$$R_{ir}J^{is}R_{js}J^{jr} = R_i{}^r J^{is} R_{jr} J_s{}^j = -R_{ir}R^{ir}. \tag{5.13}$$

On the other hand, since

$$J^{rs}(R_{rpks} + R_{rkps}) = J^{rs}R_{rpks} + J^{sr}R_{skpr} = 0,$$

we have

$$J^{rs}R_{rpks} = -J^{rs}R_{rkps}. \tag{5.14}$$

From (5.8) and (5.14) it follows that

$$R_{hk} = J_h{}^r J^{is} R_{ikrs} = -J_k{}^r J^{is} R_{irks}. \tag{5.15}$$

Multiplying (5.1) by $J_j{}^h$ gives

$$J^{is}R_{ikjs} = J_k{}^r J_j{}^h J^{is} R_{irhs}. \tag{5.16}$$

Moreover, using (5.8), (3.4), (4.12) we readily obtain

$$|R_{ij}|^2 = |J^{is}R_{ikrs}|^2. \tag{5.17}$$

The following lemma is an immediate consequence of (5.11), Theorem 5.1 and Lemma 3.1.

Lemma 5.2. *If an $AH_{2,4}$ $2n$-manifold M is of constant holomorphic sectional curvature H at each point, then M is Kählerian.*

The Bochner curvature tensor $B = (B_{hijk})$ of an almost Hermitian m-manifold M with an Hermitian mertic g and an almost Hermitian structure $J_i^{\ j}$ is defined as follows:

$$B_{hijk} = R_{hijk} - \frac{1}{2(n+2)} A_{hijk} + \frac{R}{4(n+1)(n+2)} G_{hijk}, \tag{5.18}$$

where the components of the tensors $G = (G_{hijk})$ and $A = (A_{hijk})$ are given respectively by (3.11) and

$$A_{hijk} = a_{hijk} + b_{hijk} - 2c_{hijk}, \tag{5.19}$$

with

$$a_{hijk} = g_{hk}R_{ij} - g_{hj}R_{ik} + g_{ij}R_{hk} - g_{ik}R_{hj}, \tag{5.20}$$

$$b_{hijk} = J_{ij}J_h^{\ r}R_{rk} - J_{ik}J_h^{\ r}R_{rj} + J_{hk}J_i^{\ r}R_{rj} - J_{hj}J_i^{\ r}R_{rk}, \tag{5.21}$$

$$c_{hijk} = J_{jk}J_h^{\ r}R_{ri} + J_{hi}J_j^{\ r}R_{rk}. \tag{5.22}$$

Now we have the following crucial lemma.

Lemma 5.3. *For an AH_4 $2n$-manifold M,*

$$|B_{hijk}|^2 = |R_{hijk}|^2 - \frac{8}{n+2}|R_{ij}|^2 + \frac{2}{(n+1)(n+2)}R^2. \tag{5.23}$$

Proof. From (5.18) it follows that

$$\begin{aligned} |B_{hijk}|^2 =& |R_{hijk}|^2 - \frac{1}{n+2} R_{hijk}A^{hijk} \\ &+ \frac{R}{2(n+1)(n+2)} R_{hijk}G^{hijk} + \frac{1}{4(n+2)^2}|A_{hijk}|^2 \\ &- \frac{R}{4(n+1)(n+2)^2} A_{hijk}G^{hijk} + \frac{R^2}{16(n+1)^2(n+2)^2}|G_{hijk}|^2. \end{aligned} \tag{5.24}$$

By (3.1), (3.4) and other equations of this section some elementary but complicated computations give the following:

$$|a_{hijk}|^2 = 8(n-1)|R_{ij}|^2 + 4R^2,$$

$$(5.25)\qquad \begin{aligned} a_{hijk}b^{hijk} &= 8|R_{ij}|^2 \qquad \text{by (5.13)},\\ a_{hijk}c^{hijk} &= -8|R_{ij}|^2 \qquad \text{by (5.13)},\\ |b_{hijk}|^2 &= 8(n-1)|R_{ij}|^2 + 4R^2,\\ b_{hijk}c^{hijk} &= -8|R_{ij}|^2 \qquad \text{by (5.13)},\\ |c_{hijk}|^2 &= 4n|R_{ij}|^2 + 2R^2; \end{aligned}$$

