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TWO WEIGHTED Φ–INEQUALITIES FOR THE FRACTIONAL MAXIMAL OPERATOR

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Abstract. Sufficient conditions are given for the fractional maximal operator to send a weighted Orlicz class into another one. As an application, an Orlicz version of the famous Fefferman-Stein inequality is obtained.

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1. Introduction and Results

The fractional maximal operator M_α of order α, $0 \le \alpha < n$, is defined as

$$(M_\alpha f)(x) = \sup\Big\{|Q|^{\frac{\alpha}{n}-1}\int_Q |f(y)|dy;\ \text{where } Q \text{ is a cube containing } x\Big\}.$$

Here n is a nonnegative integer and the cubes considered have sides parallel to the coordinates axis. So $M = M_0$ is the well-known Hardy-Littlewood maximal operator. In this paper $\Phi_1(.)$, $\Phi_2(.)$ and $\Phi(.)$ denote some continuous and increasing functions defined on $[0,\infty[\to[0,\infty[$ which take the value 0 at 0 and tend to ∞ when $t\to\infty$; and $z(.)$, $z_1(.)$, $z_2(.)$, $u(.)$ and $v(.)$ are weights, i.e. nonnegative locally integrable functions on $\mathbb{R}^n$.

Our purpose is to give a sufficient condition on these weights which guarantees $M_\alpha : z_1L_v^{\Phi_1} \to z_2L_u^{\Phi_2}$. This boundedness means there is $C>0$ for which

(1.1)
$$\int_{\mathbb{R}^n}\Phi_2\Big(z_2(x)(M_\alpha f)(x)\Big)u(x)dx \le \Phi_2\Phi_1^{-1}\Big[\int_{\mathbb{R}^n}\Phi_1\Big(Cz_1(x)f(x)\Big)v(x)dx\Big]$$

for all nonnegative functions $f(.)$. Interest in study of (1.1) comes from the fact that such an integral inequality is more and more used (particularly by Italian schools) to tackle problems in P.D.E..

Inequality (1.1) is a generalization of the classical two weight inequality

$$(1.2)\qquad \int_{\mathbb{R}^n}(M_\alpha f)^q(x)u(x)dx \le C\Big(\int_{\mathbb{R}^n} f^p(x)v(x)dx\Big)^{\frac{q}{p}}.$$

The consideration of $z_1(.)$ and $z_2(.)$, as in [Bl-Ke] and [Go-Ko2], is motivated by the fact that the weights cannot be combined as in the Lebesgue case where $\int_{\mathbb{R}^n} f^p(y)v(y)dy = \int_{\mathbb{R}^n}(fv^{\frac{1}{p}})^p(y)dy$.

Problem (1.1) for particular α and weight functions was considered by many authors [Ke-To], [Bl-Ke], [Go-Ko1], [Go-Ko2], [Su], [Ch] and [Qi]. But this inequality has not been studied in full generality, as we will do in this work. Indeed the boundedness $M_0 : L_v^\Phi \to L_v^\Phi$ was characterized by Bloom and Kerman [Bl-Ke] and also independently by Gogatishvili and Kokilashvili [Go-Ko1]. A significant approach of the two weight inequality $M_0 : L_v^\Phi \to L_u^\Phi$ was given by Sun [Su] and also by Chen [Ch]. And a solution for $M_\alpha : \frac{1}{v}L_v^{\Phi_1} \to L_u^{\Phi_2}$, with $0 \leq \alpha < n$, was presented by Qinsheng [Qi].