$$(5.26)\qquad \begin{aligned} &R_{hijk}a^{hijk} = 4|R_{ij}|^2, \quad R_{hijk}b^{hijk} = 4|R_{ij}|^2 \qquad \text{by (5.15)},\\ &R_{hijk}c^{hijk} = -4|R_{ij}|^2 \qquad \text{by (5.10)}; \end{aligned}$$

$$(5.27)\qquad R_{hijk}G^{hijk} = 8R \qquad \text{by (5.9), (5.11)};$$

$$(5.28)\qquad \begin{aligned} a_{hijk}a^{hijk} &= 8(n-1)|R_{ij}|^2 + 4R^2,\\ a_{hijk}b^{hijk} &= 8|R_{ij}|^2 \qquad \text{by (5.13)},\\ a_{hijk}c^{hijk} &= -8|R_{ij}|^2 \qquad \text{by (5.13)},\\ |b_{hijk}|^2 &= 8(n-1)|R_{ij}|^2 + 4R^2,\\ b_{hijk}c^{hijk} &= -8|R_{ij}|^2 \qquad \text{by (5.13)},\\ |c_{hijk}|^2 &= 4n|R_{ij}|^2 + 2R^2; \end{aligned}$$

$$(5.29)\qquad \begin{aligned} a_{hijk}G^{hijk} &= 8(n+1)R,\\ b_{hijk}G^{hijk} &= 8(n+1)R,\\ c_{hijk}G^{hijk} &= -8(n+1)R; \end{aligned}$$

$$(5.30)\qquad |G_{hijk}|^2 = 32n(n+1),$$

From (5.19), (5.25), $\cdots$, (5.29) we obtain

$$\begin{aligned} R_{hijk}A^{hijk} &= 16|R_{ij}|^2,\\ |A_{hijk}|^2 &= 32(n+2)|R_{ij}|^2 + 16R^2,\\ A_{hijk}G^{hijk} &= 32(n+1)R,\\ |G_{hijk}|^2 &= 16n(n+2). \end{aligned}$$

Substituting (5.27) and (5.31) in (5.24) yields (5.23) immediately. □

Assume that M is compact. Now we can express the coefficient $a_{2,p}$ of formula (4.5) in terms of $|B_{hijk}|^2$ and

$$(5.32)\qquad |T_{ij}|^2 = |R_{ij}|^2 - \frac{1}{2n}R^2,$$

which follows from (4.3) readily.

Lemma 5.4. *For a compact AH_4 $2n$-manifold M,*

$$a_{2,p} = \int_M [b_1(n,p)|B_{hijk}|^2 + b_2(n,p)|T_{ij}|^2 + b_3(n,p)R^2]dM, \tag{5.33}$$

where

$$\begin{aligned} b_1(n,p) &= c_3(m,p), \\ b_2(n,p) &= c_2(m,p) + \frac{8}{n+2}c_3(m,p), \\ b_3(n,p) &= c_1(m,p) + \frac{1}{2n}c_2(m,p) + \frac{2}{n(n+1)}c_3(m,p), \end{aligned} \tag{5.34}$$

and $m = 2n$.

Proof. The lemma is an immediate consequence by substituting (5.23) and (5.32) in (4.8). □

Lemma 5.5. *An $AH_{2,4}$ $2n$-manifold M for $n \geq 2$ is of constant holomorphic sectional curvature if and only if the tensors B and $T = (T_{ij})$ are zero.*

Proof. Suppose M to be of constant holomorphic sectional curvature H. Then (3.10), (3.11), (3.13) and (3.14) hold by Theorem 5.1. Substituting (3.10), (3.11), (3.13) and (3.14) in (5.18) shows readily that $B_{hijk} = 0$. $T_{ij} = 0$ follows from (4.3), (3.13), (3.14).

Conversely, suppose that $B = 0$ and $T = 0$ which implies that M is an Einstein space, so that R is constant for $2n \geq 4$. Substituting $R_{ij} = \dfrac{R}{2n}g_{ij}$ in (5.18) and using (3.1) and (2.6), we obtain

$$R_{hijk} = \frac{R}{4n(n+1)}G_{hijk}.$$

Hence, by Theorem 5.1. M is of constant holomorphic sectional curvature $H = \dfrac{R}{n(n+1)}$. □

6. The main theorem

The main results of this paper are listed in the following theorem.