A characterization of (1.2), with $1 < p \leq q < \infty$, was due to Sawyer [Sa]. However the right necessary and sufficient condition is expressed in terms of the maximal operator M_α itself, so in general it is difficult to decide whether a given pair of weight functions is convenient for (1.2). Consequently people, who were interested in problems of weighted inequalities, investigated simpler conditions not necessarily a characterizing condition. Observe that (1.2) implies

$$(1.3)\qquad |Q|^{\frac{\alpha}{n}+\frac{1}{q}-\frac{1}{p}}\Big(\frac{1}{|Q|}\int_Q u(y)dy\Big)^{\frac{1}{q}}\Big(\frac{1}{|Q|}\int_Q\Big[\frac{1}{v(y)}\Big]^{p'}v(y)dy\Big)^{\frac{1}{p'}} \leq A$$

for all cubes Q and for a fixed constant $A > 0$. Here $p' = \frac{p}{p-1}$. Conversely Pérez [Pe] proved that, for $1 < p \leq q < \infty$, (1.2) is true whenever there is $\varepsilon > 1$ such that

$$(1.4)\qquad |Q|^{\frac{\alpha}{n}+\frac{1}{q}-\frac{1}{p}}\Big(\frac{1}{|Q|}\int_Q u(y)dy\Big)^{\frac{1}{q}}\Big(\frac{1}{|Q|}\int_Q\Big[\frac{1}{v(y)}\Big]^{\varepsilon p'}v^\varepsilon(y)dy\Big)^{\frac{1}{\varepsilon p'}} \leq A.$$

for all cubes Q. Clearly by the Hölder inequality condition (1.4) implies (1.3).

So the natural question, answered in this paper, is to find an analogue of this Pérez's result for the problem (1.1) without using standard assumptions like $\triangle_2$-condition on $\Phi_1(.)$ and $\Phi_2(.)$ [see [Ke-To]] nor Muckenhoupt A_∞-condition on the weight functions.

Following Bloom and Kerman [Bl-Ke], if $\Phi_1(.)$ is a N-function, then a necessary condition for the boundedness $M_\alpha : z_1 L_v^{\Phi_1} \to z_2 L_u^{\Phi_2}$ is

$$(1.5)\qquad \int_Q \Phi_1^*\Big[\frac{|Q|^{\frac{\alpha}{n}-1}}{A\lambda z_1(x)v(x)}\Theta(\lambda,Q)\Big]v(x)dx \leq \Theta(\lambda,Q) < \infty \quad \text{for all } \lambda > 0 \text{ and all cubes } Q,$$

where $A > 0$ is a fixed constant. Here

$$\Theta(\lambda,Q) = \Phi_1\Phi_2^{-1}\Big[\int_Q \Phi_2\Big(\lambda z_2(y)\Big)u(y)dy\Big],$$

and $\Phi_1^*(.)$ is the complementary function to $\Phi_1(.)$. Recall that $\Phi(.)$ is a N-function whenever it is a convex function with $\lim_{s\to 0}\frac{\Phi(s)}{s}=\lim_{s\to\infty}\frac{s}{\Phi(s)}=0$, and its complementary function $\Phi^*(.)$ is defined as $\Phi^*(t)=\sup_{s\geq 0}\{st-\Phi(s)\}$. Condition (1.5) is the substitute of (1.3) in the Orlicz setting. And the assumption "$p\leq q$" will be expressed by the growth condition

$$\sum_k \Phi_2\Phi_1^{-1}(t_k)\leq \Phi_2\Phi_1^{-1}\Big[c_0\sum_k t_k\Big]\quad \text{for all } t_k>0, \tag{1.6}$$

where $c_0>0$ is a fixed constant. Our main result can be stated as follows.

Theorem 1. *Suppose the condition* (1.6) *is satisfied and* $\Phi_1^{\frac{1}{t}}(.)$ *is a N-function for some* $t>1$. *Then* $M_\alpha : z_1L_v^{\Phi_1}\to z_2L_u^{\Phi_2}$ *whenever for a constant* $A>0$

$$\int_Q (\Phi_1^{\frac{1}{t}})^*\left[\frac{|Q|^{\frac{\alpha}{n}-1}}{A\lambda z_1(x)v^{\frac{1}{t}}(x)}\Theta_t(\lambda,Q)\right]v^{\frac{1}{t}}(x)dx \\ \leq \Theta_t(\lambda,Q)<\infty \quad \textit{for all } \lambda>0 \textit{ and all cubes } Q, \tag{1.7}$$

where

$$\Theta_t(\lambda,Q)=|Q|\left\{\frac{1}{|Q|}\Phi_1\Phi_2^{-1}\left[\int_Q \Phi_2\Big(\lambda z_2(y)\Big)u(y)dy\right]\right\}^{\frac{1}{t}}.$$

Observe that (1.7), a substitute of (1.4), is reduced to (1.5) when $t=1$. And the Pérez's result, quoted above, is covered by Theorem 1 by taking $\Phi_1(s)\approx s^p$ and $t=\frac{p}{(\varepsilon p')'}$, where $\varepsilon>1<p<\infty$ and $r'=\frac{r}{r-1}$ for each $r>1$. Note that $\Phi_1^{\frac{1}{t}}(s)\approx s^{(\varepsilon p')'}$, $(\Phi_1^{\frac{1}{t}})^*(s)\approx s^{(\varepsilon p')}$ and $v^{-(\varepsilon p')\frac{1}{t}+\frac{1}{t}}(.)=v^{\frac{-\varepsilon}{p-1}}(.)=\left[\frac{1}{v(.)}\right]^{\varepsilon p'}v^{\varepsilon}(.)$.

In the classical Lebesgue setting, many problems in Analysis are involved by the famous Fefferman-Stein inequality

$$\int_{\mathbb{R}^n}(M_\alpha f)^p(x)u(x)dx\leq C\int_{\mathbb{R}^n} f^p(x)(M_{\alpha p}u)(x)dx\quad \text{for all } f(.)\geq 0. \tag{1.8}$$

Here $0\leq\alpha<\frac{n}{p}$ and $C>0$ is a fixed constant independant of $u(.)$. As an application of Theorem 1, we obtain an Orlicz version of (1.8).

Proposition 2. *Assume that* $\Phi^{\frac{1}{t}}(.)$ *is a N-function for some* $t>1$. *Then*

$$\int_{\mathbb{R}^n}\Phi\Big(z(x)(M_\alpha f)(x)\Big)u(x)dx\leq \\ \int_{\mathbb{R}^n}\Phi\Big(Cf(x)\Big)(\mathcal{M}_{\alpha,t,\Phi,z}u)(x)dx\quad \textit{for all } f(.)\geq 0. \tag{1.9}$$

Here $C > 0$ does not depend on the weight functions $u(.)$ and $z(.)$. The maximal operator $\mathcal{M}_{\alpha,t,\Phi,z}$ is defined by

$$(\mathcal{M}_{\alpha,t,\Phi,z}g)(x) = \sup_{Q\ni x}\sup_{\lambda>0}\Big\{\Big[\frac{\lambda^{-1}|Q|^{\frac{\alpha}{n}}}{S^{-1}_{\Phi,t}(\lambda|Q|^{-\frac{\alpha}{n}})}\Big]^t \frac{1}{|Q|}\int_Q \Phi\Big(\lambda z(y)\Big)|g(y)|dy\Big\}$$

where $S_{\Phi,t}(s) = s^{-1}(\Phi^{\frac{1}{t}})^*(s)$.

For $z(.) = 1$, $\Phi(s) \approx s^p$ and $t = \frac{p}{(\varepsilon p')'}$ with $\varepsilon > 1 < p < \infty$, then $\mathcal{M}_{\alpha,t,\Phi,z} = M_{\alpha p}$, so inequality (1.8) is covered by (1.9). For $z(.) = 1$ and $\alpha = 0$, elementary arguments lead to a similar inequality as (1.9) with $\mathcal{M}_{\alpha,t,\Phi,z}$ replaced by $M = M_0$. Thus the real significance of Proposition 2 appears when $z(.) \neq 1$ or $\alpha \neq 0$.