Theorem 6.1. *Let (M, g, J) and (M', g', J') be compact $AH_{2,4}$ $2n$-manifolds with almost Hermitian structures J and J', Hermitian mertics g and g'. Let $(\mathbb{C}P^n, g_0, J_0)$ be the complex n-dimensional projective space $\mathbb{C}P^n$ with the Fubini-Study metric g_0 and the standard complex structure J_0. Consider the following statements:*

1. *(*) (M, g, J) is of constant holomorphic sectional curvature H if and only if (M', g', J') is of constant holomorphic sectional curvature H', and $H = H'$;*
2. *(**) (M, g, J) is Kählerian and holomorphically isometric to $(\mathbb{C}P^n, g_0, J_0)$.*

Then we have the following:

- **(i)** *(*) is true if* $Spec^0(M, g) = Spec^0(M', g')$ *and* $2n \leq 10$.
- **(ii)** *(**) is true if* $Spec^0(M, g) = Spec^0(\mathbb{C}P^n, g_0)$ *and* $2n \leq 10$.
- **(iii)** *(*) is true if* $Spec^1(M, g) = Spec^1(M', g')$ *and* $2n = 2$ *or* $16 \leq 2n \leq 102$.
- **(iv)** *(**) is true if* $Spec^1(M, g) = Spec^1(\mathbb{C}P^n, g_0)$ *and* $2n = 2$ *or* $16 \leq 2n \leq 102$.
- **(v)** *(*) is true if* $Spec^2(M, g) = Spec^2(M', g')$ *and* $2n = 2, 6, 8, 14$ *or* $18 \leq 2n \leq 188$.
- **(vi)** *(**) is true if* $Spec^2(M, g) = Spec^2(\mathbb{C}P^n, g_0)$ *and* $2n = 2, 6, 8, 14$ *or* $18 \leq 2n \leq 188$.
- **(vii)** *(*) is true if* $Spec^p(M, g) = Spec^p(M', g')$ *for* $p = 0$ *and 1.*
- **(viii)** *(**) is true if* $Spec^p(M, g) = Spec^p(\mathbb{C}P^n, g_0)$ *for* $p = 0$ *and 1.*

Remark. For almost L manifolds M and M' Theorem 6.1 is due to Hsiung and Wu [8]. When (M, g, J) and (M', g', J') are Kählerian manifolds, parts (**i**)-(**vi**) of Theorem 6.1 are reduced to the known results in ([14],[15],[16]) mentioned before.

Proof of Theorem 6.1. It is clear that (*) and (**) hold in the case of $n = 1$. So we assume $n \geq 2$.

(i) From (4.6), (4.7), (4.9), ..., (4.11) for $m = 2n$, and (5.33), (5.34) we obtain

$$a_{0,0} = \int_M dM = Vol\ M, \tag{6.1}$$

$$a_{1,0} = \frac{1}{6} \int_M R\ dM, \tag{6.2}$$

$$a_{2,0} = \frac{1}{360} \int_M \left[2|B_{hijk}|^2 + \frac{2(6-n)}{n+2} |T_{ij}|^2 + \frac{5n^2 + 4n + 3}{n(n+1)} R^2 \right] dM. \tag{6.3}$$

Since the roles of (M, g, J) and (M', g', J') in the theorem are the same, we need only to prove the "if" part, and the "only if" part can be proved in the same way as by interchanging the roles of (M, g, J) and (M', g', J'). So we assume that (M', g', J') has constant holomorphic sectional curvature H'. Then $R' = 2H'$ by Theorem 5.1, and therefore R' is constant. Thus from $a_{0,0} = {a'}_{0,0}$ and $a_{1,0} = {a'}_{1,0}$ it follows that $\int_M R^2 dM \geq \int_{M'} {R'}^2 dM'$. In fact, using the Schwarz inequality we have

$$\begin{aligned}\left(\int_M dM\right)\left(\int_M R^2 dM\right) &\geq \left(\int_M R\, dM\right)^2 = \left(\int_{M'} R'\, dM'\right)^2 \\ &= {R'}^2(Vol\ M')^2 \\ &= {R'}^2 Vol\ M \cdot Vol\ M' \\ &= Vol\ M \cdot \int_{M'} {R'}^2 dM.\end{aligned} \tag{6.4}$$

On the other hand, applying Lemma 5.5 to (M', g', J') gives

$$|{B'}_{hijk}|^2 = |T_{ij}|^2 = 0. \tag{6.5}$$

Thus from $a_{2,0} = {a'}_{2,0}$, (6.4), (6.3) and its corresponding equation $(6,3)'$ for (M', g', J') and $2n \leq 10$, it follows that

$$|B_{hijk}|^2 = |T_{ij}|^2 = 0, \quad \int_M R^2 dM = \int_{M'} {R'}^2 dM'. \tag{6.6}$$

Hence by Lemma 5.5, (M, g, J) is of constant holomorphic sectional curvature $H = H'$.

(ii) Since $(\mathbb{C}P^n, g_0, J_0)$ is of constant holomorphic sectional curvature $c > 0$, (M, g, J) is of constant holomorphic sectional curvature $c > 0$ by part (**i**). From Lemma 5.2 it follows that (M, g, J) is Kählerian. Hence (M, g, J) is holomorphically isomertic to $(\mathbb{C}P^n, g_0, J_0)$.

(iii) As in part (**i**) we have

$$a_{0,1} = 2n \int_M dM = 2n\ Vol\ M, \tag{6.7}$$

$$a_{1,1} = \frac{n-3}{3} \int_M R\ dM, \tag{6.8}$$

$$\begin{aligned} a_{2,1} = \frac{1}{360} \int_M &\{2(2n-15)|B_{hijk}|^2 + \frac{4}{n+2}[n(51-n)+30]|T_{ij}|^2 \\ &+ \frac{2}{n(n+1)}[n^2(5n-26)+18n+15]R^2\} dM. \end{aligned} \tag{6.9}$$

For $8 \leq n \leq 51$, all coefficients of $|B_{hijk}|^2$, $|T_{ij}|^2$ and R^2 in (6.9) are positive. Also as before we may assume that (M', g', J') is of constant holomorphic sectional curvature H' so that R' is constant and (6.5) holds. We also have (6.4). Thus from $a_{2,1} = a'_{2,1}$ and (6.9), (6.9)$'$ follows (6.6). Hence (M, g, J) is of constant holomorphic sectional curvature $H = H'$.

(**iv**) follows from (**iii**) and Lemma 5.2.

(**v**) As before we have

$$a_{0,2} = n(2n-1)\int_M dM = n(2n-1)Vol\ M, \tag{6.10}$$

$$a_{1,2} = \frac{1}{6}(2n^2 - 13n + 12)\int_M R\ dM. \tag{6.11}$$

$$a_{2,2} = \frac{1}{360}\int_M (A_1|B_{hijk}|^2 + A_2|T_{ij}|^2 + A_3R^2)dM, \tag{6.12}$$

where

$$A_1 = 2(2n^2 - 31n + 120)$$

$$A_2 = \frac{2}{n+2}(2n^3 - 193n^2 + 426n + 120), \tag{6.13}$$

$$A_3 = \frac{1}{n(n+1)}(10n^4 - 117n^3 + 362n^2 - 183n - 60).$$

Thus for $n = 3, 4$ or 7, or $9 \le n \le 94$, A_1, A_2, and A_3 are positive. The remaining part of the proof is completely similar to that in (**i**) or (**iii**).

(**vi**) follows from (**v**) and Lemma 5.2.

(**vii**) Multiplying (6.3) by $(2n-15)$ and subtracting the resulting equation from (6.9) we obtain

$$a_{2,1} - (2n-15)a_{2,0} = 15\int_M \left[10|T_{ij}|^2 + \frac{1}{n}(n+5)R^2\right] dM. \tag{6.14}$$

Assume that (M', g', J') is of constant holomorphic curvature H'. Then we have (6.5) and (6.4) which together with $a_{2,1} - (2n-15)a_{2,0} = a'_{2,1} - (2n-15)a'_{2,0}$, (6.14) and (6.14)$'$ implies that $|T_{ij}|^2 = 0$. Thus from $a_{2,0} = a'_{2,0}$, (6.3) and (6.3)$'$ it follows that $|B_{hijk}|^2 = 0$. Hence (M, g, J) is of constant holomorphic sectional curvature $H = H'$.

(**viii**) follows from (**vii**) and Lemma 5.2. □

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