2. Proof of Proposition 2

Following Theorem 1, it remains to get

(2.1)
$$\begin{aligned}&\int_Q (\Phi^{\frac{1}{t}})^*\Big[\frac{\lambda^{-1}|Q|^{\frac{\alpha}{n}-1}}{v^{\frac{1}{t}}(x)}\Theta_t(\lambda,Q)\Big]v^{\frac{1}{t}}(x)dx\\ =&\lambda^{-1}|Q|^{\frac{\alpha}{n}-1}\Theta_t(\lambda,Q)\int_Q S_{\Phi,t}\Big[\lambda^{-1}|Q|^{\frac{\alpha}{n}-1}\Theta_t(\lambda,Q)\frac{1}{v^{\frac{1}{t}}(x)}\Big]dx\\ \leq&\Theta_t(\lambda,Q) < \infty \quad \text{for all } \lambda > 0 \text{ and all cubes } Q,\end{aligned}$$

where $v(x) = \Big(\mathcal{M}_{\alpha,t,\Phi,z}u\Big)(x)$ and $\Theta_t(\lambda,Q) = |Q|\Big\{\frac{1}{|Q|}\int_Q \Phi\Big(\lambda z(y)dy\Big)u(y)dy\Big\}^{\frac{1}{t}}$. By the definition of $v(.)$, then

(2.2)
$$\begin{aligned}&\lambda^{-1}|Q|^{\frac{\alpha}{n}}\Big(\frac{1}{|Q|}\int_Q \Phi\Big(\lambda z(y)\Big)u(y)dy\Big)^{\frac{1}{t}}\\ &\leq S^{-1}_{\Phi,t}\Big(\lambda|Q|^{-\frac{\alpha}{n}}\Big) \times v^{\frac{1}{t}}(x) \quad \text{for all } \lambda > 0 \text{ and all cubes } Q \ni x.\end{aligned}$$

Condition (2.1) will appear once it is proved that

(2.3)
$$\lambda^{-1}|Q|^{\frac{\alpha}{n}-1}\int_Q S_{\Phi,t}\Big[\lambda^{-1}|Q|^{\frac{\alpha}{n}-1}\Theta_t(\lambda,Q)\frac{1}{v^{\frac{1}{t}}(x)}\Big]dx \leq 1.$$

For doing, call $\mathcal{I}(Q,t,\lambda)$ the left member of (2.3). Using the definition of $\Theta_t(\lambda,Q)$ and (2.2), then inequality (2.3) will follow since

$$\begin{aligned}&\mathcal{I}(Q,t,\lambda) =\\ &\lambda^{-1}|Q|^{\frac{\alpha}{n}-1}\int_Q S_{\Phi,t}\Big[\lambda^{-1}|Q|^{\frac{\alpha}{n}}\Big(\frac{1}{|Q|}\int_Q \Phi\Big(\lambda z(y)\Big)u(y)dy\Big)^{\frac{1}{t}}\frac{1}{v^{\frac{1}{t}}(x)}\Big]dx\\ &\leq \lambda^{-1}|Q|^{\frac{\alpha}{n}}S_{\Phi,t}S^{-1}_{\Phi,t}\Big(\lambda|Q|^{-\frac{\alpha}{n}}\Big) \leq 1.\end{aligned}$$

3. Proof of Theorem 1

The result is based on two lemmas.

Lemma 3. *The above condition* (1.7) *implies*

$$\int_Q \Phi_2\Big(z_2(x)\Big[|Q|^{\frac{\alpha}{n}-1}\int_Q f(y)dy\Big]\Big)u(x)dx \leq \Phi_2\Phi_1^{-1}\Big[|Q|\Big(\frac{1}{|Q|}\int_Q \Phi_1^{\frac{1}{t}}\Big(2Az_1(x)f(x)\Big)v^{\frac{1}{t}}(x)dx\Big)^t\Big] \tag{2.4}$$

for all cubes Q and all $f(.) \geq 0$.

Lemma 4. *Suppose $f(.)$ is a bounded nonnegative function with a compact support. Let $a > 2^n$ and $\Omega_k = \{x;\, (M_\alpha f)(x) > a^k\}$ for each integer k. Then one can find non overlapping maximal dyadic cubes satisfying the following:*

$$\Omega_k \subset \bigcup_j (3Q_{jk}); \tag{2.5}$$

$$4^{-n}a^k < |Q_{jk}|^{\frac{\alpha}{n}-1}\int_{Q_{jk}} f(y)dy \leq 2^{-n}a^k; \tag{2.6}$$

$$\Big(1-\frac{2^n}{a}\Big)|Q_{jk}| < |E_{jk}| \tag{2.7}$$

for some disjoints sets $E_{jk} \subset Q_{jk}$, and so

$$\sum_k \sum_j \mathbb{I}_{E_{jk}}(.) \leq 1. \tag{2.8}$$

This is a sort of discretization of M_α by means of Calderón-Zygmund, whose details of proof can be seen in [Pe] (p. 678, 681 and 682).

By the monotone convergence theorem and since the estimates do not involve the bound of $f(.)$, then it can be assumed that this functions is nonnegative, bounded and has a compact support. Therefore the chain of computations which yields to the conclusion in Theorem 1 is as follows

$$\int_{\mathbb{R}^n} \Phi_2\Big(z_2(x)(M_\alpha f)(x)\Big)u(x)dx = \sum_k \int_{\Omega_k\setminus\Omega_{k+1}} \Phi_2\Big(z_2(x)(M_\alpha f)(x)\Big)u(x)dx$$

$$\leq \sum_k \sum_j \int_{(3Q_{jk})} \Phi_2\Big(z_2(x)a^{k+1}\Big)u(x)dx \quad \text{by (2.5) and the definition of } \Omega_{k+1}$$

$$\leq \sum_k \sum_j \int_{(3Q_{jk})} \Phi_2 \Big[c_1 z_2(x) \Big(|3Q_{jk}|^{\frac{\alpha}{n}-1} \int_{3Q_{jk}} f(y)dy \Big) \Big] u(x)dx \text{ by (2.6)}$$

$$\leq \sum_k \sum_j \Phi_2 \Phi_1^{-1} \Big[|3Q_{jk}| \Big(\frac{1}{|3Q_{jk}|} \int_{(3Q_{jk})} \Phi_1^{\frac{1}{t}} \Big(2c_1 A z_1(x) f(x) \Big) v^{\frac{1}{t}}(x)dx \Big)^t \Big]$$

by Lemma 3

$$\leq \Phi_2 \Phi_1^{-1} \Big[c_2 \sum_k \sum_j |3Q_{jk}| \Big(\frac{1}{|3Q_{jk}|} \int_{(3Q_{jk})} \Phi_1^{\frac{1}{t}} \Big(2c_1 A z_1(x) f(x) \Big) v^{\frac{1}{t}}(x)dx \Big)^t \Big]$$

by the growth condition (1.6)

$$\leq \Phi_2 \Phi_1^{-1} \Big[c_3 \sum_k \sum_j |E_{jk}| \Big(\frac{1}{|3Q_{jk}|} \int_{(3Q_{jk})} \Phi_1^{\frac{1}{t}} \Big(2c_1 A z_1(x) f(x) \Big) v^{\frac{1}{t}}(x)dx \Big)^t \Big]$$

by property (2.7)

$$\leq \Phi_2 \Phi_1^{-1} \Big[c_3 \sum_k \sum_j \int_{E_{jk}} \Big[M \Phi_1^{\frac{1}{t}} \Big(2c_1 A z_1(.) f(.) \Big) v^{\frac{1}{t}}(.) \Big]^t (x)dx \Big]$$

recall that M is the Hardy-Littlewood maximal operator

$$\leq \Phi_2 \Phi_1^{-1} \Big[c_3 \int_{\mathbb{R}^n} \Big[M \Phi_1^{\frac{1}{t}} \Big(2c_1 A z_1(.) f(.) \Big) v^{\frac{1}{t}}(.) \Big]^t (x)dx \Big]$$

by property (2.8)

$$\leq \Phi_2 \Phi_1^{-1} \Big[c_4 \int_{\mathbb{R}^n} \Big[\Phi_1^{\frac{1}{t}} \Big(2c_1 A z_1(x) f(x) \Big) v^{\frac{1}{t}}(x) \Big]^t dx \Big]$$

since $t > 1$ and $M : L_1^t \to L_1^t$, and here $c_4 > 1$

$$\leq \Phi_2 \Phi_1^{-1} \Big[\int_{\mathbb{R}^n} \Phi_1 \Big(2c_1 c_4 A z_1(x) f(x) \Big) v(x)dx \Big]$$

since $\Phi_1^{\frac{1}{t}}(.)$ is convex function and $c_4 > 1$.

To achieve the proof of Theorem 1, it remains to give

Proof of Lemma 3.

The conclusion in Lemma 3 is equivalent to

$$\Theta_t(\lambda, Q) \leq \int_Q \Phi_1^{\frac{1}{t}} \Big(2A z_1(x) f(x) \Big) v^{\frac{1}{t}}(x)dx = \mathcal{B}(Q) \tag{2.9}$$

where $\lambda = |Q|^{\frac{\alpha}{n}-1} \int_Q f(y)dy$. On the other hand by condition (1.7) then

$$\mathcal{A}(Q) = \int_Q (\Phi_1^{\frac{1}{t}})^* \Big[\frac{\lambda^{-1} |Q|^{\frac{\alpha}{n}-1}}{A z_1(x) v^{\frac{1}{t}}(x)} \Theta_t(\lambda, Q) \Big] v^{\frac{1}{t}}(x)dx \leq \Theta_t(\lambda, Q) < \infty. \tag{2.10}$$

Estimate (2.9) can be obtained by using the Young inequality [which asserts that $s_1 s_2 \leq (\Phi_1^{\frac{1}{t}})^*(s_1) + \Phi_1^{\frac{1}{t}}(s_2)$] and (2.10) as follows

$$\begin{aligned}
2\Theta_t(\lambda, Q) &= \int_Q \lambda^{-1}|Q|^{\frac{\alpha}{n}-1} \times 2\Theta_t(\lambda, Q) f(y) dy \\
&= \int_Q \left[\frac{\lambda^{-1}|Q|^{\frac{\alpha}{n}-1}}{A z_1(y) v^{\frac{1}{t}}(y)} \Theta_t(\lambda, Q) \right] \times \Big[2A z_1(y) f(y) \Big] v^{\frac{1}{t}}(y) dy \\
&\leq \mathcal{A}(Q) + \mathcal{B}(Q) \leq \Theta_t(\lambda, Q) + \mathcal{B}(Q).
\end{aligned}$$

References

[Bl-Ke] S. Bloom, R. Kerman, *Weighted Orlicz space integral inequalities for the Hardy-Littlewood maximal operator*, Studia Math. **110** (1994), 149-167.

[Ch] J.C. Chen, *Weights and L_Φ-boundedness of the Poisson integral operator*, Israel J. Math. **81** (1993), 193-202.

[Go-Ko1] A. Gogatishvili, V. Kokilashvili, *Criteria of weighted inequalities in Orlicz classes for maximal functions defined on homogeneous type spaces*, Georgian Math. J. **1** (1994), 641-673.

[Go-Ko2] A. Gogatishvili, V. Kokilashvili, *Necessary and sufficient conditions for weighted Orlicz class inequalities for maximal functions and singular integrals. II*, Georgian Math. J. **2** (1995), 361-384.

[Ke-To] R. Kerman, A. Torchinsky, *Integral inequalities with weights for the Hardy maximal function*, Studia Math. **71** (1981), 277-284.

[Pe] C. Pérez, *Two weighted inequalities for potential and fractional type maximal operators*, Indiana Univ. Math. J. **43** (1994), 663-683.

[Qi] L. Qinsheng, *Two weighted mixed Φ-inequalities for the Hardy operator and the Hardy-Littlewood maximal operator*, J. London Math. Soc. **46** (1992), 301-318.

[Sa] E. Sawyer, *A charcterization for two weight norm inequality for maximal operators*, Studia Math. **75** (1982), 1-11.

[Su] Q. Sun, *Weighted norm inequalities on spaces of homogeneous type*, Studia Math. **101** (1992), 241-251.

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